NOTES ON CRYSTALS AND ALGEBRAIC $\mathcal{D}$-MODULES

Let $X$ be a smooth manifold, and let $V$ be a vector bundle on $X$ equipped with a flat connection

$$\nabla : V \to V \otimes \Omega_X.$$

Then the flat sections of $V$ determine a local system $L$ on $X$. For every point $x \in X$, the fiber of the local system $L_x$ can be identified with the fiber $V_x$. Given a path $p : [0,1]$ from $x = p(0)$ to $y = p(1)$, there is a map $p_! : L_x \to L_y$ given by parallel transport along $p$, using the connection $\nabla$; moreover the map $p_!$ depends only on the homotopy class of the path $p$. This construction is entirely reversible: the local system $L$ determines the vector bundle $V$ and its connection up to canonical isomorphism. In other words, the category of vector bundles with flat connection on $X$ is equivalent to the category of local systems of vector spaces on $X$.

Now suppose that $X$ is a smooth algebraic variety over a field $k$ of characteristic zero (fixed through the remainder of this lecture). There is a purely algebraic notion of a vector bundle with flat connection on $X$: that is, an algebraic vector bundle $V$ on $X$ equipped with a map of sheaves

$$\nabla : V \to V \otimes \Omega_X$$

which satisfies the Leibniz rule. If $k$ is the field of complex numbers, then the set of $k$-valued points $X(k)$ is endowed with the structure of a smooth (complex) manifold, so that $V$ determines a local system on $X(k)$ as above. However, the relationship between vector bundles with connection to local systems is essentially transcendental. There is no algebraic notion of a path from a point $x \in X$ to another point $y \in X$, and hence no algebraic theory of parallel transport along paths.

Let us return for the moment to a case of a general manifold $X$. Every point $x \in X$ has a neighborhood $U$ which is homeomorphic to a Euclidean space $\mathbb{R}^n$. Consequently, for every point $y$ which is sufficiently close to $x$ (so that $y \in U$), we can choose a path from $x$ to $y$ which is contained in $U$: moreover, this path is uniquely determined up to homotopy. Consequently, parallel transport along some connection from $x$ to $y$ does not depend on a choice of path, provided that path lies in $U$. We can summarize this informally as follows: if $x$ and $y$ are nearby points of $X$ and $V$ is a vector bundle with connection on $X$, then we get a canonical isomorphism $V_x \simeq V_y$.

If $X$ is an algebraic variety, then it typically does not have a basis consisting of “contractible” Zariski-open subsets (for example, if $X$ is a smooth curve of genus $> 0$, then it has no simply-connected open subsets at all). However, Grothendieck’s theory of schemes provides a good substitute: namely, the notion of infinitesimally close points.

**Definition 0.1.** Let $X$ be a scheme over $k$, let $R$ be a $k$-algebra. We let $X(R) = \text{Hom}(\text{Spec } R, X)$ be the set of $R$-valued points of $X$. Let $I$ denote the nilradical of $R$: that is, the ideal in $R$ consisting of nilpotent elements. We say that two $R$-valued points $x, y \in X(R)$ are *infinitesimally close* if $x$ and $y$ have the same image under the map $X(R) \to X(R/I)$.

**Remark 0.2.** Note that if $x, y : \text{Spec } R \to X$ are infinitesimally close points, then they induce the same map of topological spaces from $\text{Spec } R$ into $X$: the only difference is what happens with sheaves of functions. This is one sense in which $x$ and $y$ really can be regarded as “close”.

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Using this notion of “infinitesimally close” points, we can formulate what it means for a sheaf \( \mathcal{F} \) on a scheme \( X \) to have a good theory of “parallel transport along short distances”:

**Definition 0.3.** [Grothendieck] Let \( X \) be a smooth scheme over \( k \). A *crystal of quasi-coherent sheaves on \( X \) consists of the following data:

1. A quasi-coherent sheaf \( \mathcal{F} \) on \( X \). For every \( R \)-valued point \( x : \text{Spec} \, R \to X \), the pullback \( x^*(\mathcal{F}) \) can be regarded as a quasi-coherent sheaf on \( \text{Spec} \, R \); that is, as an \( R \)-module.

   We will denote this \( R \)-module by \( \mathcal{F}_x \).

