Let $V$ be a vector space of dimension $n+1$ over a field $k$, and consider the scheme $X = \mathbb{P}V \cong \mathbb{P}^n k \cong \text{Proj}(k[x_0, \ldots, x_n])$.

Consider $\mathcal{F}$ a quasi-coherent sheaf over $X$. We can examine its Cech cohomology - which coincides with its sheaf (non-etale) cohomology because $X$ is Noetherian and separated (cf. Hartshorne, ch. III theorem 4.5.). Particularly, $H^0(X, \mathcal{F}) \cong \Gamma(\mathcal{F}, X)$. However $\Gamma(\mathcal{F}, X) \cong \text{Hom}_X(\mathcal{O}_X, \mathcal{F})$. Indeed, the identification is obtained as follows: if $s \in \Gamma(\mathcal{F}, X)$ we define $\phi: \mathcal{O}_X \to \mathcal{F}$ by $\phi_U: \Gamma(\mathcal{O}_X, U) \to \Gamma(\mathcal{F}, U)$ given by $\phi_U(x) = x \cdot \text{Res}_Us$. Conversely, if $\phi: \mathcal{O}_X \to \mathcal{F}$ gets sent to $\phi(1_X) \in \Gamma(\mathcal{F}, X)$. Moreover this identification is clearly an isomorphism of $k$-vector spaces.

We can therefore consider a $k$-bilinear map $H^0(X, \mathcal{F}) \otimes_k H^k(\mathcal{F}', \mathcal{F}') \to H^k(X, \mathcal{F} \otimes_X \mathcal{F}')$ defined as follows for any $s \in H^0(X, \mathcal{F})$ identified with $\phi: \mathcal{O}_X \to \mathcal{F}$ as above, we have the map $\mathcal{F}' \cong \mathcal{O}_X \otimes_X \mathcal{F}' \to \mathcal{F} \otimes_X \mathcal{F}'$ induced by $\phi$, and hence by functoriality of homology we have the map $\phi_s: H^k(\mathcal{F}', \mathcal{F}') \to H^k(X, \mathcal{F} \otimes_X \mathcal{F}')$ whence we have the map $H^0(X, \mathcal{F}) \otimes_k H^k(\mathcal{F}', \mathcal{F}') \to H^k(X, \mathcal{F} \otimes_X \mathcal{F}')$ given by $s \otimes t \to \phi_s(t)$. Take now $\mathcal{F} = \mathcal{O}(i)$ and $\mathcal{F}' = \mathcal{O}(-n - 1 - i)$. Since $H^n(X, \mathcal{O}(-n - 1)) \cong k$, the above bilinear map reads

$$H^0(X, \mathcal{O}(i)) \otimes H^n(X, \mathcal{O}(-n - 1 - i)) \to k$$

**Theorem (Serre duality for projective space).** The map

$$H^0(X, \mathcal{O}(i)) \otimes H^n(X, \mathcal{O}(-n - 1 - i)) \to k$$

is a perfect pairing of vector spaces.

**Corollary.** There is a natural isomorphism

$$H^n(X, \mathcal{O}(-n - 1 - i)) \cong \text{Sym}^i(V)$$

for $i \geq 0$

We will give two proofs for the theorem and the corollary. The first follows an idea used in the computation of cohomology of projective space in class, and the second performs explicit computations used in the proof of theorem 5.1. ch. III of Hartshorne.

**Proof 1.**

We will perform two nested inductions: in $n$ and in $i$. Notice that we do not need to perform the base case for $i$: if $i < 0$ then we know that $H^0(X, \mathcal{O}(i)) = 0$ and $H^n(X, \mathcal{O}(-n - 1 - i)) = 0$ (problem 1 PS 10) and therefore the pairing is perfect. The base case for $n$ is the degenerate case $n = 0$. In that setting $V \cong k$ and $\mathbb{P}V \cong \ast \cong \text{Spec}(k)$. Then every sheaf on $\mathbb{P}V$ is simply a vector space over $k$ and since $\mathcal{O}(i)$ is a line bundle, $\mathcal{O}(i)$ is a 1-dimensional vector space over $k$.

Therefore the above pairing is simply the pairing $k \otimes k \to k$. This pairing will be perfect unless it is degenerate. It’s not: if we look at 1 in the first component, then the associated map $k \to k$ is the identity map (this is straightforward from the definition).

Therefore we only need to worry about the induction step. Also we only need to consider $i \geq 0$.

