THE DIRAC OPERATOR OF A GRAPH

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Jun 5
ILAS 2013
LEONARD EULER

1707-1783
GRAPHS

finite  simple
no loops
no multiple connections

1736

Wednesday, June 5, 13
GUSTAV KIRCHHOFF

1824-1887
ADJACENCY MATRIX

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
\end{pmatrix}
\]
\[ L = B - A = \]

\[ B = \text{Degree matrix} \]

\[ \sigma(L) = \{0, 1, 3, 4\} \geq 0 \]
HENRI POINCARE

1854-1912
\[ \text{oriented incidence matrix} \]

\[
\begin{array}{ccccc}
a & -1 & 1 & 0 & 0 \\
b & 0 & -1 & 0 & 1 \\
c & 0 & -1 & 1 & 0 \\
d & 0 & 0 & -1 & 1 \\
\end{array}
\]

\[ d_0 = \]

\[ d \ f = \text{gradient (f)} \]
CURL

\[ \mathbf{d} = \mathbf{b} \]

\[ \mathbf{d}_1 = \begin{bmatrix} 0 & 1 & -1 & -1 \end{bmatrix} \]

\[ \mathbf{d} \mathbf{F} = \text{curl} \ (\mathbf{F}) \]
\[ \int \text{curl}(F) \, dS = \int_{C} F \, dr \]
**CURL OF GRAD**

\[ d_1 \begin{array}{lll} d_0 \end{array} = \]

\[
\begin{array}{cccc}
0 & 1 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
= \]

\[ d \cdot d = 0 \]
curl(F)=0
but F not equal to grad(f)
DIVERGENCE

$$d_0^* =$$

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
\[ d_1 = \begin{array}{c|ccc}
   & 0 & 1 & 1 \\
\hline
   a & 0 & -1 & -1 \\
   b & -1 & 0 & -1 \\
   c & -1 & -1 & 0 \\
   d & -1 & -1 & -1 \\
\end{array} \]
$\text{DIV(\text{GRAD}) = LAPLACE}$

$L_0 =$

\[
\begin{align*}
1 & \quad 0 & \quad 0 & \quad 0 \\
-1 & \quad -1 & \quad -1 & \quad 0 \\
0 & \quad 0 & \quad 1 & \quad -1 \\
0 & \quad 1 & \quad 0 & \quad 1
\end{align*}
\]

$d_0^* \quad d_0 =$

\[
\begin{align*}
1 & \quad -1 & \quad 0 & \quad 0 & \quad 0 \\
-1 & \quad 3 & \quad -1 & \quad -1 \\
0 & \quad -1 & \quad 2 & \quad -1 \\
0 & \quad -1 & \quad -1 & \quad 2
\end{align*}
\]

Diagram:

- Node 1
- Node 2
- Node 3
- Node 4

Connections:
- 1 to 2
- 2 to 3
- 3 to 4
- 4 to 1
\[ L_k = d_k^* d_k + d_{k-1}^* d_{k-1} \]

\[ L = (d + d^*)^2 \]
PAUL DIRAC

1902-1984
DIRAC OPERATOR

\[ D = \frac{d}{2} + \frac{d^*}{2} \]

\[ v_0 = 4 \]

\[ v_1 = 4 \]

\[ v_2 = 1 \]

\[ \chi(G) = 1 \]
LAPLACE-BELTRAMI MATRIX

\[ L = (d + d) \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \]

is block diagonal.
AGAIN:

vertices  edges  triangles

1 2 3 4 5 6  a  b  c  d  e  f  g  A  B

\[ D^2 = L \]
ENRICO BETTI

1823-1892
\[ H^k(G) = \ker(d_k)/\text{im}(d_{k-1}) \]

k'th cohomology group

\[ b_k(G) = \dim(H^k(G)) \]

Betti number

\[ b_0 - b_1 + b_2 \ldots = \chi(G) \]

cohomological Euler characteristic
$v_0 = 8 \quad \chi = -4 \quad \text{dim} = 1$

$v_1 = 12$

$b_0 = 1 \quad b_1 = 5$
EULER CHARACTERISTIC

\[ \chi(G)_{\text{simp}} = v - v + v - \ldots = 8 - 11 + 2 \]

\[ \chi(G)_{\text{coho}} = b_0 - b_1 + b_2 - \ldots = 1 - 2 \]

Example: no tetrahedra, then \[ v - e + f = l - g \]
Proof. $C_m$ space of $m$ forms

$Z_m = \ker(d_m)$ space of $m$ cocycles

$R_m = \text{ran}(d_m)$ space of $m$ coboundaries

$z_m = \nu_m - r_m$ rank-nullity theorem

$b_m = z_m - r_{m-1}$ definition of cohomology

$$\sum_m (-1)^m (\nu_m - b_m) = \sum_m (-1)^m (r_m - r_{m-1}) = 0$$ QED
WILLIAM HODGE

1903-1975
Hodge Theorem

\[ H^m (G) \cong \text{Ker}(L_m) \]

**Proof:**

\( L f = 0 \) is equivalent to \( df = d^* f = 0 \)

\[ <f, Lf> = <d^* f, d^* f> + <df, df> \]

\( \text{im}(d) + \text{im}(d^*) + \text{ker}(L) = \mathbb{R}^n \)

\[ <dg, d^* h> = 0 \). Also \( \text{ker}(L) \) and \( \text{im}(L) \) are perp.

