SEMINAR NOTES: G-BUNDLES, STACKS AND BG (SEPT. 15, 2009)

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1. G-bundles

1.1. Let $G$ be an affine algebraic group over a ground field $k$. We assume that $k$ is algebraically closed.

Definition 1.2. A $G$-bundle $P$ over a scheme $X$ is a sheaf on the category $\text{Sch}/X$ (i.e., the category of schemes over $X$ with a flat topology) which is a torsor for the sheaf of groups $\Gamma(Y/X) \to \text{Hom}(Y,G)$, in the flat topology.

Let $Y/X$ be a scheme, such that $\Gamma(Y,P) \neq \emptyset$. Choosing a section, we obtain a Čech 1-cocycle on $\phi: Y \times_X Y$ with values in $G$.

Definition 1.3. A $G$-bundle over $X$ is a scheme $\tilde{X}$ over $X$, acted on by $G$, such that, locally in the flat topology, $\tilde{X}$ is isomorphic to the product, i.e., there exists a faithfully flat morphism $Y \to X$, such that $\tilde{Y} := Y \times_X \tilde{X} \cong Y \times G$ as $G$-schemes.

Let’s see why the two definitions are equivalent. Having $\tilde{X}$ we define $P$ to be the sheaf $Y/X \mapsto \text{Hom}(Y,\tilde{X}) = \text{Hom}_Y(Y,\tilde{Y})$. Going in the opposite direction, let $Y \to X$ be a faithfully flat cover such that $\Gamma(Y,P) \neq \emptyset$. Set $\tilde{Y} := Y \times G$. The Čech cocycle $\phi$ introduced above defines a descent datum for $\tilde{Y}$ with respect to $Y \to X$.

The same construction also establishes the following: given a $G$-bundle $\mathcal{P}$ and a scheme $Z$ acted on by $G$, we can form a scheme $Z_P := G \backslash (\tilde{X} \times Z)$ over $X$. We call it "the fiber-bundle over $X$ associated to $\mathcal{P}$ and the $G$-scheme $Z$".

Remark 1.4. Technically speaking, for this construction to work we need to assume something about $Z$. E.g., if $Z$ affine is always OK. More generally, we can take $Z$ projective or quasi-projective endowed with a polarization (i.e., an ample line bundle), which is $G$-equivariant. This assumption will always be satisfied in our example, where $Z$ is of the form $G/G_1$ for a subgroup $G_1 \subset G$.

Here is yet one more equivalent definition:

Definition 1.5. A $G$-bundle over $X$ is a scheme $\tilde{X}$ over $X$, acted on by $G$, such that $\tilde{X} \to X$ is faithfully flat, and the morphism $G \times \tilde{X} \to \tilde{X} \times \tilde{X}$ (where the first component is the projection $\tilde{X}$, and the second component is the action map), is an isomorphism.

It is easy to see that if $\tilde{X}$ satisfies Definition 1.3, then it also satisfies Definition 1.5: indeed, it’s enough to check the corresponding properties after a faithfully flat base change $Y \to X$, when the statement is evident b/c we are in the product situation.

Vice versa, having $\tilde{X}$ satisfying Definition 1.5, we can take $Y := \tilde{X}$, and it satisfies Definition 1.3.

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1.6. Assume now that $G$ is smooth (always true if we are in char. 0). In this case we claim that every $G$-torsor is locally trivial in the smooth (and, hence, the étale) topology.

Indeed, the map $\tilde{X} \to X$ is smooth (b/c it becomes such after a faithfully flat base change). So, $Y := \tilde{X} \to X$ is the sought-for smooth map, over which $\tilde{Y}$ is isomorphic to the product $Y \times G$.

To get the étale triviality, having a smooth map $Y \to X$, locally in $Y$, we can factor it as $Y \to X \times \mathbb{A}^n \to X$, corresponding to any point $a \in \mathbb{A}^n$.

1.7. Let $G = GL_n$. We claim that a $GL_n$-bundle is the same as a rank-$n$ vector bundle.

In one direction, having a $GL_n$-bundle $P$, we define the vector bundle $E := E_0^P$, where $E_0$ is the standard $n$-dimensional representation of $GL_n$, and the subscript $P$ means the associated bundle construction introduced above.

In the other direction, given a vector bundle $E$, we define a $GL_n$-torsor that assigns to $Y \to X$ the set $\text{Isom}_Y(E_0^Y, E_Y)$, where $E_0^Y$ is the trivial rank-$n$ bundle.