Consider $n \geq 1$ and let $\xi \in V^*$. Define $V_\xi$ be the subspace annihilated by $\xi$, we also know that $\mathbb{P}V_\xi$ is naturally identified with the locus of vanishing of $l_\xi$ and we have a closed embedding $i: \mathbb{P}V_\xi \hookrightarrow \mathbb{P}V$. Recall the exact sequence of line bundles from class:

$$0 \to \mathcal{O}(m - 1) \xrightarrow{i} \mathcal{O}(m) \to i_*\mathcal{O}(m)_{\mathbb{P}V_\xi} \to 0$$

We apply it for $m = i$ and take the long exact sequence of cohomology. Note that $H^1(X, \mathcal{O}(i - 1)) = 0$ - this fact follows from problem 1 PS 10 (one needs to distinguish the cases $n = 1$ and $n > 1$ but both follow directly). Moreover, since $i$ is an
affine map (being a closed embedding), we have \( H^k(X, i_! \mathcal{O}(P \mathcal{V}_\xi)) \cong H^k(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(i)_! \mathcal{V}_\xi) \). We deduce the following short-exact sequence of \( k \)-vector spaces:

\[
0 \to H^0(X, \mathcal{O}(i - 1)) \to H^0(X, \mathcal{O}(i)) \to H^0(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(i)_! \mathcal{V}_\xi) \to 0
\]

Now take \( m = -n - i \) and apply the long exact sequence of cohomology. Because \( \mathbb{P} \mathcal{V}_\xi \) can be covered by \( n \) affines, \( H^n(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(i)_! \mathcal{V}_\xi) = 0 \). Also, \( H^{n-1}(X, \mathcal{O}(-n - i)) = 0 \) according to problem 1 PS 10. Therefore we extract the following short exact sequence:

\[
0 \to H^{n-1}(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(-n - i)) \to H^n(X, \mathcal{O}(-n - 1 - i)) \to H^n(X, \mathcal{O}(-n - i)) \to 0
\]

Observe that we can form three pairings of vector spaces from the two short exact sequences obtained: \( H^0(X, \mathcal{O}(i - 1)) \otimes H^n(X, \mathcal{O}(-n - i)) \to H^n(X, \mathcal{O}(-n - 1)) \cong k, H^0(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(i)_! \mathcal{V}_\xi) \otimes H^{n-1}(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(-n - i)) \to H^{n-1}(\mathbb{P} \mathcal{V}_\xi, \mathcal{O}(-n)_! \mathcal{V}_\xi) \cong k \) and \( H^0(X, \mathcal{O}(i)) \otimes H^n(X, \mathcal{O}(-n - 1 - i)) \to H^n(X, \mathcal{O}(-n - 1)) \cong k \). The first two pairings are perfect by the induction hypothesis. The following lemma will ensure that the third pairing is perfect as well, finishing the proof:

**Lemma.** Assume that \( 0 \to U_1 \to U_2 \to U_3 \to 0, 0 \to V_1 \to V_2 \to V_3 \to 0 \) are short exact sequences of finite-dimensional \( k \)-vector spaces. Suppose that there exist perfect pairings \( U_1 \otimes V_3 \to k, U_3 \otimes V_1 \to k \). Assume that there is a \( k \)-bilinear pairing \( U_2 \otimes V_2 \to k \) that makes the following diagrams commute:

\[
\begin{array}{ccc}
U_1 \otimes V_2 & \to & U_1 \otimes V_3 \\
\downarrow & & \downarrow \\
U_2 \otimes V_2 & \to & k
\end{array}
\]

\[
\begin{array}{ccc}
U_2 \otimes V_2 & \to & U_3 \otimes V_1 \\
\downarrow & & \downarrow \\
U_2 \otimes V_2 & \to & k
\end{array}
\]

Then the pairing \( U_2 \otimes V_2 \to k \) is perfect, too.

