MATH 233B, FLATNESS AND SMOOTHNESS.

The discussion of smooth morphisms is one place were Hartshorne doesn’t do a very good job. Here’s a summary of this week’s material. I’ll also insert some (optional) exercises that I recommend that you have do in order to better understand the material.

1. A refresher on flatness

1.1. Let $A$ be a commutative ring. Recall that an $A$-module $M$ is flat if and only if the functor $M \otimes_A -$ is exact.

**Exercise 1.1.1.** (1) Show that a module is flat if and only if its localization at any prime is flat.
(2) Show that $M$ is flat if and only if $\text{Tor}_1(M, A/I) = 0$ for any ideal $I \subset A$.

**Exercise 1.1.2.** Assume that $A$ is Noetherian.
(1) Show that a module $M$ is flat if and only if $\text{Tor}_1(M, A/p) = 0$ for any prime $p$.
(2) Show that a module $M$ is flat if and only if $\text{Tor}_i(M, k_p) = 0$ for any prime $p$ and $i \in \mathbb{N}$.
(3) Assume in addition that $M$ is f.g. over $A$. Show that in (2) it’s enough to check only the maximal ideals and $\text{Tor}_1$.
(4) Show that the f.g. assumption above is necessary.

1.2. Flat morphisms.

**Definition 1.2.1.** We say that a morphism $f : X \to Y$ is flat if, locally on $X$, the structure sheaf $\mathcal{O}_X$ is flat as an $\mathcal{O}_Y$-module.

**Definition 1.2.2.** We say that a morphism $f : X \to Y$ is flat at $x$, if the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$.

**Easy exercise 1.2.3.** A morphism is flat if and only if it is flat at any point.

More generally, we can define the notion of $Y$-flatness (resp., $Y$-flatness at a point $x$), a.k.a. flatness with respect to $f$ (resp., flatness with respect to $f$ at a point $x$) for $\mathcal{F} \in \text{QCoh}(X)$.

**Definition 1.2.4.** A map is faithfully flat if it is flat and surjective.

**Exercise 1.2.5.**
(1) Show that a map $f$ if faithfully flat if and only if the functor $f^*$ is exact and conservative.
(2) Deduce that the notion of faithful flatness is stable under base change.
(3) Show that a flat map is open.

*Date: April 5, 2010.*
1.3. Here are some basic facts about flatness in the Noetherian situation:

**Exercise 1.3.1.** Assume that $Y$ is Noetherian, and $\mathcal{F} \in \text{QCoh}(X)$.

1. Show that $\mathcal{F}$ is $Y$-flat if and only if, $\text{Tor}_i^{O_Y}(\mathcal{F}, k_y)$ vanishes (as a q.c. sheaf on $X$) for all $y \in Y$ and $i \in \mathbb{N}$.

2. Give a (stupid) counterexample of how the above fails if we only check the closed points.

3* Assume that both $X$ and $Y$ of finite type over a field, and assume that $\mathcal{F}$ is coherent on $X$. Show that in this case, it’s enough to check only the closed points and $\text{Tor}_1$.

Here’s an important generic flatness theorem:

**Proposition 1.3.2.** Let $f : X \to Y$ be a morphism of finite type with $Y$ integral and Noetherian. Let $\mathcal{F}$ be a coherent sheaf on $X$. Then there exists a non-empty open subscheme $\hat{Y} \subset Y$, such that $\mathcal{F}|_{f^{-1}(\hat{Y})}$ is $Y$-flat.

For the proof, google ”generic flatness fantechi” and go to the first link (Gothendieck’s FGA is explained).

Here is a relationship between the notions of flatness and flatness at a point:

**Exercise 1.3.3.** Let $f : X \to Y$ be a morphism of finite type with $Y$ Noetherian. Let $\mathcal{F}$ be a coherent sheaf on $X$. Let $y \in Y$ be a point such that for every $x \in X$ with $f(x) = y$, the sheaf $\mathcal{F}$ is $Y$-flat at $x$. Show that $y$ admits a Zariski neighborhood over which $\mathcal{F}$ is flat.

