Lecture 15: Review for first midterm

Major points

A function is **continuous**, if whenever \(x, y\) are close, also \(f(x), f(y)\) are close. Formally, for every \(a\) there exists \(b = f(a)\) such that \(\lim_{x \to a} f(x) = b\) for every \(a\). The Intermediate value theorem: \(f(a) > 0, f(b) < 0\) implies \(f\) having a root in \((a, b)\).

If \(f'(x) = 0\) and \(f''(x) > 0\) then \(x\) is a local minimum. If \(f'(x) = 0\) and \(f''(x) < 0\) then \(x\) is a local maximum. To find **global extrema**, compare local extrema and boundary values.

If \(f' > 0\) then \(f\) is increasing, if \(f' < 0\) it is decreasing. If \(f''(x) > 0\) it is **concave up**, if \(f''(x) < 0\) it is **concave down**. If \(f'(x) = 0\) then \(f\) has a horizontal tangent.

Hopital’s theorem tells that limits \(\lim_{x \to a} f(x)/g(x)\), where \(f(p) = g(p) = 0\) or \(f(p) = g(p) = \infty\) with \(g'(p) \neq 0\) are given by \(f'(p)/g'(p)\).

With \(Df(x) = (f(x + h) - f(x))/h\) and \(S(x) = h(f(h) + f(2h) + ... f(kh))\) we have \(SDf(kh) = f(kh) - f(0)\) and \(DS(f(kh)) = f(kh)\). This is a preliminary fundamental theorem of calculus.

Roots of \(f(x)\) with \(f(a) < 0, f(b) > 0\) can be obtained by the dissection method by applying the Newton map \(T(x) = x - f(x)/f'(x)\) again and again.

Algebra reminders

**Healing:** \((a + b)/a - b = a^2 - b^2\) or \(1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)\)

**Denominator:** \(1/a + 1/b = (a + b)/(ab)\)

**Exponential:** \(e^{a+b} = e^a e^b, a^b = e^b \log a\)

**Logarithm:** \(\log(ab) = \log(a) + \log(b), \log(a^b) = b \log(a)\)

**Trig functions:** \(\cos^2(x) + \sin^2(x) = 1, \tan(2x) = 2 \sin(x) \cos(x)\)

**Square roots:** \(\sqrt{a^2} = \sqrt{a}, a^{1/2} = 1/\sqrt{a}\)

Important functions

\[
\begin{align*}
\text{Polynomials} & \quad x^3 + 2x^2 + 3x + 1 \\
\text{Rational functions} & \quad (x + 1)/(x^3 + 2x + 1) \\
\text{Trig functions} & \quad 2 \cos(3x)
\end{align*}
\]

\[
\begin{align*}
\text{Exponential} & \quad 5e^{3x} \\
\text{Logarithm} & \quad \log(3x) \\
\text{Inverse trig functions} & \quad \arctan(x)
\end{align*}
\]

Important derivatives

\[
\begin{align*}
f(x) & \quad f'(x) \\
f(x) = x^n & \quad nf^{n-1} \\
f(x) = e^{ax} & \quad ae^{ax} \\
f(x) = \cos(ax) & \quad -a \sin(ax) \\
f(x) = \log(x) & \quad 1/x
\end{align*}
\]

Differentiation rules

| Addition rule | \((f + g)' = f' + g'\) |
| Scaling rule | \((cf)' = cf'\) |
| Quotient rule | \((f/g)' = (f'g - fg')/g^2\) |
| Chain rule | \((fg)' = f'g + fg'\) |
| Product rule | \((cf) = cf\) |
| Easy rule | simplify before deriving |

Extravagant problems

1. Build a fence of length \(2x + 2y = 12\) with dimensions \(x, y\) with maximal area \(A = xy\).
2. Find the largest area \(A = 4xy\) of a rectangle with vertices \((x, y), (-x, y), (-x, -y), (x, -y)\) inscribed in the ellipse \(x^2 + 2y^2 = 1\).
3. Which isosceles triangle of height \(h\) and base \(2x\) and area \(xh = 1\) has minimal circumference \(2x + 2\sqrt{x^2 + h^2}\)?
4. Where is the distance \(\sqrt{x^2 + y^2}\) of the parabola \(y = x^2 - 2\) to the point \((0, 0)\) minimal?
5. A cone of height \(h = 1 + x\) and radius \(r = \sqrt{1-x^2}\) is tightly enclosed by a unit sphere centered at height \(x\). Maximize the volume \(\pi r^2 h/3\) of the cone.
6. Maximize \(f(x) = \sin(x)\) on \([0, \pi]\).

Limit examples

\[
\begin{align*}
\lim_{x \to 0} \sin(x)/x & \quad \text{Hopital 0/0} \\
\lim_{x \to 0} (1 - \cos(x))/x^2 & \quad \text{Hopital 0/0 twice} \\
\lim_{x \to \infty} \exp(x)/(1 + \exp(x)) & \quad \text{Hopital} \\
\lim_{x \to 0} x \log(x) & \quad \text{Ho}\text{l}it\text{a} \infty/\infty \\
\lim_{x \to -\infty} -x + 1/(x + 5) & \quad \text{no work necessary}
\end{align*}
\]

Important things

Summation and taking differences is at the heart of calculus.

The 3 major types of discontinuities are jump, oscillation, infinity.

Dissection and Newton methods are algorithms to find roots. Dissection needs continuity, Newton needs differentials.

The fundamental theorem of trigonometry is \(\lim_{x \to 0} \sin(x)/x = 1\).

The derivative is the limit \(Df(x) = [f(x + h) - f(x)]/h\) as \(h \to 0\). It is called rate of change.

The rule \(D(1 + h)^{1/2} = (1 + h)^{1/2}\) leads to \(\exp(x) = \exp(x)\).

More Examples

1. Is \(\log|z|\) continuous at \(x = 0\). Answer: yes
2. Is \(\log(1/|z|)\) continuous at \(x = 0\). Answer: no
3. Find \(\lim_{x \to 1} (x^3 - 1)/(x^{1/4} - 1)\). Answer: 4/3.
4. Find \(\lim_{x \to 1} 5(x - 5)/(x - 1)\). Answer: 5.
5. Find \(\lim_{x \to a} -\sqrt{1 - 2x^2}\). Answer -1/6.
6. Find \(\arcsin(5x^2)\). Answer: \(10x(1 - 25x^4)^{-1/2}\).