**Proof of lemma:** Because \( U_1, V_3 \) and \( U_3, V_1 \) are dual to each other, they have equal dimensions, hence \( U_2 \) and \( V_2 \) have equal dimensions since their dimensions are \( \dim(U_1) + \dim(U_3) \) respectively \( \dim(V_1) + \dim(V_3) \). To prove the pairing is perfect, it suffices to prove that there is no non-zero \( u \in U_2 \) such that \( < u, v > = 0 \) for all \( v \in V_2 \). Indeed, assume we have such a non-zero \( u \). Particularly \( < u, v_1 > = 0 \) for all \( v_1 \in V_1 \) (we regard \( V_1 \) as a subspace of \( V_2 \)) and using the second diagram we conclude that the projection of \( u \) to \( U_3 \) must pair to 0 with everything in \( V_1 \), so it must be 0 because the pairing \( U_3 \otimes V_1 \to k \) is perfect. Thus \( u \in U_1 \), and \( < u, v_2 > = 0 \) for all \( v_2 \in V_2 \). Using the first diagram, we conclude that \( u \) must pair to 0 with everything in \( V_3 \), hence it must be 0 because the pairing \( U_1 \to V_3 \) is perfect. \( \Box \)

An alternative proof, following the suggestion of Kaloyan, is to observe that the commutativity of the diagrams implies that the three pairings induce a map of short exact sequences:

\[
\begin{array}{ccc}
0 & \to & U_1 \\
\downarrow & & \downarrow \\
0 & \to & V_1^* \\
\downarrow & & \downarrow \\
0 & \to & V_3^*
\end{array}
\]

The outer vertical maps are isomorphisms hence the middle vertical map is an isomorphism by the short five lemma.

To finish the proof, we need to show that the conditions of the lemma are satisfied, i.e. that the following diagrams are commutative:

\[
\begin{array}{ccc}
H^0(X, \mathcal{O}(i - 1)) \otimes H^n(X, \mathcal{O}(-n - 1 - i)) & \to & H^0(X, \mathcal{O}(i - 1)) \otimes H^n(X, \mathcal{O}(-n - i)) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{O}(i)) \otimes H^n(X, \mathcal{O}(-n - 1 - i)) & \to & k
\end{array}
\]
Theorem 1.1

Consider again $H^0(X,\mathcal{O}(i)) \otimes H^{n-1}(PV,\mathcal{O}(-n-i)_{PV}) \rightarrow H^0(PV,\mathcal{O}(i)_{PV}) \otimes H^{n-1}(PV,\mathcal{O}(-n-i)_{PV}) \rightarrow k$

The first diagram can be rewritten as

$$H^0(X,\mathcal{O}(i)) \otimes H^n(X,\mathcal{O}(-n-1-i)) \rightarrow H^n(X,\mathcal{O}(-n-1-i))$$

In this setting, the commutativity follows from the functoriality. Indeed, if we choose some $h \in Hom_X(\mathcal{O}(i-1),\mathcal{O}(i-1))$ then $h \otimes a$ will go under the bottom path to $H^n(l_\xi \circ h)(a)$ and under the upper path to $H^n(l_\xi)(H^n(h)(a))$ but these two expressions are equal because $H^n(l_\xi) \circ H^n(h) = H^n(l_\xi \circ h)$ by functoriality of $H^n$.

For the second diagram, rewrite it as

$$Hom_X(\mathcal{O}(i),\mathcal{O}(i)) \otimes H^{n-1}(X,i_*\mathcal{O}(-n-i)_{PV}) \rightarrow Hom_X(\mathcal{O}(i),\mathcal{O}(i)_{PV}) \otimes H^{n-1}(X,i_*\mathcal{O}(-n-i)_{PV}) \rightarrow k$$

Let $g_m : \mathcal{O}(m) \rightarrow i_*\mathcal{O}(m)_{PV}$ denote the corresponding map.

Consider again $h \in Hom_X(\mathcal{O}(i),\mathcal{O}(i))$ and $a \in H^{n-1}(X,i_*\mathcal{O}(-n-i))$. To show that the diagram commutes, we need to show that $H^{n-1}(q_1 \circ h)(a) \subset H^{n-1}(X,i_*\mathcal{O}(-n-i))$ equals $H^n(h)(\partial_{n-1}a) \subset H^n(X,\mathcal{O}(-n-1))$ once $H^{n-1}(X,i_*\mathcal{O}(-n)_{PV})$ and $H^n(X,\mathcal{O}(-n-1))$ are identified with $k$. However note that $H^{n-1}(X,i_*\mathcal{O}(-n)_{PV})$ and $H^n(X,\mathcal{O}(-n-1))$ are isomorphic to each other via the map $\partial_{n-1}$ of the corresponding exact sequence. Therefore we need to show that $\partial_{n}H^{n-1}(q_1 \circ h) = H^n(h)\circ \partial_{n-1}$ (see remark at the end). This follows from the cohomology axioms. Indeed, it’s easy to see that the following diagram is commutative and induces an map of short exact sequences:

$$0 \rightarrow \mathcal{O}(-n-i-1) \rightarrow \mathcal{O}(-n-i) \rightarrow i_*\mathcal{O}(-n-i)_{PV} \rightarrow 0$$

