Homework 10: Coordinates

This homework is due on Wednesday, February 24, respectively on Thursday, February 25, 2016.

1. What are the $\mathcal{B}$-coordinates of the vector $\vec{v}$ in the basis $\mathcal{B}$.

   $$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

2. What is the matrix $B$ for the transformation $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ in the basis $\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$.

3. Chose a suitable basis to solve the following two problems:
   a) Find the matrix $A$ which belongs to a reflection at the plane $x + y + 2z = 0$.
   b) Find the matrix $A$ which belongs to the reflection at the line spanned by $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

4. Find the matrix $A$ corresponding to the orthogonal projection onto the plane spanned by the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

5. The whole plane is covered with regular hexagons "Graphene", where the first basis vector is $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. a) Find $w$ so that $\mathcal{B} = \{ v, w \}$ is the basis as seen in the picture.
b) What are the standard coordinates of the vector given in the \( B \) basis as \[
\begin{bmatrix}
2 \\
-1
\end{bmatrix}
\]?

c) Is the point with \( B \) coordinate \[
\begin{bmatrix}
17 \\
21
\end{bmatrix}
\] a vertex of a hexagon or the center of one?

Source: CNN

Coordinates

Given a basis \( B = \{\vec{v}_1, \ldots, \vec{v}_n\} \) of a linear space \( V \), every \( \vec{w} \) in \( V \) can be written as \( \vec{w} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \), where \( c_i \) are the coordinates of \( \vec{v} \). The basis defines a matrix \( S = \begin{bmatrix}
| & | & | \\
\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n
\end{bmatrix} \). Since \( S\vec{c} = \vec{w} \) we get \( \vec{c} = S^{-1}\vec{w} \).

If \( A \) is a matrix given in the standard basis \( e_1, \ldots, e_n \) and \( B \) is the matrix written in the basis \( B \), then \( B = S^{-1}AS \). We say \( B \) is similar to \( A \). Why do we want to change basis? Because it is convenient: for example if \( \vec{v}_1, \vec{v}_2 \) are non-parallel vectors in a plane and \( \vec{v}_3 \) is perpendicular to the plane then a projection onto the plane is the matrix \[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]. The matrix in the standard basis is then \( A = SBS^{-1} \).