1.8. Our goal is to prove the following:

**Proposition 1.9.** A $G$-bundle on $X$ is the same as a tensor (=braided monoidal) exact functor $\text{Rep}(G) \to \text{Vect}_X$, where $\text{Rep}(G)$ is the tensor category of finite-dimensional representations of $G$, and $\text{Vect}_X$ is the tensor category of vector bundles on $X$.

**Proof.** In one direction, having a $G$-bundle $P$, we define a functor $F : \text{Rep}(G) \to \text{Vect}_X$ by $V \mapsto V_P$ (again, the associated bundle construction).

In the opposite direction, let $F$ be a tensor functor as above. We are going to produce a $G$-scheme $\tilde{X}$ over $X$. Consider the ring of functions on $G$, denoted $\text{Reg}(G)$, viewed as a (infinite-dimensional) representation of $G$ with respect to the left-regular action of $G$. We have $\text{Reg}(G) \simeq \lim V_i$, where $V_i$ are finite-dimensional representations. It is easy to see that the commutative ring structure on $\text{Reg}(G)$ endows the quasi-coherent sheaf $A := F(\text{Reg}(G)) := \lim F(V_i)$ with a commutative multiplication. Set

$$\tilde{X} := \text{Spec}_X(A).$$

The right-regular action of $G$ on $\text{Reg}(G)$ defines a $G$-action on $\tilde{X}$ as a scheme over $X$.

Since all $F(V_i)$ are vector bundles, and, in particular, flat, so is $A$. By the exactness of $F$, we have a short exact sequence of quasi-coherent sheaves on $X$:

$$0 \to \mathcal{O}_X \to A \to F(\text{Reg}(G)/k) \to 0,$$

where $k$ denotes the trivial representation. Since $F(\text{Reg}(G)/k)$ is $\mathcal{O}_X$-flat (by the same argument as above), for any point $x \in X$, we obtain that the map $k \to A_x$ is injective. In particular, $A_x \neq 0$. Hence, $\tilde{X}$ is faithfully flat over $X$.

Finally,

$$A \otimes A = F(\text{Reg}(G)) \otimes F(\text{Reg}(G)) \simeq F(\text{Reg}(G) \otimes \text{Reg}(G)).$$

However, for any $V \in \text{Rep}(G)$,

$$V \otimes \text{Reg}(G) \simeq \text{Reg}(G) \otimes V.$$
as $G$-representations, where $V$ is the vector space underlying the representation $V$. In particular, $\text{Reg}(G) \otimes \text{Reg}(G) \simeq \text{Reg}(G) \otimes \text{Reg}(G)$, and, hence,

$$A \otimes A \simeq A \otimes \text{Reg}(G),$$

compatible with the algebra structure. So $\tilde{X} \times_X \tilde{X} \simeq \tilde{X} \times G$. Moreover, the latter isomorphism respects the actions of $G$ on both sides, and hence, $\tilde{X}$ satisfies Definition 1.5.

\[\square\]

2. A refresher on stacks

Here we'll just repeat the main points from Toly's talk.

2.1. Unless specified otherwise, we'll consider the category of affine schemes of finite type $\text{Aff}^{ft}$, which is the same as the opposite category of finitely generated $k$-algebras.

We consider presheaves of groupoids on $\text{Aff}^{ft}$, i.e., assignments $S \in \text{Aff}^{ft} \mapsto F(S)$, where $F(S)$ is a groupoid; for every $\alpha : S_1 \to S_2$ a functor $F(\alpha) : F(S_2) \to F(S_1)$, for any $\alpha : S_1 \to S_2$ and $\beta : S_2 \to S_3$ a natural transformation (automatically an isomorphism of functors, as we’re dealing with groupoids)

$$F(\alpha, \beta) : F(\alpha) \circ F(\beta) \Rightarrow F(\beta \circ \alpha),$$

such that the natural condition holds for 3-fold compositions. Sometimes we’ll write $\alpha^*$ instead of $F(\alpha)$, as we think of it as the pull-back.