0 \rightarrow \mathcal{O}(-n-1) \rightarrow \mathcal{O}(-n) \rightarrow i_*\mathcal{O}(-n)_{PV} \rightarrow 0

The identity $\partial_{n}H^{n-1}(q_1 \circ h) = H^n(h)\circ \partial_{n-1}$ is then a consequence of the cohomology axiom d) (cf. Hartshorne ch. III Theorem 1.1A) and the proof of the theorem is finished.

Remark: the fact that $H^{n-1}(X,i_*\mathcal{O}(-n)_{PV})$ and $H^n(X,\mathcal{O}(-n-1))$ are isomorphic via $\partial_{n}$ does not imply yet that their identifications with $k$ commute with $\partial_{n}$ (although it’s true). This is not a problem however: even if they are different, they differ by a factor in $k^*$. Then simply scaling the pairing with the inverse of this factor would make the diagram commutative, and we would be able to apply the lemma.

**Proof of Corollary.** The corollary follows immediately from the theorem. Indeed, we know $H^0(P(V),\mathcal{O}(i)) = \Gamma(P(V),\mathcal{O}(i)) \cong Sym^i(V^*)$. The perfect pairing described above implies a canonical isomorphism between $H^n(P(V),\mathcal{O}(-n-1-i))$ and $(Sym^i(V^*))^*$. According to problem 2 PS 7, $(Sym^i(V^*))^* \cong Sym^i(V)$.

**Proof 2.** We use a modified version of Cech cohomology used in Hartshorne. Namely, cover $X$ by a set of affines $(U_a)_{a \in I}$ and assume that $I$ is well-ordered. We then construct the Cech complex $C((U_a),\mathcal{F})$ with

$$C^n(U,\mathcal{F}) = \bigoplus_{i_0 < i_1 < \ldots < i_n} \Gamma(\mathcal{F},U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_n})$$

1
The differentials $\partial_n$ are given by

$$(\partial_n \alpha)_{i_0, \ldots, i_{n+1}} = \sum_{j=0}^{n} (-1)^j \alpha_{i_0, \ldots, i_j, \ldots, i_{n+1}} | U_{i_0, \ldots, i_{n+1}}$$

The Čech cohomology of the sheaf is defined to be the cohomology of the complex.

The difference between this definition of Čech cohomology and the Čech cohomology used in class is that the indices $i_k$ are required to be different.

We will show that the two definitions of cohomology coincide for quasi-coherent sheaves on $X = \mathbb{P} V$ (in fact, on all separated Noetherian schemes). Note that this fact follows from the results of ch. III.4. of Hartshorne, by showing that both versions coincide with the cohomology defined in III.2. We will use a slightly different approach, following ch. III. 1. instead.

**Definition.** Let $C_1, C_2$ be abelian categories. A (covariant) $\delta$-functor from $C_1 \to C_2$ is a collection of (covariant) functors $F = (F^i)_{i \geq 0}$ together with morphisms $\delta^i: F^i(A_3) \to F^{i+1}(A_1)$ for every short exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ satisfying the following properties:

(i) There is a long exact sequence

$$0 \to F^0(A_1) \to F^0(A_2) \to F^0(A_3) \xrightarrow{\delta^0} F^1(A_1) \to \ldots \to F^i(A_1) \to F^i(A_2) \to F^i(A_3) \xrightarrow{\delta^i} F^{i+1}(A_1) \to \ldots$$

(ii) for each morphism of the exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ into the short exact sequence $0 \to B_1 \to B_2 \to B_3 \to 0$, the following diagram commutes:

$$
\begin{array}{ccc}
F^i(A_3) & \xrightarrow{\delta^i} & F^{i+1}(A_1) \\
\downarrow & & \downarrow \\
F^i(B_3) & \xrightarrow{\delta^i} & F^{i+1}(B_3)
\end{array}
$$

**Definition.** The $\delta$-functor $F = (F^i): C_1 \to C_2$ is called universal if for any other $\delta$-functor $F' = (F'^i): C_1 \to C_2$ with a natural transformation $f^0: F^0 \to F'^0$, there exists a unique set of natural transformations $f^i: F^i \to F'^i$ starting with $f_0$, which commute with the boundary morphisms.

