1. Let $X$ and $Y$ be complexes of objects of some additive category $A$. Define the complex $\text{Hom}^\bullet(X, Y)$ by
$$\text{Hom}^n(X, Y) := \prod_i \text{Hom}(X^i, Y^{i+n})$$
with the differential $d(f) = d_Y \circ f - (-1)^n \cdot f \circ d_X$ for $f \in \text{Hom}^n(X, Y)$. We denote the resulting DG category by $\text{Comp}_{DG}(A)$.

Recall that if $\mathcal{C}$ is a DG category, we consider the additive category $Z(\mathcal{C})$ with objects the same as those of $\mathcal{C}$ and $\text{Hom}_{Z(\mathcal{C})}(X, Y) = \{ \alpha \in \text{Hom}_\mathcal{C}(X, Y), d(\alpha) = 0 \}$ and the category $\text{Ho}(\mathcal{C})$ (called the homotopy category of $\mathcal{C}$) again with the same objects but with $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) = H^0(\text{Hom}_\mathcal{C}(X, Y))$.

Identify $Z(\text{Comp}_{DG}(A))$ with the usual category $\text{Comp}(A)$ (whose objects are complexes an morphisms are maps commuting with the differential). Identify $\text{Ho}(\text{Comp}_{DG}(A))$ with $K(A)$ (the category of comlexes with morphisms taken up to homotopy).

2. Let $\mathcal{C}$ be a DG category. For $X, Y \in \mathcal{C}$ we’ll denote by $\text{Hom}^\bullet_{\mathcal{C}}(X, Y)$ the corresponding complex. We’ll also regard $\mathcal{C}$ as a plain (additive) category by setting $\text{Hom}^i_{\mathcal{C}}(X, Y) = \bigoplus \text{Hom}^i(X, Y)$.

(a) For an object $X \in \mathcal{C}$ and an integer $k$ we let $X[k]$ be an object (if it exists) endowed with an isomorphism $X \to X[k]$ in $\mathcal{C}$ regarded as a plain category, such that $\alpha \in \text{Hom}_{\mathcal{C}}^{k-1}(X, X[k])$ and $d(\alpha) = 0$. Show that if $X[k]$ exists, then it’s unique up to a unique DG-isomorphism. 1 Show that for $Y \in \mathcal{C}$ we have a canonical isomorphism of complexes
$$\text{Hom}_{\mathcal{C}}^\bullet(Y, X[k]) \simeq \text{Hom}_{\mathcal{C}}^\bullet(Y, X)[k]$$
such that for $Y_1, Y_2$, the square
$$\begin{array}{ccc}
\text{Hom}_{\mathcal{C}}^\bullet(Y_2, X[k]) \otimes \text{Hom}_{\mathcal{C}}^\bullet(Y_1, Y_2) & \longrightarrow & \text{Hom}_{\mathcal{C}}^\bullet(Y_2, X)[k] \otimes \text{Hom}_{\mathcal{C}}^\bullet(Y_1, Y_2) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{C}}^\bullet(Y_1, X[k]) & \longrightarrow & \text{Hom}_{\mathcal{C}}^\bullet(Y_1, X)[k]
\end{array}$$
commutes. Formulate and prove the corresponding ”mapping out” property.

(b) Let $f : X \to Y$ be a closed morphism of degree 0. We let $\text{Cone}(f)$ be an object $Z$ of $\mathcal{C}$ (if it exists) endowed with an isomorphism $Z \simeq X \oplus Y$ in $\mathcal{C}$ (regarded as a plain additive category) such that

1. The morphism $i : Y \to Z$ is closed of degree 0.

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1"DG-morphism":="closed morphism of degree 0"; "DG-isomorphism":="isomorphism+DG morphism"; "homotopy equivalence":="a DG morphism which induces an isomorphism in $\text{Ho}(\mathcal{C})$".
• (ii) \( p : Z \to X \) is closed of degree 1.

• (iii) The morphism \( j : X \to Z \) (which is automatically of degree \(-1\)) is such that \( d(j) \in \text{Hom}^0(X, Z) \) (which automatically is a cycle and belongs to \( \text{Hom}^0(X, Y) \)), equals \( f \).

Show that if \( \text{Cone}(f) \) exists, then it's unique up to a unique DG-isomorphism. Formulate and prove the "mapping in" and "mapping out" properties for \( \text{Cone}(f) \), similar to point (a) above.

3. Let \( \mathcal{C} \) be a DG category.

(a) Let \( f : X \to Y \) be a DG-morphism, which is a split embedding when \( \mathcal{C} \) is considered as a plain additive category. Assuming that \( \text{Cone}(f) \) exists, construct a homotopy equivalence between \( \text{Cone}(f) \) and \( \text{coker}(f) \).

