Homological algebra

1. Let $\mathcal{A}$ be an abelian category.
   (a) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence. Consider the "hut"
   
   \[ Z \leftarrow (X \to Y) \to X[1], \]
   
   where $(X \to Y)$ is considered as a complex in degree $-1$ and $0$. Show that this
   construction establishes a bijection between isomorphism classes of extensions as
   above and $\text{Ext}^1(X,Y) := \text{Hom}_D(\mathcal{A})(Z,X[1])$.
   
   (b) Let $X \to Y$ be a morphism in $\mathcal{A}$; let $K,I,J$ denote its kernel, image and
   cokernel, respectively. We obtain the short exact sequences
   
   \[ 0 \to K \to X \to I \to 0 \quad \text{and} \quad 0 \to I \to Y \to J \to 0, \]
   
   along with the corresponding elements in
   
   $\text{Ext}^1(J,I) \simeq \text{Hom}_D(\mathcal{A})(J,I[1])$ and $\text{Ext}^1(I,K) \simeq \text{Hom}_D(\mathcal{A})(I,K[1])$.
   
   Their composition (a.k.a. cup product) gives an element in
   
   $\text{Ext}^2(J,K) := \text{Hom}_D(\mathcal{A})(J,K[2])$.
   
   Consider the "hut"
   
   \[ J \leftarrow (K \to X \to Y) \to K[2]. \]
   
   Show that it represents the same class in $\text{Hom}_D(\mathcal{A})(J,K[2])$.
   
   (c) Show that in the situation of (b) the resulting element of $\text{Hom}_D(\mathcal{A})(J,K[2])$ vanishes if and only if there exists an object $Z \in \mathcal{A}$ endowed with maps $K \hookrightarrow Z; Z \twoheadrightarrow J$, and isomorphisms $Z/K \simeq Y$ (compatibly with the projection to $J$) and $X \simeq \ker(Z \to J)$, compatibly with the embedding of $K$.

2. Let $\mathcal{A}$ be an abelian category, and consider its derived category $D(\mathcal{A})$. For
   an integer $n$ consider the full subcategory $D(\mathcal{A})^{\leq n}$ consisting of objects $X^\bullet$ such
   that $H^i(X^\bullet) = 0$ for $i > n$, and the full subcategory $D(\mathcal{A})^{\geq n}$ consisting of objects $X^\bullet$ such that $H^i(X^\bullet) = 0$ for $i < n$. (Observe that $D(\mathcal{A})^- = \bigcup_{n \in \mathbb{Z}} D(\mathcal{A})^{\leq n}$ and $D(\mathcal{A})^+ = \bigcup_{n \in \mathbb{Z}} D(\mathcal{A})^{\geq n}$.)
   
   (a) Show that for $X \in D(\mathcal{A})^{\leq n}$ and $Y \in D(\mathcal{A})^{\geq n+1}$ we have $\text{Hom}_D(\mathcal{A})(X,Y) = 0$.
   
   (b) Assume that an object $X \in D(\mathcal{A})$ fits into a distinguished triangle
   
   \[ X^{\leq n} \to X \to X^{\geq n+1} \to X^{\leq n}[1] \]
with $X^{\leq n} \in D(A)^{\leq n}$ and $X^{\geq n+1} \in D(A)^{\geq n+1}$, and similarly for an object $Y$. Show that any map $X \to Y$ fits into a *unique* map of triangles

$$
\begin{array}{cccccccc}
X^{\leq n} & \longrightarrow & X & \longrightarrow & X^{\geq n+1} & \longrightarrow & X^{\leq n}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y^{\leq n} & \longrightarrow & Y & \longrightarrow & Y^{\geq n+1} & \longrightarrow & Y^{\leq n}[1].
\end{array}
$$

Deduce that a decomposition as above, if exists, is canonical. Show that $H^i(X^{\leq n}) \simeq H^i(X)$ for $i \leq n$ and is 0 otherwise, and $H^i(X^{\geq n+1}) \simeq H^i(X)$ for $i \geq n + 1$ and is 0 otherwise.

(c) Show that a decomposition as in (b) always exists. Hint: for a complex $X^\bullet$ of objects of $A$ define $X^{\leq n}$ to be the subcomplex, which coincides with $X^\bullet$ in degrees $< n$, to be $\ker(X^n \to X^{n+1})$ in degree $n$, and zero in degrees $> n$. Set $X^{\geq n+1}$ be the quotient complex.

Combining points (b) and (c), we obtain that we have well-defined functors

$$(X) \mapsto \tau^{\leq n}(X) := X^{\leq n} \text{ and } X \mapsto \tau^{\geq n+1}(X) := X^{\geq n+1},$$

which are the left and right adjoints to the embeddings $D(A)^{\leq n} \to D(A)$ and $D(A)^{\geq n+1} \to D(A)$, respectively. These functors are called the truncation functors.

3. *(optional).* Let $D$ be a triangulated category. Let $D^{\leq 0}$ and $D^{\geq 1}$ be two full subcategories satisfying:

- $D^{\leq 0}[1] \subset D^{\leq 0}$, $D^{\geq 1}[-1] \subset D^{\geq 1}$.
- For $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$, $\text{Hom}(X, Y) = 0$.
- For any $X \in D$ there exists a distinguished triangle

$$X^{\leq 0} \to X \to X^{\geq 1} \to X^{\leq 0}[1].$$

(a) Prove the analog of point (b) of Problem 2. Show that the categories $D^{\leq 0}$ and $D^{\geq 1}$ determine each other by means of considering $\perp$.

