Let me know if you need an extension on Problems 7, 8 and 9.

1. Let $X$ be a scheme and let $E$ be a locally free sheaf on it. Consider the following functor on the category of schemes, denoted $P_X(E)$. It assigns to a scheme $Y$ the set of triples $(\Phi, L, \alpha)$, where $\Phi$ is a map of schemes $Y \to X$, $L$ is a line bundle on $Y$ (a.k.a. a local free sheaf of rank 1), and $\alpha$ is a map $\alpha : L \to \Phi^*(E)$, which satisfies the equivalent conditions of Problem 2. Show that $P_X(E)$ is representable.

2. Let $A$ be a commutative ring, and $M$ a locally free $A$-module. Define $\widetilde{\text{Sym}}^k(M)$ as

$$\text{ker} \left( M^\otimes k \to \bigoplus_{i,j} M^\otimes k - 2 \otimes \Lambda^2(M) \right),$$

where $i \neq j$ and run through the set $\{1, ..., n\}$.

(a) Construct a natural isomorphism $\widetilde{\text{Sym}}^k(M) \simeq \left( \text{Sym}^k(M^\vee) \right)^\vee$.

(b) Construct a natural isomorphism $\widetilde{\text{Sym}}^k(M_1 \oplus M_2) \simeq \bigoplus_{k_1 + k_2 = k} \widetilde{\text{Sym}}^{k_1}(M_1) \otimes \widetilde{\text{Sym}}^{k_2}(M_2)$.

(c) Show that $\widetilde{\text{Sym}}^k(M)$, viewed as a submodule of $M^\otimes k$, is spanned by elements of the form $m^\otimes k$, $m \in M$.

(d) Show that symmetrization gives a map $M^\otimes k \to \text{Sym}^k(M)$ that factors through a map $\text{Sym}^k(M) \to \text{Sym}^k(M)$.

(e) Note that we also have a tautological map $\widetilde{\text{Sym}}^k(M) \hookrightarrow M^\otimes k \to \text{Sym}^k(M)$. Show that the compositions $\text{Sym}^k(M) \hookrightarrow \text{Sym}^k(M)$ are both given by multiplication by $k!$.

(f) Show, however, that if $M$ is locally free of rank 1, then the above map $\widetilde{\text{Sym}}^k(M) \hookrightarrow M^\otimes k \to \text{Sym}^k(M)$ is an isomorphism.

3. Let $V$ be a vector space (over some field), and let $p_k$ be an element of $\text{Sym}^k(V)$. Let $V(p_k)$ be the corresponding closed subscheme of $\mathbb{P}(V)$. Show that an $S$-point $(\mathcal{L}, \alpha : \mathcal{L} \to \mathcal{O}_S \otimes V)$ factors through $V(p_k)$ if and only if the map $\widetilde{\text{Sym}}^k(\alpha) : \mathcal{L}^\otimes k \to \mathcal{O}_S \otimes \widetilde{\text{Sym}}^k(V)$ vanishes.

4. Write down complete proofs of the fact that the Segre map $\mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$ and the Verosese map $\mathbb{P}(V) \to \mathbb{P}(\widetilde{\text{Sym}}^k(V))$ are closed embeddings.
5. Write the complete proof of the fact that the Plücker map $\text{Gr}^m(V) \to \mathbb{P}(\Lambda^m(V))$ is a closed embedding.

6. Let $A$ be a commutative ring, $M$ an $A$-module, and let $f_1, \ldots, f_n \in A$ be elements generating the unit ideal. Consider the Čech complex of the sheaf $\text{Loc}_A(M)$, corresponding to the covering $\text{Spec}(A)$ by the $U_{f_i}$'s:

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i_1, i_2} M_{f_{i_1}, f_{i_2}} \to \bigoplus_{i_1, i_2, i_3} M_{f_{i_1}, f_{i_2}, f_{i_3}} \to \cdots$$

Show that it is exact.

NB: Suggested strategy: consider first the case when $f_1 = 1$, and construct an explicit null-homotopy. Reduce the general case to the one just considered by localizing with respect to one $f_i$ at a time.

7. Let $j : U \hookrightarrow X$ be as in Problem 10(a) on PS 5. Let $i : V \hookrightarrow X$ be the embedding of the complement of $U$.

(a) Show that for $\mathcal{F}_U \in \text{Sh}(U)$ the adjunction arrow $j^* j_! (\mathcal{F}_U) \to \mathcal{F}_U$ is an isomorphism. Deduce that the functor $j_! : \text{Sh}(U) \to \text{Sh}(V)$ is fully faithful.

(b) Show that the pair of categories and functors $j_! : \text{Sh}(U) \rightleftarrows \text{Sh}(X) : j^*$ satisfies the conditions of Problem 3(b) on PS 2.

(c) Deduce from the adjunction property satisfied by $j_!$ that for any $\mathcal{F}_U \in \text{Sh}(U)$, we have $i^*(j_!(\mathcal{F}_U)) = 0$. From here deduce that the functor $j_!$ is exact.