2. For every pair of infinitesimally close points \( x, y \in X(R) \), an isomorphism of \( R \)-modules \( \eta_{x,y} : \mathcal{F}(x) \to \mathcal{F}(y) \). These isomorphisms are required to be functorial in the following sense: let \( R \to R' \) be any map of commutative rings, so that \( x \) and \( y \) have images \( x', y' \in X(R') \). Then

   \[
   \eta_{x',y'} : \mathcal{F}(x') \simeq \mathcal{F}(x) \otimes_R R' \to \mathcal{F}(y) \otimes_R R' \simeq \mathcal{F}(y')
   \]

   is obtained from \( \eta_{x,y} \) by tensoring with \( R' \).

3. Let \( x, y, z \in X(R) \). If \( x \) is infinitesimally close to \( y \) and \( y \) is infinitesimally close to \( z \), then \( x \) is infinitesimally close to \( z \); we require that \( \eta_{x,z} \simeq \eta_{y,z} \circ \eta_{x,y} \). In particular (taking \( x = y = z \)), we see that \( \eta_{x,x} \) is the identity on \( \mathcal{F}(x) \), and (taking \( x = z \)) that \( \eta_{x,y} \) is inverse to \( \eta_{y,x} \).

There is another way to formulate Definition 0.3. Let \( X \) be an arbitrary functor from commutative rings to sets, not necessarily a functor which is representable by a scheme. A *quasi-coherent sheaf* \( \mathcal{F} \) on \( X \) consists of a specification, for every \( R \)-point \( x \in X(R) \), of an \( R \)-module \( \mathcal{F}(x) \), which is compatible with base change in the following sense:

- (a) If \( R \to R' \) is a map of commutative rings and \( x' \in X(R') \) is the image of \( x \), we are given an isomorphism \( \alpha_{x,x'} : \mathcal{F}(x') \simeq \mathcal{F}(x) \otimes_R R' \).

- (b) Given a pair of maps \( R \to R' \to R'' \) and a point \( x \in X(R) \) having images \( x' \in X(R') \) and \( x'' \in X(R'') \), the map \( \alpha_{x,x''} \) is given by the composition

\[
\mathcal{F}(x) \otimes_R R'' \to (\mathcal{F}(x) \otimes_R R') \otimes_R R'' \xrightarrow{\alpha_{x,x'}} \mathcal{F}(x') \otimes_R R'' \xrightarrow{\alpha_{x',x''}} \mathcal{F}(x'').
\]

If \( X \) is a scheme, then this definition recovers the usual notion of a quasi-coherent sheaf on \( X \). We define \( X^{\text{dR}} \), the deRham stack of \( X \), to be the functor given by the formula \( X^{\text{dR}}(R) = X(R/I) \), where \( I \) is the nilradical of \( R \). If \( X \) is a smooth scheme, then the map \( X(R) \to X(R/I) \) is surjective, so that \( X^{\text{dR}}(R) \) can be described as the quotient of \( X(R) \) by the relation of “infinitesimal closeness”. Unwinding the definitions, we see that a crystal of quasi-coherent sheaves on \( X \) is essentially the same thing as a quasi-coherent sheaf on \( X^{\text{dR}} \).

The main point of introducing these definitions is the following result:

**Theorem 0.4.** Let \( X \) be a smooth scheme over \( k \). Then the category of crystals of quasi-coherent sheaves on \( X \) is equivalent to the category of quasi-coherent \( \mathcal{D}_X \)-modules.

The equivalence of Theorem 0.4 is compatible with the forgetful functor to quasi-coherent sheaves. In other words, we are asserting that if \( \mathcal{F} \) is a quasi-coherent sheaf on \( X \), then equipping \( \mathcal{F} \) with a flat connection \( \nabla : \mathcal{F} \to \mathcal{F} \otimes \mathcal{D}_X \) is equivalent to endowing \( \mathcal{F} \) with the structure of a crystal. This can be regarded as an algebro-geometric version of the equivalence of categories mentioned at the beginning of this lecture.

