1. Let $\mathcal{C}$ be an abelian category, and recall the context of Problem 3 on PS 2.
(a) Let $i : \mathcal{C}' \to \mathcal{C}$ be as in [PS 2, Problem 3(b)]. Show that $\mathcal{C}'' := \ker(s)$ is a Serre subcategory of $\mathcal{C}$. Show that the embedding $j : \mathcal{C}' \to \mathcal{C}$ is fully faithful and admits a right (resp., left), adjoint, and the triple $(j : \mathcal{C}' \to \mathcal{C})$ satisfies the condition of [PS 2, Problem 3(c)].
(b) Let $i : \mathcal{C}' \to \mathcal{C}$ be as in [PS 2, Problem 3(c)], and assume that $\mathcal{C}'$ is a Serre subcategory of $\mathcal{C}$. Assume that the projection $\mathcal{C} \to \mathcal{C}/\mathcal{C}' := \mathcal{C}''$ admits a right (resp., left) adjoint. Show that the resulting functor $t : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$ is fully faithful, and the triple $(t : \mathcal{C}'' \to \mathcal{C})$ satisfies condition of [PS 2, Problem 3(c)].
(c) Let $\mathcal{C}_1$ be a Serre subcategory of $\mathcal{C}$. Show that the embedding $\mathcal{C}_1 \to \mathcal{C}$ admits a left (resp., right) adjoint if the projection $\mathcal{C} \to \mathcal{C}/\mathcal{C}_1$ admits a right (resp., left) adjoint.
(d) Is $\text{Sh}^{Ab}(X)$ a Serre subcategory of $\text{PSh}^{Ab}(X)$?
(e) Is $\text{QCoh}(X)$ a Serre subcategory of $\text{Sh}(X)$-$\text{mod}$?
(f) Let $X, U, V$ be as in Problems 7-8 on PS 7. Show that $j_1 : \text{Sh}(U) \to \text{Sh}(X)$, $i_* : \text{Sh}(V) \to \text{Sh}(X)$ are embeddings of Serre subcategories.
(g) Let $X$ be as in Problem 9 on PS 7. Does $i_* : \text{QCoh}(V) \to \text{QCoh}(X)$ define an embedding of a Serre subcategory?
(h) Let $i : V \hookrightarrow X \twoheadrightarrow U : j$ be as above. Show that $\ker(j^*)$ is a Serre subcategory of $\text{QCoh}(X)$.
(i) What is the difference between the subcategory in point (h) and the image of $\text{QCoh}(V)$ under $i_*$ in point (g)? Describe these categories explicitly in terms of the corresponding ideals when $X = \text{Spec}(A)$ and $V = V(I)$, where $I$ is a finitely generated ideal.

2. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories, and set $\mathcal{C}_1 := \ker(F)$.
(a) Show that $\mathcal{C}_1$ is a Serre subcategory of $\mathcal{C}$. Denote $\mathcal{C}_2 := \mathcal{C}/\mathcal{C}_1$.
(b) Show that $F$ induces an exact functor $F_2 : \mathcal{C}_2 \to \mathcal{D}$.
(c) Give an example that $F_2$ is not an equivalence.
(d) Show that if $F$ admits an adjoint (either left or right), which is fully faithful, then $F_2$ is an equivalence.

3. Let $X$ be a scheme, and let
$$0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \mathcal{I}_3 \to 0$$
be a short exact sequence of sheaves of $\mathcal{O}_X$-modules.
(a) Assume that $X$ is affine and $\mathcal{I}_1$ is quasi-coherent. Show that in this case the map $\Gamma(X, \mathcal{I}_2) \to \Gamma(X, \mathcal{I}_3)$ is surjective. (Hint: use Problem 6 on PS 7.)

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(b) Deduce that if \( \mathcal{F}_1 \) and \( \mathcal{F}_3 \) are qc, then so is \( \mathcal{F}_2 \).

4*. Let be \( X \) be a quasi-compact and quasi-separated scheme, such that for every short exact sequence of qc sheaves

\[
0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0
\]

the map \( \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \) is surjective. Show that \( X \) is affine. \(^2\)

Suggested strategy: First, show that the functor of global sections induces an equivalence \( \text{QCoh}(X) \to A\text{-mod} \), where \( A := \Gamma(X, \mathcal{O}_X) \). Then show that you can read off the topology of \( X \) from the category \( \text{QCoh}(X) \): the closed subschemes of \( X \) are in bijection with sub-objects of \( \mathcal{O}_X \).

5. Consider the scheme \( \tilde{V} := \text{Tot}_{\mathcal{O}(V)}(\mathcal{O}(-1)) \).

(a) Interpret is functorially, and deduce the existence of a natural map \( p : \tilde{V} \to V \).

(b) Show that \( p \) induces an isomorphism over \( V - 0 \).

(c) Show that \( \tilde{V} \times V \to \mathbb{P}(V) \), where \( pt \to V \) corresponds to 0.

