Homework 14: Orthogonal transformations

This homework is due on Monday, March 7, respectively on Tuesday, March 8, 2016.

1. Determine from each of the following matrices whether they are orthogonal:
   a) \[
   \begin{bmatrix}
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 & 1 \\
   \end{bmatrix}
   \]
   /2, b) \[
   \begin{bmatrix}
   1 & 1 & 1 & 1 \\
   1 & -1 & 1 & -1 \\
   1 & 1 & -1 & -1 \\
   1 & -1 & -1 & 1 \\
   \end{bmatrix}
   \]
   /2, c) \[
   \begin{bmatrix}
   1 & 0 & 0 & 0 \\
   0 & -1 & 0 & 0 \\
   0 & 0 & -1 & 0 \\
   0 & 0 & 0 & 1 \\
   \end{bmatrix}
   \]
   d) \[
   \begin{bmatrix}
   0 & 0 & 1 & 0 \\
   0 & -1 & 0 & 0 \\
   1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 \\
   \end{bmatrix}
   \]
   e) \[
   \begin{bmatrix}
   \cos(1) & \sin(1) & 0 & 0 \\
   -\sin(1) & \cos(1) & 0 & 0 \\
   0 & 0 & \cos(2) & \sin(2) \\
   0 & 0 & \sin(2) & -\cos(2) \\
   \end{bmatrix}
   .
   \]

2. If \( A, B \) are orthogonal, then
   a) Is \( A + B \) is orthogonal? b) is \( 3A \) is orthogonal? c) Is \( A^T \) is orthogonal? d) \( B^{-1} \) is orthogonal? e) Is \( B^{-1}AB \) orthogonal?

3. a) Matrices of the form \[
   \begin{bmatrix}
   a & b \\
   -b & a \\
   \end{bmatrix}
   \]
   can be multiplied and the result is of the same form. These rotation dilation matrices are also called “complex numbers”! Which of these matrices plays the role of \( i = \sqrt{-1} \), that is, which of them has the property that \( A^2 = -1 \) (where \(-1 \) means \(-I_2\))? 
   b) Figure out the formula for the multiplication \((a + ib)(c + id)\) of complex numbers by looking at the product \[
   \begin{bmatrix}
   a & b \\
   -b & a \\
   \end{bmatrix}
   \begin{bmatrix}
   c & d \\
   -d & c \\
   \end{bmatrix}
   .
   \]
   c) If you draw complex numbers \( a + ib, \ and \ c + id \) as vectors, what is the multiplication geometrically?

4. Mathematicians for a long time looked for higher dimensional analogues of the complex numbers. Matrices of the form \( A(p, q, r, s) = \)

\[
\begin{bmatrix}
p & -q & -r & -s \\
q & p & s & -r \\
r & -s & p & q \\
s & r & -q & p \\
\end{bmatrix}
\]
are called quaternions. They were invented by Hamilton.

a) Find a basis for the set of all the matrices above.

b) Check that every unit sphere \( p^2 + q^2 + r^2 + s^2 \) in the four dimensional space of quaternions corresponds to an orthogonal matrix.

5 a) Explain why the identity matrix is the only \( n \times n \) matrix that is orthogonal, upper triangular and has positive entries on the diagonal.  
b) Show that the \( QR \) factorization of an invertible \( n \times n \) matrix \( A \) is unique. That is, if \( A = Q_1R_1 \) and \( A = Q_2R_2 \) are two factorizations, argue why \( Q_1 = Q_2 \) and \( R_1 = R_2 \).

**Orthogonal transformations**

The transpose \( A_{ij}^T = A_{ji} \) satisfies \( (AB)^T = B^T A^T \) and \( (A^T)^T = A \). The rank of the transpose is the same as the rank of \( A \). An \( n \times n \) matrix \( A \) is orthogonal if \( A^T A = 1 = 1_n \). The linear transformation of an orthogonal matrix is called an **orthogonal transformation**. It preserves length and angle. The column vectors of an orthogonal matrix forms an orthonormal basis. The product of two orthogonal matrices is orthogonal. The inverse \( A^{-1} \) is orthogonal and given by \( A^T \). Examples of orthogonal transformations are rotations or reflections or the identity matrix.