1. Let $A$ be a commutative Noetherian ring and let $M$ be a f.g. $A$-module.
   (a) Show that $p$ is an associated prime of $M$ if and only if $M_p$ has depth 0.
   (b) Let $p_i, i \in I$ be the set of associated primes of $M$. Show that the map
   $$M \to \bigoplus_{i \in I} M_{p_i}$$
is injective.
   (c) Let $p$ be a prime and $Y_p$ the corresponding closed subscheme of $X := \text{Spec}(A)$. Let $U_p \hookrightarrow X$ be the complement. Show that the map $M \to j_! j^*(M)$ is injective when localized at $p$ if and only if $p$ isn’t an associated prime of $M$.
   (d) Let $Y \subset X$ be an arbitrary closed subscheme and $U \hookrightarrow X$ its complement. Deduce that $Y$ doesn’t contain primes $p$ such that $\text{depth}(M_p) = 0$ if and only if the map $M \to j_! j^*(M)$ is an injection.
   (e) Deduce that if $A$ is $S_1$, and $X$ is reduced at the generic point of each of its irreducible components, then it’s reduced.

2. We retain the assumptions and notation from Problem 1.
   (a) Let $p, Y_p, U_p$ be as in Problem 1(c). Show that the map $M \to j_! j^*(M)$ is an isomorphism when localized at $p$ if and only if $M_p$ has depth $\geq 2$.
   (b) Assume that $M$ satisfies $S_2$ and that it has support on multiple irreducible components of $X$, whose intersections are subschemes of codimension $\geq 2$. Show that $M$ is a direct sum of modules, each of which is supported on its own irreducible component.
(c) Assume now that $A$ is integral, and let $K$ denote its field of fractions. Show that $M$ can be described as the $A$-submodule of $M \otimes_K A$ equal to the intersection of the localizations $M_p$, where $p$ runs over the set of height 1 primes of $A$.

4. Let $A$ be a Noetherian ring.

(a) Show that if $A$ itself is $S_1 + S_2$ and is reduced at each generic point, then it is a disjoint union of integral schemes.

(b) Show that if $A$ is $R_1$ and $S_2$, then it’s also $S_1$.

(c) Show that if $A$ satisfies $S_2$ and $R_1$ and $X$ is connected, then $A$ is integrally closed.

(d) Show that the $S_2$-condition on a module $M$ can be rephrased as follows: $M$ doesn’t have submodules supported on subschemes which are not unions of irreducible components, and any extension

$$0 \to M \to M' \to T \to 0$$

with $\operatorname{codim}(\operatorname{supp}(T)) \geq 2$, splits.

(e) Let $A$ be a domain. Show that the normality of $A$ is equivalent to the following property: for any $A \hookrightarrow A'$ such that $A'$ a domain and finite as an $A$-module and $K \to A' \otimes_A K$ is an isomorphism, we have $A = A'$. Argue that $S_2 + R_1 \Rightarrow "normal"$ from this point of view using (d).

5. Show that if $A$ is an integrally closed Noetherian domain, then it satisfies $S_2$.

Suggested strategy: Argue by contradiction. Reduce the assertion to the situation when $A$ is local, and we have an extension

$$0 \to A \to M' \to T \to 0$$

of $A$-modules, where $M'$ is torsion-free and $T$ is supported at the maximal ideal. Thus, we can think of $M'$ as an $A$-submodule of $K := \operatorname{Frac}(A)$. Let $t \in T$ be an $\mathfrak{m}$-torsion element; let $f$ be some its lift to an element of $M'$. Show that the action of $f$ on $A$ (perceived as an $A$-submodule of $K$) sends $\mathfrak{m}$ to itself, or we'll obtain a contradiction to Krull’s Haupstellensatz.

6. Let $A$ be a Noetherian ring of finite cohomological dimension. Show that for a f.g. $A$-module $M$, its projective dimension is the minimal integer $i$ such that for $i' > i$ we have $\operatorname{Ext}^i_A(M, A) = 0$. (E.g., $M$ is projective iff all the higher Ext's vanish.) Hint: if $N$ is any $A$-module, use a left (!) resolution of $N$ by projective $A$-modules to express $\operatorname{Ext}^i_A(M, N)$ via $\operatorname{Ext}^i_A(M, A)$.

7. Let $A$ be a Noetherian ring of finite cohomological dimension.

(a) Show that any locally free coherent sheaf over $A$ is $S_k$ for any $k$.

(b) Reprove that $\operatorname{Ext}^i(N, A) = 0$ for a f.g. $A$-module $N$ with $\operatorname{codim}(\operatorname{supp}(M)) > i$.

(c) Deduce that for a f.g. $A$-module, its projective dimension is $\geq$ the codimension of its support.

8. Let $A$ be a Noetherian ring, such that for every maximal idea $\mathfrak{m}$ the localization $A_{\mathfrak{m}}$ is regular of dimension $n$. Let $M$ be a f.g. $A$-module, which is $S_n$. Show that $M$ is locally free.