(d) Show that the pair of categories and functors $j_! : \text{Sh}(U) \rightleftarrows \text{Sh}(X) : j^*$ satisfies the conditions of Problem 3(c) on PS 2.

(e) Show that for any $\mathcal{F}_V \in \text{Sh}(V)$ the adjunction map $\mathcal{F}_V \to i_* (i^*(\mathcal{F}_V))$ is an isomorphism, or, equivalently, that the functor $i_*$ is fully faithful.

(f) Show that the pair $i^* : \text{Sh}(X) \rightleftarrows \text{Sh}(V) : i_*$ satisfies the conditions of Problem 3(c) on PS 2.

(g) Show that for any $\mathcal{F}_V \in \text{Sh}(V)$, we have $j^*(i_!(\mathcal{F}_V)) = 0$. Deduce that the functor $i_!$ is exact.

(h) Show that the pair $i^* : \text{Sh}(X) \rightleftarrows \text{Sh}(V) : i_*$ satisfies the conditions of Problem 3(b).

(i) Show that for $\mathcal{F} \in \text{Sh}(X)$ the adjunction maps define a short exact sequence:

$$0 \to j_!(j^*(\mathcal{F})) \to \mathcal{F} \to i_!(i^*(\mathcal{F})) \to 0.$$

(j) Show that $\mathcal{F} \in \text{Sh}(X)$ belongs to the essential image of $j_!$ (resp., $i^*$) if and only if $i^*(\mathcal{F})$ (resp., $j^*(\mathcal{F})$) vanishes.

(k) Deduce that the abelian category $\text{Sh}(U)$ (resp., $\text{Sh}(U)$) viewed as a subcategory of $\text{Sh}(X)$ by means of $j_!$ (resp., $i_*$) has the following property: for a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

$\mathcal{F}_2$ belongs to the subcategory if and only if both $\mathcal{F}_1$ and $\mathcal{F}_3$ do.

(l) Let now $X$ be a scheme, and let us understand $i_*$ and $i^*$ as (a mutually adjoint pair of) functors $\text{QCoh}(X) \rightleftarrows \text{QCoh}(V)$. Show that points (e), (f), (g) still hold in this context, however (h), (j) and (k) fail.

8. Let us return to the set-up of the previous problem.

(a) Show that for $\mathcal{F}_U \in \text{Sh}(U)$, the adjunction map $j^*(j_!(\mathcal{F}_U)) \to \mathcal{F}_U$ is an isomorphism. Deduce that he functor $j_! : \text{Sh}(U) \to \text{Sh}(V)$ is fully faithful.
(b) Show that the pair $j^* : Sh(X) \cong Sh(U) : j_*$ satisfies the condition of Problem 3(b) on PS 2, but not, in general, that of Problem 3(c).

(c) For $\mathcal{F} \in Sh(X)$ consider the sheaf $\ker(\mathcal{F} \to j_*(j^*(\mathcal{F})))$, where $\mathcal{F} \to j_*(j^*(\mathcal{F}))$ is the adjunction map. Show that this kernel is annihilated by the functor $j^*$. Deduce that it belongs to the essential image of the functor $i^!$.

(d) We define the functor $i^! : Sh(V) \to Sh(X)$ to be such that $i^!(i^*(\mathcal{F})) \cong \ker(\mathcal{F} \to j_*(j^*(\mathcal{F})))$. It is well-defined by point (c) above. Show that $i^!$ is the right adjoint of the functor $i_* : Sh(V) \to Sh(X)$.

(e) Give the following interpretation of $i^!(\mathcal{F})$: for an open $U' \subset X$, $\Gamma(V \cap U', i^!(\mathcal{F})) = \{ f \in \Gamma(U', \mathcal{F}) \mid f|_{V \cap U'} = 0 \}$; show that is only depends on $U' \cap V$, but not $U'$ itself.

(f) Show that the pair $i_* : Sh(V) \cong Sh(X) : i^!$ satisfies the condition of Problem 3(c) on PS 2, but not, in general, that of Problem 3(b).

9. Let us be in the context of the previous problem, where $X$ is a scheme, and assume that $j : U \to X$ is quasi-compact. Let us view $i_!, j^!, j_*$ as functors between the corresponding quasi-coherent categories.

(a) Show that points (a) and (b) of Problem 7 remain valid in the present context.

(b*) Show that the functor $i^! : \text{QCoh}(X) \to \text{QCoh}(V)$, right adjoint to $i_*$, exists, but it differs from the functor $i^!$ defined in the context of just sheaves (i.e., the formation of $i^!$ doesn’t commute with the forgetful functor $\text{QCoh}(\cdot) \to Sh(\cdot)$).

Hint: Show that for $X = \text{Spec}(A)$ and $V = V(I)$ for some ideal $I \subset A$, for an $A$-module $M$, we have $i^!(\text{Loc}_A(M)) = \text{Loc}_{A/I}(N)$, where $N = \{ m \in M \mid I \cdot m = 0 \}$. 