**Proposition.** If $F, G$ are universal $\delta$-functors and $F^0 \cong G^0$ then $F, G$ are isomorphic.

**Proof:** If $f^0: F^0 \to G^0$ provides the isomorphism, then it can be extended to morphisms $f^i: F^i \to G^i$. Similarly, if $g^0: G^0 \to F^0$ is the inverse map it can be extended to morphisms $g^i: G^i \to F^i$. Then $f^i \circ g^i$ is a sequence of morphisms extending $f^0 \circ g^0 = Id_{G^0}$, and by the uniqueness clause we get $f^i \circ g^i = Id_{C^0}$. Similarly, $g^i \circ f^i = Id_{F^0}$, hence $F \cong G$.

**Definition.** An additive functor $F: C_1 \to C_2$ is called effaceable if every object $A \in C_1$ if there is a monomorphism $i: A \hookrightarrow I$ such that $F(i) = 0$.

**Theorem.** (Grothendieck) If $F = (F^i)$ is a covariant $\delta$-functor with all $F^i$ effaceable for $i > 0$, then $F$ is universal.

**Corollary.** If every object can be injected into an acyclic object, i.e. there for every object $A \in C_1$ there is a monomorphism $i: A \hookrightarrow I$ into an object $I$ with $F^i(I) = 0$ for $i > 0$, then $F$ is universal.

**Theorem.** Let $X$ be a quasi-compact separated scheme. Then the Čech cohomology (both definitions) is a universal $\delta$-functor.

**Proof:** We will work with the definition given in class, the other definition being done similarly. The property i) of the $\delta$-functor has already been shown, and ii) is trivial from the definition. It remains to show that every object can be injected into an acyclic object. Indeed, let $\mathcal{F}$ be a sheaf. Consider $X = \cup \text{Spec}(A_i)$ a finite open covering, and let $M_i = \Gamma(\mathcal{F}, \text{Spec}(A_i))$. 

4
If \( j_i \) is the open embedding of \( \text{Spec}(A_i) \) into \( X \), then \( (j_i)_*((\text{locSpec}(A_j))(M_i))) \) is quasi-coherent because \( X \) is separated. \( \mathcal{F} \) injects into \( \bigoplus (j_i)_*((\text{locSpec}(A_j))(M_i))) \) since the morphism of stalks \( (\mathcal{F})_x \to ((j_i)_*((\text{locSpec}(A_j))(M_i)))_x \cong (M_i)_x \) is an isomorphism for \( x \in \text{Spec}(A_i) \). Finally \( \bigoplus (j_i)_*((\text{locSpec}(A_j))(M_i))) \) is acyclic: every \( \bigoplus (j_i)_*((\text{locSpec}(A_j))(M_i))) \) is acyclic since its cohomologies coincide with the cohomologies of \( (\text{locSpec}(A_j))(M_i)) \) as shown in class, and the higher cohomologies of the latter sheaf are 0 because the underlying scheme is affine.

**Corollary.** The two versions of Cech cohomology of quasi-coherent sheaves on quasi-compact separated schemes are equal.

**Proof:** It suffices to show that the zeroth cohomologies are the same. Indeed, both are isomorphic to the global sections by the gluing axiom.

Now that we have shown that the two definitions of cohomology are equivalent, we use the Hartshorne version of cohomology to write out the Serre duality in explicit form.

Choose a basis \( x_0, x_1, \ldots, x_n \) of \( V^* \) (dual to some basis \( v_0, \ldots, v_n \) of \( V \)). We have the affine subsets \( U_x \cong \text{Spec}(k[\frac{x}{x}]) \) and note that \( U_{x} \cap U_{x_k} = \text{Spec}(k[\frac{x}{x_k}, \frac{x_k}{x}]) \). We choose \( U_{x_k} \) as an open cover.

Note that \( \Gamma(O(i), U_{x} \cap U_{x_k} \cap \ldots \cap U_{x_k}) \) consists of elements in \( \text{Spec}(k[x, 1\frac{x}{x}]) \) of degree \( i \), such that the degree of \( x_r \) for \( r > k \) is non-negative in each monomial.