Morphisms between presheaves are defined naturally: for two presheaves $\mathcal{F}_1, \mathcal{F}_2$ a morphism $f$ is a datum for every $S \in \text{Aff}^{ft}$ of a functor $f(S) : \mathcal{F}_1(S) \to \mathcal{F}_2(S)$, and for every $\alpha : S_1 \to S_2$ of a natural transformation

$$f(S_2) \circ F_1(\alpha) \Rightarrow F_2(\alpha) \circ f(S_1),$$

compatible with the data of $\mathcal{F}_1(\alpha, \beta), \mathcal{F}_2(\alpha, \beta)$. Morphisms $\mathcal{F}_1 \to \mathcal{F}_2$ form a category. Isomorphisms should also be understood naturally:

$$f : \mathcal{F}_1 \to \mathcal{F}_2 : g$$

are mutually inverse iff $f \circ g$ and $g \circ f$ are isomorphic to the identity self-functors of of $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively.

For three presheaves and morphisms $f : \mathcal{F}_1 \to \mathcal{F}_2 \leftarrow \mathcal{F}_3 : g$ we form the Cartesian product $\mathcal{F}_1 \times \mathcal{F}_3$ naturally: $(\mathcal{F}_1 \times \mathcal{F}_3)(S)$ is the category of triples

$$\{a_1 \in \mathcal{F}_1(S), a_3 \in \mathcal{F}_3(S), \gamma : f(a_1) \simeq g(a_3) \in \mathcal{F}_3(S)\}.$$

Morphisms between such triples are defined naturally: they must respect the data of $\gamma$.

2.2. Note that every presheaf of sets can be viewed as a presheaf of groupoids. For a scheme $X$ we define the presheaf $\underline{X}$ to be one corresponding to the presheaf of sets $S \mapsto \text{Hom}(S, X)$.

Yoneda’s lemma says that for an affine scheme $S$, the category $\text{Hom}(S, \mathcal{F})$ is naturally equivalent to $\mathcal{F}(S)$.

We say that $\mathcal{F}$ is schematic if it is equivalent to a presheaf of the form $\underline{X}$ where $X$ is a scheme.
2.3. We say that a map of presheaves \( f : \mathcal{F}' \to \mathcal{F} \) is schematic if "its fibers are schemes". We formalize this idea as follows: we require that for every \( S \in \text{Aff}^{ft} \) and \( a \in \mathcal{F}(S) \), thought of as a map of presheaves \( S \to \mathcal{F} \), the Cartesian product

\[
S \times_{\mathcal{F}} \mathcal{F}'
\]

is a schematic presheaf. Again, by Yoneda, the map of presheaves

\[
S \times_{\mathcal{F}} \mathcal{F}' \to S
\]

corresponds to a map of schemes \( S' \to S \), where \( S' \) is such that \( S' \simeq S \times_{\mathcal{F}} \mathcal{F}' \).

For a schematic map of presheaves it makes sense to require that it be an open embedding/closed embedding/affine/projective/flat/smooth, etc. In fact, any property of morphisms stable with respect to the base make sense. By definition, this means that the corresponding property holds for the map of schemes \( S' \to S \) above for any \( S \) with a map to \( \mathcal{F} \).

2.4. Let \( S \) be an affine scheme, \( \mathcal{F} \) a presheaf, and \( a_1, a_2 \) be two objects of \( \mathcal{F}(S) \). Consider the presheaf

\[
\text{Isom}_S(a_1, a_2) := S \times_{\mathcal{F}} \mathcal{F}.
\]

By definition, for \( S' \in \text{Aff}^{ft} \), the category \( \text{Isom}_S(a_1, a_2)(S') \) is discrete (i.e., equivalent to a set) and consists of a data \( \alpha : S' \to S \) and an isomorphism \( \alpha^*(a_1) \simeq \alpha^*(a_2) \). (This explains the name "Isom".)

2.5. We say that a presheaf \( \mathcal{F} \) is a sheaf if the following two conditions are satisfied.

First, we require that for every \( S, a_1, a_2 \in \mathcal{F}(S) \), the presheaf of sets on \( \text{Aff}^{ft}/S \) given by \( \text{Isom}_S(a_1, a_2) \) be a sheaf in the flat topology.

Secondly, we require that for a faithfully flat map \( \alpha : S' \to S \in \text{Aff}^{ft} \), we have descent for \( \mathcal{F}(S') \) with respect to \( \alpha \). To formulate what this means think of the example \( \mathcal{F}(S) = \text{QCoh}(S) \), and formulate the assertion in abstract terms.

2.6. We say that a sheaf of groupoids is an algebraic stack if the following two additional conditions hold.

Condition 1 says that the diagonal map \( \mathcal{F} \to \mathcal{F} \times \mathcal{F} \) be schematic. This is tautologically equivalent to the following: for any \( S \in \text{Aff}^{ft} \) and \( a_1, a_2 \in \mathcal{F}(S) \), the presheaf \( \text{Isom}_S(a_1, a_2) \) (which is a sheaf by the assumption that \( \mathcal{F} \) is a sheaf of groupoids) must be schematic.