(This is a non-trivial exercise, and the proof that I know uses generic flatness.)

1.4. Here is a basic fact about the interaction between the notions of flatness and dimension.

If $X$ is a Noetherian scheme and $x \in X$ is a point, we denote by $\dim(X)_x$ the "dimension of $X$ at $x"$", i.e., the dimension of the Noetherian local ring $\mathcal{O}_{X,x}$.

**Proposition 1.4.1.** Let $f : X \to Y$ be a morphism between Noetherian schemes. Let $x \in X$ be a point and $y = f(x)$ its image in $Y$. Let $X_y$ be the fiber of $X$ over $y$, i.e., $X \times_Y k_y$.

1. We have the inequality: $\dim(X_y)_x + \dim(Y)_y \geq \dim(X)_x$.

2. If $f$ is flat at $x$, then the above inequality is an equality.

3. If $Y$ is regular at $y$ and $X$ is CM at $x$, then the assertion of (2) is "if and only if".

1.5. Let $X \to Y$ be a faithfully flat map with both $X$ and $Y$ locally Noetherian. Let’s see how some favorable properties of $X$ imply those for $Y$.

**Proposition 1.5.1.** If $X$ is (i) reduced, (ii) integral, (iii) regular, (iv) $R_n$, (v) $S_n$, (vi) locally factorial, then the same is true for $Y$.

**Proof.** The proof of (i) and (ii) is easy, while (iii) was done in class. To prove (iv) let’s localize $Y$ at a point $y$ of height $m \leq n$. Consider the scheme $X_y$, and let $x$ be one of its generic points. By Proposition 1.4.1, $\dim(X)_x = \dim(Y)_y$, so $X$ is regular at $x$ by assumption. Hence, $Y$ is regular at $y$ by (iv).
Point (v) was done in class, but let’s repeat it nonetheless. The assertion is local, so we can assume that both $X$ and $Y$ are affine. Property $S_n$ means that for $F \in \text{Coh}(Y)$ with $\text{codim}(\text{supp}(F)) \geq n$, we have $\text{Ext}^k_{\mathcal{O}_Y}(F, \mathcal{O}_Y) = 0$ for $k \leq n-1$. By faithful flatness, it’s enough to show that $f^*(\text{Ext}^k_{\mathcal{O}_Y}(F, \mathcal{O}_Y)) = 0$, while the latter, by flatness and the fact that $F$ is coherent, is isomorphic to $\text{Ext}^k_{\mathcal{O}_X}(f^*(F), \mathcal{O}_X)$. By Proposition 1.4.1, $\text{codim}(\text{supp}(X)(f^*(F))) = \text{codim}(\text{supp}(Y)(F))$, so the latter group vanishes since $X$ was $S_n$.

Let’s prove (vi). Let $j : \overset{\circ}{Y} \hookrightarrow Y$ be an open subset whose complement is of codim $\geq 2$. Let $L$ be a line bundle over $\overset{\circ}{Y}$. We need to show that it admits an extension to a line bundle on the entire $Y$. We claim that $j_* (L)$ does the job. Indeed, since $f$ is faithfully flat, it’s enough to show that $f^*(j_*(L))$ is a line bundle on $X$. Since $f$ is flat, by the trivial case of the projection formula, we have: $f^*(j_*(L)) \cong \tilde{j}_*(f^*(L))$, where $\tilde{j} : \overset{\circ}{X} \hookrightarrow X$, where $\overset{\circ}{X} := f^{-1}(\overset{\circ}{Y})$. However, by Proposition 1.4.1, $\text{codim}(X - \overset{\circ}{X}) = \text{codim}(Y - \overset{\circ}{Y})$, so we know that $f^*(L)$ admits an extension to a line bundle $L'$ on $X$. Since $X$ is normal, we also know that this extension must coincide with $\tilde{j}_*(f^*(L))$. \qed

2. Smoothness over a field

2.1. Before we proceed, let’s give a refresher on the behavior of dimension for schemes of finite type over a field.

Here are two important facts:

**Proposition 2.1.1.** Let $X$ be a scheme of finite type over a field $k$. Then:

(i) If $X$ irreducible, then $\dim(X)_{x_1} = \dim(X)_{x_2}$ for any two closed points $x_1, x_2 \in X$.