(b) Let \( f : X \to Y \) be an arbitrary DG-morphism. Construct an object \( \tilde{Y} \) and DG-morphisms \( \tilde{f} : X \to \tilde{Y} \) and \( g : \tilde{Y} \to Y \), so that \( f = g \circ \tilde{f} \), \( g \) is a homotopy equivalence, and \( \tilde{f} \) is a split embedding.

(c) Prove analogs of (a) and (b) with "split embedding" replaced by "split surjection".

4. Let \( \mathcal{D} \) be an additive category satisfying the axioms \( \text{Tr1}, \text{Tr2}, \text{Tr3} \).

(a) Let \( X \xrightarrow{f} Y \to Z \to X[1] \) and \( X' \xrightarrow{f'} Y' \to Z' \to X'[1] \) be distinguished triangles, and let \( u : X \to X' \), \( v : Y \to Y' \) be maps such that \( u \circ f = f' \circ v \). Let \( w : Z \to Z' \) be a map completing \( u,v \) to a map of distinguished triangles. Show that the ambiguity for \( w \) is given by an element of \( \text{Hom}_{\mathcal{D}}(X[1], Y') \).

(b) Let \( f : X \to Y \) be a morphism. Show that any two ways of completing \( f \) to a distinguished triangle are isomorphic, but this isomorphism is not unique and the ambiguity is given by an element of \( \text{Hom}_{\mathcal{D}}(X[1], Y) \).

5. Let \( \mathcal{D} \) be an additive category satisfying the axioms \( \text{Tr1}, \text{Tr2}, \text{Tr3} \). Let \( f : X \to Y \), \( g : Y \to Z \) be morphisms, and let

\[
X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1] \quad \text{and} \quad Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]
\]

be distinguished triangle. Choose a distinguished triangle

\[
X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{\tilde{d}'} Y[1].
\]

Let \( \theta = u[1] \circ d' : Z/Y \to X/Y[1] \).

Choose maps \( \phi : Y/X \to Z/X \) and \( \psi : Z/X \to Z/Y \) so that we have maps of distinguished triangles

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\text{id}} & & \downarrow{g} \\
X & \xrightarrow{g \circ f} & Z \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
Y/X & \xrightarrow{d} & X[1] \\
\downarrow{\phi} & & \downarrow{\text{id}} \\
Z/X & \xrightarrow{\tilde{d}'} & X[1] \\
\end{array}
\]

\[\text{We say that a morphism } f : X \to Y \text{ is a split embedding if there exists a object } X' \text{ and a morphism } g : X' \to Y \text{ such that } X \oplus X' \to Y \text{ is an isomorphism. We require that } g \text{ be of degree 0 (but not necessarily a cycle), but that the projection } Y \to X' \text{ be a cycle; it identifies } X' \text{ with the cokernel of } f.\]
and

\[
\begin{array}{cccccc}
X & \xrightarrow{g \circ f} & Z & \xrightarrow{w} & Z/X & \xrightarrow{d''} & X[1] \\
\downarrow{f} & & \downarrow{id} & & \downarrow{\psi} & & \downarrow{f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{v} & Z/Y & \xrightarrow{d'} & Y[1] \\
\end{array}
\]

(which exist by Tr3). Consider the triangle

\[(2)\]

\[
\begin{array}{cccc}
Y/X & \xrightarrow{\phi} & Z/X & \xrightarrow{\psi} & Z/Y & \xrightarrow{\theta} & Y/X[1].
\end{array}
\]

Show that the following are equivalent:

• (i) For any choice of the triangle (1), there exists a choice of \(\phi\) and \(\psi\), so that the triangle (2) is distinguished.
• (ii) There exists a choice of a triangle (1), \(\phi\) and \(\psi\), so that the triangle (2) is distinguished.

Show that condition (ii) above is the same as the "octahedron" axiom, Gelfand-Manin, page 240.

6. Let \(A\) be an additive category. Set

\[
\mathcal{D} := K(A) := \text{Ho(Comp}_{DG}(A)).
\]

Define distinguished triangles as those isomorphic to one of the form

\[
X \xrightarrow{f} Y \rightarrow \text{Cone}(f) \rightarrow X[1]
\]

for a DG-morphism \(f : X \rightarrow Y\). Show that \(\mathcal{D}\) satisfies the axioms of triangulated category.

Hint: Axioms Tr1 and Tr3 should be straightforward. For Tr2 and Tr4 use Problem 3.

7. (optional). Let \(\mathcal{D}\) be a triangulated category. Show that a direct sum of two triangles is distinguished if and only if each of them is distinguished.

8. (optional). Let \(\mathcal{D}\) be a triangulated category, which also happens to be abelian. Show that this abelian category is semi-simple, and that every distinguished is isomorphic to a direct sum of (shifts of) triangles of the form \(X \xrightarrow{id} X \rightarrow 0 \rightarrow X[1]\).