Condition 1 can be reformulated (less tautologically) as follows: for any \( S \in \text{Aff}^{ft} \) and \( a \in \mathcal{F}(S) \) the corresponding morphism \( S \to \mathcal{F} \) is schematic. By definition, this is equivalent to requiring that for \( S_1, S_2 \in \text{Aff}^{ft} \) and \( a_i \in \mathcal{F}(S_i) \), the Cartesian product

\[
S_1 \times_{\mathcal{F}} S_2
\]

be schematic.

The equivalence of the two versions of Condition 1 is established as follows. For \( \mathcal{F} \) satisfying the first version, and \( S_1, S_2, a_i \in \mathcal{F}(S_i) \) we have:

\[
S_1 \times_{\mathcal{F}} S_2 \simeq \mathcal{F} \times_{\mathcal{F} \times \mathcal{F}} (S_1 \times_{\mathcal{F}} S_2).
\]

Vice versa, for \( \mathcal{F} \) satisfying the second version, \( S \in \text{Aff}^{ft} \) and \( a_1, a_2 \in \mathcal{F}(S) \), we have

\[
\text{Isom}_S(a_1, a_2) \simeq S \times_{\mathcal{F} \times \mathcal{F}} (S \times_{\mathcal{F}} S).
\]
A non-example. Show that the sheaf $\mathcal{F}(S) := \text{Coh}(S)$ doesn’t satisfy the above condition.

Condition 2 for being an algebraic stack is: there exists an affine scheme $X$ endowed with a smooth and surjective map $X \to \mathcal{F}$.

Note that by the first condition, any map $X \to \mathcal{F}$ is schematic, so the notion of smoothness and surjectivity makes sense.

Our main example of an algebraic stack is $BG$, discussed below.

3. The stack $BG$

Here again, we’ll repeat some points from Toly’s talk.

3.1. Let $G$ be an affine (smooth) algebraic group as above. We define the presheaf $BG$ as follows: for a (affine) scheme $X$, we set $BG(X)$ to be the groupoid of $G$-bundles on $X$. The fact that this presheaf is a sheaf follows from the usual descent theory.

3.2. Let’s check the first condition of algebraicity. For a scheme $X$ and two maps to $BG$, i.e., for two principal $G$-bundles $\mathcal{P}_1$ and $\mathcal{P}_2$, we have to show that the sheaf of sets $\text{Isom}_X(\mathcal{P}_1, \mathcal{P}_2)$ is representable by a scheme.

Exercise 3.3. Show that $\text{Isom}_X(\mathcal{P}_1, \mathcal{P}_2)$ is represented by the scheme $G_{\mathcal{P}_1 \times \mathcal{P}_2}$. (Here we regard $G$ as a scheme acted on by the group $G \times G$, and we are applying the associated bundle construction with respect to the $G \times G$-torsor $\mathcal{P}_1 \times \mathcal{P}_2$).

Thus, the above exercise implies that Condition 1 for being an algebraic stack holds.

3.4. Let’s check the second condition. We claim that the tautological map $pt \to BG$ corresponding to the trivial $G$-bundle on $pt$ is smooth and surjective. Let $X$ be a (affine) scheme mapping to $BG$, that is we have a $G$-bundle $\mathcal{P}$ over $X$. We need to compute the Cartesian product $X \times_{BG} pt$, which is a scheme by Condition 1, and show that its projection to $X$ is smooth.

Exercise 3.5. Deduce from Exercise 3.3 that $X \times_{BG} pt$ identifies with $\tilde{X}$—the total space of the $G$-bundle $\mathcal{P}$.

3.6. Let now $Z$ be a scheme acted on by $G$. We define the stack quotient $G \backslash Z$ as follows: $\text{Hom}(S, G \backslash Z)$ is the groupoid of pairs $(\tilde{X}, \alpha)$, where $\tilde{X}$ is a $G$-bundle on $X$, and $\alpha$ is a $G$-equivariant map $\tilde{X} \to Z$. (Note that for $Z = pt$ we recover $G \backslash pt \simeq BG$.)

It is easy the assignment $S \mapsto \text{Hom}(S, G \backslash Z)$ is a sheaf of groupoids.

Exercise 3.7. Check Condition 1 for being an algebraic stack.