A datum as above on a triangulated category is called a "t-structure".

(b*) Set $A := D^{\leq 0} \cap D^{\geq 1}[1]$. Show that this is an abelian category. This abelian category is called the "core" or "heart" of $D$ with respect to the given t-structure.

Warning: it's not in general true that for a triangulated category $D$ with a t-structure and heart $A$ we have $D \simeq D(A)$. In most situations in practice, we will have a triangulated functor $D(A)^b \to D$.

4. Let $D$ be a triangulated category that contains countable direct sums. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \ldots$ be an inductive system of objects numbered by natural numbers. We define a new object of $D$, called a homotopy direct limit of $X_i$ and denoted $\varinjlim X_i$ to be

$$\text{Cone}(\bigoplus_{i=0,1,2\ldots} X_i \xrightarrow{f_{i+1}} \bigoplus_{i=0,1,2\ldots} X_i),$$

where the map $f$ sends $X_i$ to $X_i \oplus X_{i+1}$ by means of $\id \oplus (-f_i)$.

Assume now that $D = D(A)$ for an abelian category $A$, which has countable direct sums, and in which the functor of direct limit over $\mathbb{N}$ is exact (e.g., $A = \text{Ab}$, $R$-mod, $\text{QCoh}(X)$, $\text{Sh}(X)$).

(a) Construct natural isomorphisms $H^n(\varinjlim X_i) \simeq \varinjlim H^n(X_i)$ for any $n$. 
(b) Assume that the maps $X_i \to X_{i+1}$ come from actual maps of complexes $X_i \to X_{i+1}$, and let $\lim X_i$ be their inductive limit as a complex. Construct an isomorphism in $D(A)$ between $\lim X_i$ and $\text{holim } X_i$.

(c*) Show that, in general, $\text{holim } X_i$ is not isomorphic to the categorical inductive limit of the $X_i$'s taken in $D(A)$.

Suggested strategy: let $\mathcal{A} = k[t]$-mod, where $k$ is a field. Let all $X_i = k[t]$, considered as a complex in degree 0 with the maps $f_i$ each being multiplication by $t$. Identify $\lim X_i$ with $k[t, t^{-1}]$. Show that $\text{Ext}^1(k[t, t^{-1}], k[t]) \neq 0$ although all $\text{Ext}^1(X_i, k[t]) = 0$.

(d) For an object $X \in D(A)$ set $X_i = \tau^{\leq i}(X)$, where $\tau^{\leq i}$ is as in Problem 2. Show that $X$ is isomorphic to $\lim X_i$.

Sheaves and cohomology

5. Let $\mathcal{A}$ be an abelian category with enough injectives. Let $f : K^+(\mathcal{A}) \to \text{Ab}$ be a cohomological functor, and let $Rf : D^+(\mathcal{A}) \to \text{Ab}$ be its derived functor. We say that on object $X \in \mathcal{A}$ is $f$-acyclic if $R^i f(X) := Rf(X[i]) = 0$ for $i \neq 0$.

(a) Let $X^* \in K^+(\mathcal{A})$ be a complex consisting of $f$-acyclic objects. Show that the natural morphism $f(X^*) \to Rf(X^*)$ is an isomorphism.

(b) Let $\mathcal{A} = \text{Sh}(X)$. A sheaf $\mathcal{F}$ is called "flabby" or "flasque" if for any pair of open subsets $U \subset V$, the map $\Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective.

(bi) Show that injective sheaves are flasque.

(bii) Show that if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves with $\mathcal{F}_1$ flasque, then the map $\Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3)$ is surjective.

(biii) Show that if in a short exact sequence as above $\mathcal{F}_1$ and $\mathcal{F}_2$ are flasque, then so is $\mathcal{F}_3$.

(biv) Deduce that flasque sheaves are acyclic for the functor $H^0(X, -)$, and for any direct image functor $\Phi_*$, where $\Phi : X \to Y$ is a map of topological spaces.

(c) Let $\Phi : X \to Y$ be a morphism between topological spaces. Show that $\Phi_*$ sends flasque sheaves on $X$ to flasque sheaves on $Y$.

(d) Obtain an alternative proof of the "Leray spectral sequence isomorphism": $R\Gamma_Y \circ R\Phi_* \simeq R\Gamma_X$.

5. Let $X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$ maps of topological spaces. Consider the functors $R\Phi_* : D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Y))$, $R\Psi_* : D^+(\text{Sh}(Y)) \to D^+(\text{Sh}(Z))$ and $R(\Psi \circ \Phi)_* : D^+(\text{Sh}(X)) \to D^+(\text{Sh}(Z))$. Construct a natural transformation $R(\Psi \circ \Phi)_* \to R\Psi_* \circ R\Phi_*$ and prove that it is an isomorphism.

6. For $\Phi$ as in Problem 5 and $\mathcal{F}^* \in D^+(\text{Sh}(Y))$ show that $R^i \Phi(\mathcal{F}^*)$ is isomorphic to the sheaf associated to the presheaf $U \mapsto H^i(\Phi^{-1}(U), \mathcal{F}^*)$.

7. Let $\Phi : X \to Y$ be an affine map of schemes, and $\mathcal{F} \in \text{QCoh}(X)$. Show that $R\Phi_*(\mathcal{F})$ (when $\mathcal{F}$ is considered just as a sheaf of abelian groups) is isomorphic to $\Phi_*(\mathcal{F})$, i.e., that the higher cohomologies vanish.