In this lecture, we define the integral \( \int_0^f f(t) \, dt \) if \( f \) is a differentiable function and compute it for some basic functions.

First a reminder. We have defined the Riemann sums

\[
S_f(x) = h[ f(0) + f(h) + f(2h) + \cdots + f(kh) ] ,
\]

where \( h = 1/n \) is the mesh size is used to do the integral. This converges to 0 for the salt and pepper function because for every \( h = 1/n \), let \( k \) be the largest integer smaller than \( x/h \). This converges to 0 for the salt and pepper function because rational numbers have zero probability.

Remark: Many calculus books define the Riemann integral with partitions \( x_0 < x_1 < \ldots, x_n \) of the interval \([0,x]\) such that the maximal distance \((x_{k+1} - x_k)\) between neighboring \( x_j \) goes to zero. The Riemann sum is then \( S_n = \sum_{k=1}^{n} f(y_k)(x_{k+1} - x_k) \), where \( y_k \) is arbitrarily chosen inside the interval \([x_k, x_{k+1}]\). For continuous functions, the limiting result is the same the \( S(f) \) sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated and keep a wide mesh, where the function does not change much. This leads to numerical analysis of integrals.

1. Let \( f(x) = c \) be constant everywhere. Now \( \int_0^f f(t) \, dt = cx \). We can see also that \( c/n \leq S_n(x) \leq c(n+1)/n \).

2. Let \( f(x) = cx \). The area is half of a rectangle of width \( x \) and height \( cx \) so that the area is \( cx^2/2 \). Remark: we could also have added up the Riemann sum but thats more painful:

\[
S_n(x) = \frac{1}{n} \sum_{j=1}^{k} c_j = \frac{ck(k+1)/2}{n^2} .
\]

Taking the limit \( n \to \infty \) and using that \( k/n \to x \) shows that \( \int_0^f f(t) \, dt = cx^2/2 \).

3. Let \( f(x) = x^2 \). In this case we can not see the numerical value of the area geometrically. But since we have computed \( S[x^2] \) in the first lecture of this course and seen that it is \( \frac{x^3}{3} \) and since we have defined \( S_n(f) \to \int_0^f f(t) \, dt \) for \( h \to 0 \) and \([x^2] \to x^3 \) for \( h \to 0 \), we know that

\[
\int_0^x t^2 \, dt = \frac{x^3}{3} .
\]
This example actually computes the **volume of a pyramid** which has at distance \(t\) from the top an area \(t^2\) cross section. Think about \(t^2 dt\) as a slice of the pyramid of area \(t^2\) and height \(dt\). Adding up the volumes of all these slices gives the volume.

![Pyramid](image)

**Linearity of the integral** (see homework) \(\int_a^b f(t) + g(t) \, dt = \int_a^b f(t) \, dt + \int_a^b g(t) \, dt\) and \(\int_a^b \lambda f(t) \, dt = \lambda \int_a^b f(t) \, dt\).

**Upper bound**: If \(0 \leq f(x) \leq M\) for all \(x\), then \(\int_a^b f(t) \, dt \leq Mx\).

\[ \int_0^b \sin^2(\sin(t)) / x \, dt \leq x. \] **Solution**. The function \(f(t)\) inside the interval is nonnegative and smaller or equal to 1. The graph of \(f\) is therefore contained in a rectangle of width \(x\) and height 1.

We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

**If** \(f\) and \(g\) satisfy \(|f(x) - g(x)| \leq c\), then

\[ \int_0^T |f(x) - g(x)| \, dx \leq cT. \]

We know identities like \(S_n[x]^n = \frac{\sin^{n+1}}{n+1}\) and \(S_n \exp_k(x) = \exp_k(x)\) already. Since \([x]_k^b - [x]_k^a \to 0\) we have \(S_n[x]_k^b - S_n[x]_k^a \to 0\) and from \(S_n[x]_k^b = [x]_k^b + 1\). The other equalities are the same since \(\exp_k(x) = \exp(x) \to 0\). This gives us:

\[
\int_0^b t^n \, dt = \frac{t^{n+1}}{n+1} \\
\int_0^b e^t \, dt = e^b - 1 \\
\int_0^b \cos(t) \, dt = \sin(b) \\
\int_0^b \sin(t) \, dt = 1 - \cos(b)
\]

---

**Homework**

1. a) Find the integral \(\int_3^7 x^2 + 4t^3 + e^t \, dt\).
   
   b) Calculate \(\int_0^\infty e^{-t} \, dt\).

2. c) Find \(\int_0^{\pi/2} \cos(t) \, dt\).

3. The region enclosed by the graph of \(x\) and the graph of \(x^3\) has a propeller type shape as seen in the picture. Find its (positive) area.

4. Argue geometrically with areas why the following statements hold:
   
   - If \(f(x) = \max(f(x),0)\) and \(h(x) = \max(-f(x),0)\) have the property that \(f(x) = g(x) - h(x)\) and that \(g(x) \geq 0\) and \(h(x) \geq 0\).
   
   - Draw the graphs of the two functions \(g(x), h(x)\) for \(f(x) = \cos(3x)\) where \(0 \leq x \leq 2\pi\).

5. a) Find \(\int_0^3 |x - 1| \, dx\). Distinguish cases.
   
   b) Find \(\int_0^3 f(x) \, dx\) for \(f(x) = |x - |x - 1||\). Also here, distinguish cases.