\( dg = 0 \) implies \( g = df + h \)

can match cohomology class \([g]\) - harmonic form
PDE ANALOGUES ON GRAPHS

Lu = \mathbf{j}

u' = - \mathbf{L} \mathbf{u}

u' = \mathbf{i} \mathbf{L} \mathbf{u}

u'' = - \mathbf{L} \mathbf{u}

d\mathbf{F} = 0, d^* \mathbf{F} = \mathbf{j}

Poisson

Heat

Schrödinger

Wave

Maxwell
HEAT EQUATION SOLUTION

\[ u' = -L \ u \]

\[ u(t) = e^{-tL} \ u(0) \]

Heat kernel
HEAT KERNEL

\[ \exp(-Lt) \]

projection onto harmonics
WAVE EQUATION SOLUTION

\[ u'' = -L u \]

\[ u'' = -D^2 u \]

\[ (\partial + iD)(\partial - iD)u = 0 \]

\[ u(t) = \cos(Dt)u(0) + \sin(Dt)Du'(0) \]

d'Alembert type solution
WAVE MATRIX

\[ \cos(Dt) \]
ERICH HUECKEL

1896-1980
HUECKEL THEORY

caffeine

$L_0 =$
ED WITTEN

1951-
SYMMETRY

$\lambda$ eigenvalue $\Rightarrow$ $-\lambda$ eigenvalue

Proof:

$Df = \lambda f$

$\lambda Pf = P \lambda f$

$= PDf = -DPf$

$P^2 = 1$, $D^2 = L$

Super symmetry
ANTI MATTER

Quarks

Leptons

Antiquarks

Antileptons
MC KEAN - SINGER

1930-1924-
SUPER TRACE

\[ \text{str}(A) = \text{tr}(P \ A) \]

Example:
\[ \text{str}(1) = \text{tr}(P) = \chi(G) \]
Proof. Each Boson $f$ matches a Fermion $Df$

$$\text{str}(\exp(-t \ L)) = \chi(G)$$

$$Lf = \lambda f$$

$$L^{n}Df = D^{n}Lf = D^{n}\lambda f = \lambda D^{n}f$$

$$\text{str}(L^{n}) = 0 \quad \text{for} \quad n > 0$$

$$\text{str}(\exp(-t \ L)) = \text{str}(1) = \chi(G)$$
\[ v_0 = 7 \quad v_1 = 7 \quad v_2 = 1 \]
\[ \chi = 1 \quad \text{dim} = 1.38 \]
\[ b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \]
\[ v_0 = 92 \quad v_1 = 270 \quad v_2 = 180 \]
\[ \chi = 2 \quad \text{dim} = 2. \]
\[ b_0 = 1 \quad b_1 = 0 \quad b_2 = 1 \]
\[ v_0 = 100 \quad v_1 = 300 \quad v_2 = 200 \]
\[ \chi = 0 \quad \text{dim} = 2. \]
\[ b_0 = 1 \quad b_1 = 2 \quad b_2 = 1 \]
\[ v_0 = 62 \quad v_1 = 180 \quad v_2 = 120 \]
\[ \chi = 2 \quad \text{dim} = 2. \]
\[ b_0 = 1 \quad b_1 = 0 \quad b_2 = 1 \]
\[ v_0 = 17 \quad v_1 = 52 \quad v_2 = 36 \]
\[ \chi = 1 \quad \dim = 2. \]

\[ b_0 = 1 \quad b_1 = 0 \quad b_2 = 0 \]
MARK KAC

1914-1984
HEAR THE SHAPE?

- Marc Kac: Can one hear the shape of a drum?

- Norman Biggs, Fan Chung and others: To which degree can one hear the shape of a graph?
In spectroscopy, one attempts to recover the shape or chemical composition of an object from the characteristic frequencies of sound or light emitted. Mark Kac’s question, “Can one hear the shape of a drum?” asks whether two membranes (drumheads) which vibrate at the same characteristic frequencies must have the same shape. We answer Kac’s question negatively by constructing pairs of exotic-shaped sound-alike drums. We also listen to a computer simulation, produced by Dennis DeTurck, of the sounds of these drums.

