Homework 15: Data fitting

This homework is due on Wednesday, March 11, respectively on Thursday, March 12, 2016.

1 a) Find the least square solution $x^*$ of the system $Ax = b$ with

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}, \text{ and } b = e_2.$$ 

b) What is the matrix $P$ which projects on the image of $A$?

2 Find the function $y = f(x) = ax^2 + bx^3$, which best fits the data

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

3 A curve of the form

$$y^2 = x^3 + ax + b$$

is called an elliptic curve in Weierstrass form. Elliptic curves are important in cryptography. Use data fitting to find the best parameters $(a, b)$ for an elliptic curve given the following points:

$$(x_1, y_1) = (1, 2)$$
$$(x_2, y_2) = (-1, 0)$$
$$(x_3, y_3) = (2, 1)$$
$$(x_4, y_4) = (0, 1)$$
A graphic from the Harvard Management Company Endowment Report of September 2014 is shown to the right. Assume we want to fit the growth using functions $1, x, x^2$ and assume the years are numbered starting with $1990 = 0, 1995 = 1, 2000 = 2, 2005 = 3, 2010 = 4, 2016 = 5$. What is the best parabola $a + bx + cx^2 = y$ which fits these data?

\begin{center}
\begin{tabular}{|c|c|}
\hline
quintenium & billions \\
\hline
0 & 5 \\
1 & 15 \\
2 & 19 \\
3 & 23 \\
4 & 27 \\
5 & 37 \\
\hline
\end{tabular}
\end{center}

For this fitting problem, the solution is not unique.

\begin{center}
\begin{tabular}{|c|c|}
\hline
x & y \\
\hline
0 & 1 \\
0 & 2 \\
0 & 3 \\
\hline
\end{tabular}
\end{center}

\begin{itemize}
\item[5] a) Draw the situation and find different regression lines which are optimal.
\item[5] b) Now write down the corresponding fitting problem for linear functions $f(x) = ax + c = y$ by finding the matrix $A$ and the vector $b$. What is going on?
\end{itemize}

\section*{Data fitting}

Given a system $Ax = b$. Any solution of $(A^T A)x = A^T b$ is called a \textbf{least square solution} (these always exist). (Reason: solve $A^T (Ax - b) = 0$ for $x$, assuring that $Ax - b$ is perpendicular to $\text{im}(A)$.) The least square solution is unique if $A$ has a trivial kernel. In that case $x = (A^T A)^{-1}A^T b$. The matrix $A(A^T A)^{-1}A^T$ is now the projection matrix onto $\text{im}(A)$. If the columns of $A$ are orthonormal, this simplifies to $P = AA^T$. 