1. Let $X$ be a topological space. Consider the category $\text{Open}(X)$, whose objects are open sets of $X$, and
\[
\text{Hom}(U_1, U_2) = \begin{cases} \{\ast\}, & U_1 \subset U_2 \\ \emptyset, & \text{otherwise} \end{cases}
\]
Show that a presheaf of sets (resp., abelian groups, vector spaces, rings) is the same as a contravariant functor $\text{Open}(X) \to \text{Sets}$ (resp., to $\text{Ab}$, $\text{Vect}_k$, $\text{Rings}$).

2. Complete the proof that $\text{Hom}_{\text{PSh}^{\text{Ab}}(X)}(\mathbb{Z}, F) \simeq \Gamma(X, F)$.

3. (a) Let $x \in X$ be a point, and recall the presheaf $\delta_x$. Show that it is a sheaf.
(b) Let $F$ be a presheaf of abelian groups. Show that $\text{Hom}_{\text{PSh}^{\text{Ab}}(X)}(\delta_x, F)$ identifies naturally with the set of $f \in \Gamma(X, F)$ such that $f|_{X-x} = 0$.

4. Complete the proof of the assertion that a map $\alpha : F_1 \to F_2$ in $\text{Sh}^{\text{Ab}}(X)$ is a categorical surjection if and only if for every $U \in \text{Open}(X)$ and $f \in \Gamma(U, F_2)$ there exists a covering $U = \bigcup U_i$ and elements $f_{1,i} \in \Gamma(U_i, F_1)$ such that $f_2|_U = \alpha_U(f_{1,i})$.

5. Let $I$ be a (possibly) infinite set, and $X_i, i \in I$ be objects of some category $\mathcal{C}$.
(a) Recall that a product $\prod_i X_i$ is an object of $\mathcal{C}$ (if it exists) endowed with a functorial isomorphism $\text{Hom}(Y, \prod_i X_i) \simeq \prod_i \text{Hom}(Y, X_i)$. Let $\mathcal{C}$ be the category of (i) presheaves, (ii) of sheaves of sets on a top. space $X$. Show that products exist and in both cases, and $\Gamma(U, \prod_i F_i) \simeq \prod_i \Gamma(U, F_i)$.
(b) Recall that a co-product $\sqcup_i X_i$ is an object of $\mathcal{C}$ (if it exists) endowed with a functorial isomorphism $\text{Hom}(\sqcup_i X_i, Y) \simeq \prod_i \text{Hom}(X_i, Y)$. Let $\mathcal{C}$ be the category of presheaves of (i) sets (ii) abelian groups on $X$. Show that coproducts exist in both cases, and given by $\Gamma(U, \sqcup_i F_i) \simeq \sqcup_i \Gamma(U, F_i)$ and $\Gamma(U, \sqcup_i F_i) \simeq \bigoplus_i \Gamma(U, F_i)$, respectively.
(c) Give an example that even if $F_i$ are sheaves, the presheaf coproduct constructed in point (b) may not be a sheaf.
(d) Does point (c) mean that coproducts in the category of sheaves don’t necessarily exist?