We now sketch the proof of Theorem 0.4. Fix a quasi-coherent sheaf \( \mathcal{F} \) on \( X \). We would like to understand, in more concrete terms, how to endow \( \mathcal{F} \) with the structure (2) described
in Definition 0.3. To this end, we note that a pair of \( R \)-points \( x, y \in X(R) \) can be regarded as an \( R \)-point of the product \( X \times X \). The points \( x \) and \( y \) are infinitesimally close if and only if the map \( \text{Spec} \ R/I \to \text{Spec} \ R \to X \times X \) factors through the diagonal. This is equivalent to the requirement that the map \( \text{Spec} \ R \to X \times X \) factor set-theoretically through the diagonal. In other words, it is equivalent to the requirement that \( (x, y) : \text{Spec} \ R \to X \times X \) factors through \((X \times X) \vee\), where \((X \times X) \vee\) denotes the formal completion of \( X \times X \) along the diagonal.

More concretely, let \( \mathcal{J} \) denote the ideal sheaf of the diagonal closed immersion \( X \to X \times X \).

For each \( n \geq 0 \), we let \( \mathcal{J}^{n+1} \) denote the \((n+1)\)st power of the ideal sheaf \( \mathcal{J} \), and \( X^{(n)} \subseteq X \times X \) the corresponding closed subscheme. Then \( (X \times X) \vee \) is defined to be the Ind-scheme \( \text{lim} X^{(n)} \).

At the level of points, this means that \((X \times X) \vee(R) \simeq \text{lim} X^{(n)}(R)\). This is because given an \( R \)-point \( (x, y) : \text{Spec} \ R \to X \times X \), the points \( x, y \in X(R) \) are infinitesimally close if and only if the ideal generated by \( (x, y)^* \mathcal{J} \) is contained in the nilradical of \( R \), which is equivalent to the requirement that \((x, y)^* \mathcal{J}^n \) has trivial image in \( R \) for \( n \gg 0 \).

Consequently, to supply the data described in (2), we need to give an isomorphism \( \pi_1^* \mathcal{F} \simeq \pi_2^* \mathcal{F} \), where \( \pi_1, \pi_2 : (X \times X) \vee \to X \) denote the two projections. Let \( \pi_i^{(n)} \) denote the restriction of \( \pi_i \) to \( X^{(n)} \); we need to give a compatible family of maps \( \pi_1^{(n)} \circ \mathcal{F} \to \pi_2^{(n)} \circ \mathcal{F} \) of quasi-coherent sheaves on \( X^{(n)} \). This is equivalent to giving a map of sheaves

\[
\mathcal{F} \to (\pi_1^{(n)})^* (\pi_2^{(n)})^* \mathcal{F}
\]

on \( X \). To understand this data, we need to understand the functor \( (\pi_1^{(n)})_*(\pi_2^{(n)})^* \) from the category of quasi-coherent sheaves on \( X \) to itself.

Note that the underlying topological space of \( X^{(n)} \) coincides with the underlying topological space of \( X \). We may therefore view the structure sheaf \( \mathcal{O}_{X^{(n)}} \) as a sheaf on \( X \); the projection maps \( \pi_1^{(n)} \) and \( \pi_2^{(n)} \) endow \( \mathcal{O}_{X^{(n)}} \) with two (different!) \( \mathcal{O}_X \)-module structures. The functor \( (\pi_1^{(n)})_*(\pi_2^{(n)})^* \) is given by the relative tensor product

\[
\mathcal{F} \mapsto \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F}.
\]

Let \( \mathcal{D}_X^{\leq n} \) denote the sheaf of algebraic differential operators on \( X \) of order \( \leq n \). There is a canonical pairing

\[
\langle \cdot, \cdot \rangle : \mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{(n)}},
\]

which can be described as follows. Think of sections of \( \mathcal{O}_X \) as functions \( f(x) \), and sections of \( \mathcal{O}_{X^{(n)}} \) as functions \( g(x, y) \) of two variables, defined modulo the \((n+1)\)th power of \( \mathcal{J} \). Given a differential operator \( D \) on \( X \), we can regard \( g(x, y) \) as a function of \( x \) (keeping \( y \) constant) to obtain a new function \( Dg \) of two variables. We now define \( \langle D, g \rangle(x) = \langle Dg \rangle(x, y) \). It follows that \( \langle D, g \rangle \) has order \( \leq n \), and then \( D \) carries \( \mathcal{J}^{n+1} \) into \( \mathcal{J} \), so that the resulting function on \( X \) is independent of the choice of \( y \).