Carolyn Gordon
Dartmouth College

“YOU CAN’T HEAR THE SHAPE OF A DRUM”

THURSDAY, APRIL 4, 2013
5:00PM
127 HAYES-HEALY CENTER
ISOSPECTRAL TORI

Also Dirac isospectral

Sunada: general technique
HUNGERBUHLER-HALBEISEN
ISOSPECTRAL GRAPHS

H-H Graphs are Dirac Isospectral!

proof: $L_0, L_2$, McKean Singer: $L_1$

Wednesday, June 5, 13
ISOSPECTRAL GRAPHS

Isospectral with respect to $L_0$ but not for $L_1$

Haemers-Spence
THE END

5 topics currently cooking:

1) Graph limits
2) Zeta Functions
3) Integrable Deformation
4) Cauchy-Binet
5) Pseudo Determinants
This volume is dedicated to Victor Borisovich Lidskii who died on 29 July 2008. It is a collection of papers in subject areas related to Lidskii's work, some of which are written by people who knew him well. The editors of this volume, Michael Levitin and Dmitri Vassiliev, are former students of Lidskii, both at undergraduate and PhD level.

Lidskii was born in Odessa in the Soviet Union on 4 May 1924. As with most men of his generation, his life was severely affected by the Second World War. Lidskii finished secondary school on 20 June 1941, two days before Germany invaded the Soviet Union. After his parents' divorce he remained with his father and stepmother, so at that time Lidskii was living in the city of Bobruisk (now Minsk region, Belarus). His mother lived separately and died in the Minsk ghetto in 1942.

Lidskii escaped the advancing German troops and found himself, on his own, in the city of Saratov. Shortly afterwards he was drafted into the Red Army and sent to a military intelligence school. The main reason for him being assigned to military intelligence was the fact that he knew German. Lidskii studied German at school and he knew some Yiddish as well (which was helpful in learning German). After

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The correct date of his birth was established only in the mid 1970s. Lidskii's original birth certificate was lost and until he got an authorised copy he considered 5 May to be his birthday and put 5 May in all forms and documents.
A, B selfadjoint, C=A-B
\[ \lambda_1, \lambda_2, \lambda_3 \ldots \text{eigenvalues of } A \]
\[ \mu_1, \mu_2, \mu_3 \ldots \text{eigenvalues of } B \]

\[ \sum_j |\lambda_j - \mu_j| \leq \sum_{k,l} |C_{kl}| \]

\[ \sum_i |\gamma_i| = \sum_i (-1)^{m_i} \gamma_i = \sum_{i,k,l} (-1)^{m_i} U_{ik} C_{kl} U_{il} \]

\[ \leq \sum_{k,l} |C_{kl}| \cdot \sum_i (-1)^{m_i} U_{ik} U_{il} \leq \sum_{k,l} |C_{kl}| \cdot \]
BERNHARD Riemann

1826-1866
ZETA FUNCTION

\[ \zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s} \]

\[ = (1 + \exp(i\pi s)) \sum_{\lambda > 0} \lambda^{-s} \]

(choose branch)

-\(\zeta(s)\) (\(1 + \exp(\pi s)\)) times Riemann \(\zeta\) function in Circle case (analytic!).

-\(\zeta(s)\) always analytic for graphs. Explicit in \(C_n\) case.
\[ \zeta(C_n) = \prod_{k=1}^{n-1} (1 + \exp(i \pi s)) \sum_{k=1}^{n-1} (2 \sin(\pi k/n))^{-s} \]

we see an animation with the roots in the case from

\[ n=10 \text{ to } 650 \]
ISOSPECTRAL EVOLUTIONS

\[ D' = [B, D] \quad , \quad B = d - d^* \]

\[ D(t) = d^* + d + b \]

isospectral deformation
\[ D(t) = U(t)^* D(0) U(t) \]
\[ \det(F^T G) = \sum_{P} \det(F_P^T) \det(G_P) \]

classical Cauchy-Binet (1812)
CAUCHY BINET FOR PSEUDO DETERMINANTS

\[ \text{Det}(A) = \text{product of nonzero eigenvalues} \]
Th sum is over all \( k \times k \) minors with 
\( k = \min(\text{Ran}(F^T G), \text{Ran}(F G^T)) \)

\[ \text{Det}(F^T G) = \sum_{P} \det(F_{P})\det(G_{P}) \]

counts signed trees in a double cover of simplex graph (matrix tree theorem)
PAUL ERDOS

1913-1996
STATISTICS OF \log |\text{PSEUDODET}(D(G))| |

all connected graphs of order 7

1'866'256 graphs
THE END