To check Condition 2, note that we have the tautological map $Z \to G \backslash Z$, corresponding to the trivial $G$-bundle on $Z$ and the action map $\tilde{Z} \simeq G \times Z \to Z$. We claim that this map is smooth and surjective. Indeed, fix an $X$-point $(\tilde{X}, \alpha : \tilde{X} \to Z)$ of $G \backslash Z$.

Exercise 3.8. Show that $X \times_{G \backslash Z} Z$ is canonically isomorphic to $\tilde{X}$.

3.9. Assume for a moment that the action of $G$ on $Z$ is free. By definition, this means that there exists a scheme $Y$ with a $G$-bundle $\mathcal{P}$ such that $\tilde{Y} \simeq Z$, as schemes acted on by $G$.

Exercise 3.10. Show that in the above case, the stack $G \backslash Z$ is representable by the scheme $Y$.

So, in this case it’s OK to write $Y \simeq G \backslash Z$, i.e., the two ways to understand the quotient (as a scheme and as a stack), coincide.
3.11. Assume that \( Z \) satisfies the technical assumption from the Remark 1.4. Consider the canonical map of stacks \( G \backslash Z \to BG \). We claim that this morphism is schematic. Indeed, fix an \( X \)-point \( P \) of \( BG \), and consider the Cartesian product \( X \times_{BG} G \backslash Z \).

**Exercise 3.12.** Show that \( X \times_{BG} G \backslash Z \cong Z_P \), the associated bundle.

3.13. Let us generalize the above set-up slightly. Let \( Z_1 \to Z_2 \) be a map of \( G \)-schemes. We obtain the corresponding map of stacks \( G \backslash Z_1 \to G \backslash Z_2 \). In particular, for \( Z_1 = Z \) and \( Z_2 = \text{pt} \) we recover the above map \( G \backslash Z \to BG \).

Let us make the following technical assumption: \( Z_1 \) is polarized quasi-projective over \( Z_2 \) in a \( G \)-equivariant way. This means that there exists a \( G \)-equivariant line bundle on \( Z_1 \), which is ample relative to \( Z_2 \). This assumption will be satisfied in all the example of interest, since our schemes will be "explicitly" quasi-projective.

We claim that in the above case, the map \( G \backslash Z_1 \to G \backslash Z_2 \) is schematic, and in fact quasi-projective. Indeed, fix an \( X \)-point \((\tilde{X}, \tilde{X} \to Z_2)\) of \( G \backslash Z_2 \), and consider the Cartesian product:

\[
X \times_{G \backslash Z_2} G \backslash Z_1.
\]

**Exercise 3.14.**

(a) Show that the action of \( G \) on \( \tilde{X} \times_{Z_2} \) is free. More precisely, show that one can descend \( \tilde{X} \times_{Z_2} \) viewed as a scheme over \( \tilde{X} \), to a scheme over \( X \).

(b) Show that the resulting scheme \( G \backslash (\tilde{X} \times_{Z_2} Z_1) \) identifies with \( X \times_{G \backslash Z_2} G \backslash Z_1 \).

3.15. Let \( G_1 \to G_2 \) be a homomorphism of algebraic groups. In this case we have a natural morphism of algebraic stacks \( BG_1 \to BG_2 \).

Assume now that \( G_1 \to G_2 \) is injective. We claim that in this case, the morphism \( BG_1 \to BG_2 \) is schematic (and quasi-projective).

Indeed, fix an \( X \)-point of \( BG_2 \), i.e., a \( G_2 \)-bundle \( P_2 \) on \( X \). We ask: what is the Cartesian product \( X \times_{BG_2} BG_1 \)?

For a scheme \( X' \) to map it to \( X \times_{BG_2} BG_1 \) means to fix a map \( X' \to X \) and choose a reduction of the \( G_2 \)-bundle \( P_2 := P_2|_{X'} \) to the subgroup \( G_1 \).

**Exercise 3.16.** Identify \( X \times_{BG_2} BG_1 \) with \((G_2/G_1)\cdot P_2 \) (again, the associated bundle construction), where we view the quotient \( G_2/G_1 \) as a scheme acted on by \( G_2 \).

Here is another way to view the map \( BG_1 \to BG_2 \):

**Exercise 3.17.** Identify the stack \( BG_1 \) with \((G_2/G_1)\cdot \), and the map \( BG_1 \to BG_2 \) with the map \( G_2 \backslash (G_2/G_1) \to BG_2 \). Deduce Exercise 3.16 from Exercise 3.12.