The pairing defined above is actually perfect: it identifies \( \mathcal{O}_{X^{(n)}} \) with the \( \mathcal{O}_X \)-linear dual of \( \mathcal{D}_X^{\leq n} \). We will check this in the special case where \( X \) is the affine line; the general case follows by the same reasoning, with more complicated notation. We can identify \( \mathcal{O}_X \) with the polynomial ring \( k[x] \) and \( \mathcal{O}_{X^{(n)}} \) with the algebra \( k[x, y]/(x - y)^{n+1} \).

As a module over \( k[x] \), it is free on a basis \( \{ (x, y)^k \}_{0 \leq k \leq n} \). On the other hand, we can identify \( \mathcal{D}_X^{\leq n} \) with the free \( \mathcal{O}_X \)-module generated by symbols \( \{ (x, y)^k \}_{0 \leq k \leq n} \). A simple calculation shows that these bases are dual to one another under the pairing \( \langle \cdot, \cdot \rangle \).

It follows that giving a map \( \mathcal{F} \to \mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}_X} \mathcal{F} \) is equivalent to giving a map \( \mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \). Giving a compatible family of such maps for each \( n \) is equivalent to giving a map \( \alpha : \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \). Any such map determines parallel transport morphisms \( \eta_{x,y} : \mathcal{F}(x) \to \mathcal{F}(y) \) for an arbitrary pair of infinitesimally close points \( x, y \in X(R) \).
To complete the analysis, we should spell out the meaning of condition (3) in Definition 0.3: under what conditions do we have \( \eta_{x,z} \cong \eta_{y,z} \circ \eta_{x,y} \)? The translation amounts to the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X & \xrightarrow{\eta} & \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \\
\downarrow{\beta} & & \downarrow{\alpha} \\
\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} & \xrightarrow{\alpha} & \mathcal{F},
\end{array}
\]

where \( \beta \) is induced by the multiplication on \( \mathcal{D}_X \). Similarly, the condition that \( \eta_{x,x} = \text{id} \) is equivalent to the requirement that the unit \( 1 \in \mathcal{D}_X \) act by the identity on \( \mathcal{F} \) (together with transitivity, this guarantees that \( \eta_{x,y} \) is inverse to \( \eta_{y,x} \), so that each \( \eta_{x,y} \) is invertible). This proves Theorem 0.4: endowing \( \mathcal{F} \) with the structure of a crystal is equivalent to endowing \( \mathcal{F} \) with the structure of a \( \mathcal{D}_X \)-module, compatible with the existing \( \mathcal{O}_X \)-module structure on \( \mathcal{F} \).

Theorem 0.4 provides us with two different ways to look at the same kind of structure. Each has its advantages:

(a) The definition of a crystal of quasi-coherent sheaves was somewhat abstract. The theory of \( \mathcal{D}_X \)-modules provides a much more concrete approach to the same objects, and enables us to make use of a battery of tools (such as noncommutative algebra) in their study.

(b) Definition 0.3 provides a very conceptual way of thinking about \( \mathcal{D}_X \)-modules. Given a quasi-coherent sheaf \( \mathcal{F} \) which is described in some functorial way, it might be difficult to explicitly identify a connection \( \nabla \) or a \( \mathcal{D}_X \) action on \( \mathcal{F} \). However, Definition 0.3 is easy to apply if we understand \( \mathcal{F} \) as a functor.

(c) The theory of crystals has quite a bit of flexibility. For example, differential operators are badly behaved if the variety \( X \) is not smooth. However, we can still contemplate quasi-coherent sheaves on the deRham stack \( X^{dr} \). This turns out to behave badly in general, but it behaves well if we work with complexes of sheaves rather than sheaves (it recovers the usual derived category of quasi-coherent \( \mathcal{D} \)-modules on \( X \), which can be obtained more concretely by embedding \( X \) in some smooth variety).