Particularly, \( H^n(X, O(-n-1-i)) \) is the cokernel of the map \( \partial^n: \bigoplus_j \Gamma(O(-n-1-i), \cap_{l \neq j} U_{x_j}) \to \Gamma(O(-n-1-i), \text{Spec}(k[x, 1\frac{x}{x}])) = \text{Spec}(k[x, 1\frac{x}{x}])_{-n-1-i} \). The differential map on each \( \Gamma(O(-n-1-i), \cap_{l \neq j} U_{x_j}) \to \Gamma(O(-n-1-i), \cap_{l \neq j} U_{x_j}) \) is the natural inclusion (up to a sign), and from here we see that the image of \( \Gamma(O(-n-1-i), \cap_{l \neq j} U_{x_j}) \) consists of all elements in \( \text{Spec}(k[x, 1\frac{x}{x}])_{-n-1-i} \) that only have monomials of non-negative degree in \( x_j \). Subsequently, the cokernel of \( \partial^n \) is naturally identified with the \( k \)-subspace of all elements in \( \text{Spec}(k[x, 1\frac{x}{x}])_{-n-1-i} \) that only include monomials of non-negative degree in each \( x_j \). That is, it is the submodule whose basis consists of \( "negative" \) monomials

\[ x_0^{l_0}x_1^{l_1} \cdots x_n^{l_n} \]

with \( l_i < 0, \sum l_i = -n-1-i \).

It is easy to compute combinatorially that the number of such monomials is \( \binom{n+i}{n} \) - particularly \( H^n(X, O(-n-1-i)) \) has dimension 1 with generator \( x_0^{-1}x_1^{-1} \cdots x_n^{-1} \).

On the other hand, \( H^0(X, O(i)) \) consists of all elements of degree \( i \) in \( k[x_1, \ldots, x_n] \) - with basis positive monomials \( x_0^{m_0}x_1^{m_1} \cdots x_n^{m_n} \) with \( m_i \geq 0 \), \( \sum m_i = n \) (there are \( \binom{n+i}{i} \) such monomials).

Now recall how the natural pairing was constructed. For every \( s \in H^0(X, O(i)) \), we considered the transformation \( \phi_s: O(-n-i-1) \to O(-n-1-i) - \) which is multiplication by \( s \), and took the induced map on homology. Since in Cech homology, the induced map is obtained by restricting the map to each open set that occurs in the Cech complex, we directly see that the bilinear pairing is given by

\[ (x_0^{l_0}x_1^{l_1} \cdots x_n^{l_n}) \times (x_0^{m_0}x_1^{m_1} \cdots x_n^{m_n}) = x_0^{m_0+l_0}x_1^{m_1+l_1} \cdots x_n^{m_n+l_n} \]

However note that \( H^n(X, O(-n-1-i)) \) is the quotient of \( k[x, 1\frac{x}{x}]_{-n-1} \) by monomials of non-negative degree in at least one component - whence we reduce \( x_0^{m_0+l_0}x_1^{m_1+l_1} \cdots x_n^{m_n+l_n} \) to 0 if \( m_j + l_j \geq 0 \) for some \( j \). Since \( \sum_{j=0}^n (m_j + l_j) = -n-1 \), this will happen unless \( m_j + l_j = -1 \).

Hence \( (x_0^{l_0}x_1^{l_1} \cdots x_n^{l_n}) \times (x_0^{m_0}x_1^{m_1} \cdots x_n^{m_n}) = 0 \) unless \( l_i = -1 - m_i \) in which case \( (x_0^{l_0}x_1^{l_1} \cdots x_n^{l_n}) \times (x_0^{m_0}x_1^{m_1} \cdots x_n^{m_n}) = x_0^{-1}x_1^{-1} \cdots x_n^{-1} \). If we identify \( H^n(X, O(-n-1-i)) \) with \( k \) by sending \( x_0^{-1}x_1^{-1} \cdots x_n^{-1} \) to 1, we immediately see that \( (x_0^{m_0}x_1^{m_1} \cdots x_n^{m_n}) \) and \( (x_0^{1-m_0}x_1^{-1-m_1} \cdots x_n^{-1-m_n}) \) are dual bases for these vector spaces with respect to the pairing, and hence the theorem follows. \( \square \)

Remark: in this setting, one can write out explicitly the identification of \( H^n(-n-1-i) \) with \( \text{Sym}^i(V) \) as follows: send \( x_0^{l_0}x_1^{l_1} \cdots x_n^{l_n} \) to \( \sum v_{r_1} \otimes v_{r_2} \otimes \cdots \otimes v_{r_t} \) where the sum is taken over all sequences \( (r_1, \ldots, r_t) \) that contain \(-1-l_j \) terms equal to \( j \).