NB: The scheme \( \tilde{V} \) is called "the blow up of \( V \) at 0".

6. Let \( V \) be a fin. dim. vector space, and \( 0 < m_1 < \ldots < m_k < \dim(V) \) a collection of integers. Define the following functor on the category of schemes over \( k \): we send the scheme \( S \) to the set of \( k \)-tuples of vector bundles \( M_1, \ldots, M_k \) on \( S \) with \( M_i \) of rank \( m_i \), equipped with injective bundle maps

\[
M_1 \to M_2 \to \ldots \to M_k \to V \otimes \mathcal{O}_S.
\]

Show that this functor is representable by a projective scheme.

NB: The above scheme is denoted \( \text{Fl}(V)^{m_1, \ldots, m_k} \) and is called the scheme of flags in \( V \) of type \( (m_1, \ldots, m_k) \).

7*. (a) Show that the image of the Segre embedding \( \mathbb{P}(V) \otimes \mathbb{P}(W) \to \mathbb{P}(V \otimes W) \) corresponds to the homogeneous ideal in \( \text{Sym}(V^* \otimes W^*) \) generated by the image of the canonical map \( \Lambda^2(V^*) \otimes \Lambda^2(W^*) \to \text{Sym}^2(V^* \otimes W^*) \).

(b) Show that the image of the Veronese embedding \( \mathbb{P}(V) \to \mathbb{P}(\text{Sym}^k(V)) \) corresponds to the homogeneous ideal in \( \text{Sym}^2(\text{Sym}^k(V^*)) \) generated by the image of the canonical map \( \Lambda^2(\text{Sym}^{k-1}(V^*)) \otimes \Lambda^2(V^*) \to \text{Sym}^2(\text{Sym}^k(V^*)) \).

(c) Show that the image of the Plücker map \( \text{Gr}^k(V) \to \mathbb{P}(\Lambda^k(V^*)) \) corresponds to the homogeneous ideal in \( \text{Sym}^2(\Lambda^k(V^*)) \) generated by the image of the canonical map \( \Lambda^{k-1}(V^*) \otimes \Lambda^{k+1}(V^*) \to \text{Sym}^2(\Lambda^k(V^*)) \).

NB: Before tackling points (b) and (c), explain what these relations mean in terms of linear algebra, similar to what we did in class for point (a).

8* (optional). Let \( V \) be an infinite-dimensional vector space. Consider the functor \( F_V \) on the category of schemes that sends \( S \) to the set \( \Gamma(S, \mathcal{O}_S) \otimes V \). Show that this functor is not representable by a scheme.

Suggested strategy: we'll show that there do not exist open embeddings \( \text{Spec}(A) \to F_V \).

Step 1: Show that if \( V_1 \) is a vector subspace of \( V \), there exists a closed embedding of functors \( F_{V_1} \to F_V \).

\(^2\)Chances are, we'll discuss the various approaches to this problem in class on Tuesday; so prepare to present yours.
Step 2: Show that for any scheme $S$, $\text{Hom}(S, F_V) \simeq \varinjlim F_{V_1}$, where the direct limit is taken over the poset of finite-dimensional vector subspaces $V_1 \subset V$.

Step 3: Thus, any map $\text{Spec}(A) \to F_V$ factors through a map $\text{Spec}(A) \to F_{V_1}$ for some finite-dimensional subspace $V_1 \subset V$. Let $V_2$ be a finite-dimensional subspace strictly containing $V_1$. Show that the map

$$\text{Spec}(A) \times_{F_{V_1}} F_{V_2} \to F_{V_2}$$

factors through $F_{V_1} \hookrightarrow F_{V_2}$, and, hence, cannot be an open embedding.

NB: Let $F$ be a functor on the category of schemes that can be written as $\varprojlim X_i$, where $I$ is a filtered poset, $X_i$ are schemes, and the maps $X_i \to X_j$ for $i \leq j$ are closed embeddings. Such functors are called "strict ind-schemes". So, we can say (Step 2 above) that the functor $F_V$ is a strict-indscheme.

WARNING: We have encountered two types of infinite-dimensional vector spaces in AG:

One is $\mathbb{A}_\text{thick}^\infty := \text{Spec}(k[x_1, x_2, \ldots, \cdots])$. This is a scheme, and even an affine scheme. Show that $\mathbb{A}_\text{thick}^\infty$ as a scheme/functor on the category of schemes, is isomorphic to the inverse limit $\varprojlim \mathbb{A}^n$, where the maps $\mathbb{A}^{n+1} \to \mathbb{A}^n$ forget the last coordinate.

Another is $\mathbb{A}_\text{thin}^\infty$, corresponding to $F_V$ with $V = \text{Span}(x_1^*, x_2^*, \ldots)$. As a functor on the category of schemes, $\mathbb{A}_\text{thin}^\infty$ is isomorphic to the direct limit $\varinjlim \mathbb{A}^n$, where the maps $\mathbb{A}^n \to \mathbb{A}^{n+1}$ are the natural closed embeddings.