Another advantage of Definition 0.3 is that it adapts easily to nonlinear settings. For example, we have the following:

**Definition 0.5.** Let \( S \) be a smooth scheme over \( k \). A **crystal of schemes on \( S \)** consists of the following data:

1. An \( S \)-scheme \( X \to S \). For every \( R \)-valued point \( x : \text{Spec} \, R \to S \), we will denote the pullback \( X \times_S \text{Spec} \, R \) by \( x^* X \).
2. For every pair of infinitesimally close points \( x, y \in S(R) \), an isomorphism of \( R \)-schemes \( \eta_{x,y} : x^* X \cong y^* X \). (As in Definition 0.3, we require that these isomorphisms be compatible with base change in \( R \)).
3. Let \( x, y, z \in S(R) \). If \( x \) is infinitesimally close to \( y \) and \( y \) is infinitesimally close to \( z \), then \( x \) is infinitesimally close to \( z \); we require that \( \eta_{x,z} \cong \eta_{y,z} \circ \eta_{x,y} \).

Let us now make the connection between Definition 0.5 and the theory of \( \mathcal{D} \)-schemes described earlier in the seminar. Let \( \pi : X \to S \) be a crystal of schemes over \( S \), and assume that \( \pi \) is affine. Then \( \pi_* \mathcal{O}_X \) is a crystal of quasi-coherent sheaves on \( S \), which we can identify with a quasi-coherent \( \mathcal{D}_S \)-module \( A \). However, it has more structure: namely, there is a multiplication \( \pi_* \mathcal{O}_X \otimes_{\mathcal{O}_S} \pi_* \mathcal{O}_X \to \pi_* \mathcal{O}_X \). This multiplication is a map of crystals, and translates (under the equivalence of categories of Theorem 0.4) to a map of \( \mathcal{D}_S \)-modules \( A \otimes_{\mathcal{O}_S} A \to A \). This map endows \( A \) with the structure of a quasi-coherent \( \mathcal{D}_S \)-algebra. As in Theorem 0.4, no information
is lost in the passage from \( \pi : X \to S \) to \( A \): we can recover \( X \) as the relative spectrum of \( A \), and the \( D_S \)-module structure of \( A \) exhibits \( X \) as a crystal of schemes on \( S \). We can summarize our discussion as follows:

**Theorem 0.6.** Let \( S \) be a smooth scheme over \( k \). Then the category of commutative quasi-coherent \( D_S \)-algebras is equivalent to the category of crystals of schemes \( \pi : X \to S \) such that \( \pi \) is affine.

**Remark 0.7.** Theorem 0.6 provides a concrete understanding of crystals of schemes in the affine case. However, it can be used to understand crystals of schemes in general. Assume for simplicity that the base \( S \) is separated, and suppose that \( \pi : X \to S \) is a crystal of schemes over \( S \). Let \( U \subseteq X \) be an affine open subset. We claim that \( U \to S \) is also a crystal of schemes. To prove this, we need to give a canonical isomorphism \( x^*U \simeq y^*U \) for every pair of infinitesimally close morphisms \( x, y : \text{Spec} \ R \to S \). Note that \( x^*U \) and \( y^*U \) can be identified with open subsets of the \( R \)-schemes \( x^*X \) and \( y^*X \), which are identified by virtue of our assumption that \( X \to S \) is a crystal of schemes. We claim that this identification restricts to an isomorphism \( x^*U \simeq y^*U \). This is a purely topological question. We may therefore replace \( R \) by the quotient \( R/I \), where \( I \) is the nilradical of \( R \). After this maneuver, we have \( x = y \) and the result is obvious.

Since \( S \) is separated, for every affine open subset \( U \subseteq X \) the map \( \pi|U \) is an affine map from \( U \) to \( S \), so that \( (\pi|U)_* \mathcal{O}_U \) is a sheaf of quasi-coherent \( \mathcal{O}_S \)-algebras which we will denote by \( A_U \). The above reasoning shows that, if \( X \) is a crystal of schemes over \( S \), then each \( A_U \) has the structure of \( D_S \)-algebra; moreover, this structure depends functorially on \( U \).

Conversely, suppose we are given a compatible family of \( D_S \)-algebra structures on \( A_U \), for all open affines \( U \subseteq X \). Then each affine \( U \subseteq X \) has the structure of a crystal of schemes over \( S \). We claim that \( X \) then inherits the structure of a crystal of schemes over \( S \). To prove this, we need to exhibit an isomorphism \( \eta_{x,y} : x^*X \to y^*X \) for every pair of infinitesimally close points \( x, y \in S(R) \). The underlying map of topological spaces of \( \eta_{x,y} \) is clear (since dividing out by the nilradical of \( R \) does not change these topological spaces). The problem of promoting this map of topological spaces to a map of schemes is then local: it therefore suffices to give such a map over an open covering of \( x^*X \), and such a covering is given by \( \{x^*U\} \) where \( U \) ranges over the affine open sets in \( X \).

As in the case of quasi-coherent sheaves, we can phrase the definition of crystal in terms of deRham stacks. More precisely, let \( S \) be any functor from the category of commutative \( k \)-algebras to sets. We define an \( S \)-scheme to be another functor \( X \) from commutative \( k \)-algebras to sets, equipped with a map \( \pi : X \to S \), which is relatively representable in the following sense: for any \( R \)-point \( s \in S(R) \), the fiber product \( X \times_S \{s\} \) (another functor from commutative \( k \)-algebras to sets) is representable by an \( R \)-scheme. If \( S \) is itself representable by a \( k \)-scheme, this recovers the usual notion of a scheme \( X \) with a map to \( S \). If \( S \) is a smooth \( k \)-scheme, then an \( S^{dr} \)-scheme is the same thing as a crystal of schemes over \( S \).

Let \( \pi : S' \to S \) be a map of functors. If \( X \) is an \( S \)-scheme, then the fiber product \( S' \times_S X \) is an \( S' \)-scheme, which we will denote by \( \pi^*S \). The construction \( \pi^* \) has a right adjoint \( \pi_* \), at least at the level of functors. Namely, let \( X' \to S' \) be a morphism in the category of functors from commutative \( k \)-algebras to sets. We define \( \pi_*X' \) to be the set of pairs \( (s, \phi) \), where \( s \in S(R) \) and \( \phi \) belongs to the inverse limit \( \lim_{s' \in S'(R')} X'_s(R') \), taken over all pairs \( (R', s') \) where \( R' \) is a commutative \( R \)-algebra and \( s' \in S'(R') \) lifts the image of \( s \) in \( S(R) \). The functor \( \pi_*X' \) is called the Wei restriction of \( X' \) along \( \pi \). In general, it need not be an \( S \)-scheme, even if we assume that \( X' \) is an \( S' \)-scheme.

**Example 0.8.** Let \( S \) be a separated smooth \( k \)-scheme, and let \( \pi : X \to S \) be an arbitrary map of schemes. For each \( n \geq 0 \), let \( S^{(n)} \) denote the \( n \)th order neighborhood of the diagonal
we have a bijection $\text{Hom}_{S}(Y, J^{(n)}(X)) \simeq \text{Hom}_{S}(Y \times_{S} S^{(n)}, X)$. A point of $J^{(n)}(X)$ consists of a point $x \in X$ together with an order $n$ jet of a section of $\pi$ passing through $x$.

We have forgetful maps $J^{(n+1)}(X) \to J^{(n)}(X)$ for $n \geq 0$. These maps are affine, so that the inverse limit $J(X) = \varprojlim J^{(n)}(X)$ is well-defined. We call $J(X)$ the jet-scheme of the projection $\pi$. By construction, for every $R$-valued point $x \in S(R)$, the pullback $x^{*}J(X)$ can be identified with the scheme which parametrizes sections of $\pi$ over a formal neighborhood of $x$ in $S \times \text{Spec } R$. If $x, y \in S(R)$ are infinitesimally close, then their formal neighborhoods coincide in $S \times \text{Spec } R$, so we get a canonical isomorphism of $R$-schemes $x^{*}J(X) \simeq y^{*}J(Y)$. These isomorphisms exhibit $J(X)$ as a crystal of schemes over $S$.

One can give another more abstract argument that $J(X)$ should have the structure of a crystal of schemes over $S$. Namely, we claim that $J(X)$ is given by the Weil restriction of $X$ along the quotient map $\pi : S \to S^{\text{dr}}$. More precisely, $J(X)$ is the underlying $S$-scheme of this Weil restriction: that is, it is given by $\pi^{*}\pi_{*}X$. To prove this, we observe that there is a pullback diagram

$$
\begin{array}{ccc}
(S \times S)^{\vee} & \xrightarrow{\pi_{1}} & S \\
\downarrow \pi_{2} & & \downarrow \pi \\
S & \xrightarrow{\pi} & S^{\text{dr}}.
\end{array}
$$

There is a natural transformation of functors

$$(\pi^{*}\pi_{*}X) \simeq (\pi_{2})_{*}(\pi_{1}^{*}X),$$

which can be shown to be an isomorphism in this case. Note that $(S \times S)^{\vee} \simeq \varprojlim S^{(n)}$, so that $(\pi_{2})_{*}(\pi_{1}^{*}X)$ is the inverse limit of the Weil restrictions of the fiber products $X \times_{S} S^{(n)}$. By construction, this inverse limit is given by $J(X) = \varprojlim J^{(n)}(X)$.

The argument sketched above has an additional virtue: it establishes a universal property enjoyed by the construction $X \mapsto J(X)$. Namely, we have proven the following:

**Proposition 0.9.** Let $S$ be a smooth separated $k$-scheme. Then the construction $X \mapsto J(X)$ is right adjoint to the forgetful functor from crystals of $S$-schemes to $S$-schemes. In other words, for any crystal of $S$-schemes $Y$, composition with the projection map $J(X) \to X$ induces a bijection between the set $\text{Hom}_{S^{\text{dr}}}(Y, J(X))$ of maps of crystals to the set $\text{Hom}_{S}(Y, X)$ of maps of $S$-schemes.

We now introduce a more specific example which is relevant to our study in this seminar:

**Example 0.10.** Let $X$ be an algebraic curve over $k$ and $G$ a reductive algebraic group, and let $\pi : \text{Gr}^{1} \to X$ denote the Beilinson-Drinfeld Grassmannian. More precisely, an $R$-valued point of $\text{Gr}^{1}$ is given by a triple $(x, P, \eta)$, where $x \in X(R)$ is a point of $X$, $P$ is a $G$-bundle on $X \times \text{Spec } R$, and $\eta$ is a section of $P$ over the open set $(X \times \text{Spec } R) - x(\text{Spec } R)$. Then $\pi$ exhibits $\text{Gr}^{1}$ as a crystal (of Ind-schemes) over $X$. To see this, it suffices to observe that if $x, y \in X(R)$ are infinitesimally close, then the open sets $(X \times \text{Spec } R) - x(\text{Spec } R)$ and $(X \times \text{Spec } R) - y(\text{Spec } R)$ coincide.

**Example 0.11.** Let $X$ be an algebraic curve. Given an $R$-point $x \in X(R)$, let $O_{X,x}^{\text{DR}}$ denote the ring of functions on the formal scheme given by completing $X \times \text{Spec } R$ along $x$. Then
the ordinary scheme $\Spec \mathcal{O}_{X,x}^\vee$ contains $\Spec R$ as a divisor; we will denote the difference $\Spec \mathcal{O}_{X,x}^\vee - \Spec R$ by $D_x^\circ$, and refer to it as the punctured formal disk around $x$. (If $R$ is a field, or more generally a local ring, then $D_x^\circ$ is noncanonically isomorphic to the spectrum of a Laurent power series ring $R((t))$.)

Let $Y$ be a scheme. We define a relative loop space $LY$ as follows: an $R$-valued point of $LY$ is given by a pair $(x, \phi)$, where $x \in X(R)$ and $\phi : D_x^\circ \to Y$ is a map of schemes. If $Y$ is affine, then $LY$ is an Ind-scheme, and we have an obvious projection $LY \to X$. This map exhibits $LY$ as a crystal of Ind-schemes over $X$. To see this, it suffices to observe that if $x, y \in X(R)$ are infinitesimally close, then the formal completions of $X \times \Spec R$ along $x$ and $y$ coincide. We therefore have an isomorphism of rings $\mathcal{O}_{X,x}^\vee \simeq \mathcal{O}_{X,y}^\vee$ and hence an isomorphism of affine schemes $\Spec \mathcal{O}_{X,x}^\vee \simeq \Spec \mathcal{O}_{X,y}^\vee$, which restricts to an isomorphism between the open subschemes $D_x^\circ \simeq D_y^\circ$.

In the special case where $Y$ is a reductive algebraic group $G$, the map $LG \to X$ has fibers over a rational point $x \in X(k)$ given by $G(k_x)$, $k_x$ is denotes the field of Laurent series corresponding to $x \in X$. In this case, $LG$ is a group stack over $X$, and has a natural action $LG \times_X \Gr^1 \to \Gr^1$. It is not difficult to see that this action is horizontal: that is, the preceding map is a map of crystals.