(ii) If $X$ is integral, then $\dim(X)$ equals $\text{tr. deg.}(K(X)/k)$, where $K(X)$ is the field of fractions of $X$.

2.2. Let first $k$ be an algebraically closed field, and $X$ a scheme of finite type over $k$.

**Definition 2.2.1.** We say that $X$ is smooth of dimension $n$ if it is regular as a scheme and has dimension $n$.

**Lemma 2.2.2.** Assume that every irreducible component of $X$ is of dimension $\geq n$. Then the following conditions are equivalent:

(i) $X$ is smooth of dimension $n$.

(ii) $\Omega^1_{X/k}$ is a locally free sheaf of dim $n$.

The proof follows the fact that for any closed point $x \in X$, the natural map $m_x/m_x^2 \rightarrow (\Omega^1_{X/k})_x$ is an isomorphism. \qed
2.3. A digression on the latter map:

**Exercise 2.3.1.** Let $A$ be a local ring over a field $k$, such that $k_A$, the residue field of $A$ is a finite extension of $k$.

(i) Construct the natural map $\mathfrak{m}/\mathfrak{m}^2 \to (\Omega^1_{A/k}) \otimes A$.

(ii) Give an example of how this map fails to be an isomorphism even when $A$ is regular.

(iii) Prove (by induction) that if $k_A/k$ is perfect, then for any $n$ the projection $A/\mathfrak{m}^n \to k_A$ admits a unique section (i.e., this makes $A/\mathfrak{m}^n$ into a $k_A$-algebra).

(iv) Deduce that if $k_A/k$ is perfect, then the map in (i) is an isomorphism.

2.4. Let $k$ be an arbitrary field. Let $X$ be a scheme of finite type over $X$.

**Definition 2.4.1.** We say that $X$ is smooth of dimension $n$ over $k$ if $X \times \overline{k}$ is smooth of dimension $n$ over $\overline{k}$.

From Lemma 2.2.2 we obtain:

**Corollary 2.4.2.** A scheme $X$ is smooth of dimension $n$ over $k$ if and only if each of its irreducible components has dimension $\geq n$, and $\Omega^1_{X/k}$ is locally free of rank $n$.

As we saw in class, an imperfect field extension $k'/k$ provides an example of a scheme over $k$, which is regular, but not smooth.

**Exercise 2.4.3.** Let $X$ be a scheme over $k$.

(i) Show that any scheme smooth over a field is regular.

(ii) Let $k'/k$ be a separable field extension. Show that if $X$ is regular, then so is $X' := X \times_k k'$.

(iii) Show that over a perfect field, a scheme is smooth if and only if it’s regular.

2.5. Let’s recall the following assertion about regular local rings:

**Proposition 2.5.1.** Let $A' \twoheadrightarrow A$ be a surjection of local rings with $A'$ regular of dimension $m$. Then the following conditions are equivalent:

(a) $A$ is regular of dimension $n$.

(b) The ideal $\ker(A' \twoheadrightarrow A)$ can be generated by $m - n$ elements $f_1, \ldots, f_{m-n}$, whose images in $m'/m'^2$ are linearly independent.

**Proof.** Exercise. □

We shall now discuss an analog of this assertion when “regular” is replaced by “smooth”.

**Theorem 2.5.2.** Let $X$ be a scheme of finite type over $k$. Then the following conditions are equivalent:

(a) $X$ is smooth over $k$.

(b) For any closed embedding $X \hookrightarrow X'$ with with a sheaf of ideals $\mathcal{I}$ and $X'$ smooth, the sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X'/k}|_X \to \Omega^1_{X/k} \to 0$$

is a short exact sequence of vector bundles.
Proof. The implication (b) ⇒ (a) follows from Lemma 2.4.2. For the implication in the other direction, we can base change to \( \overline{k} \). In the latter case we deduce it from Proposition 2.5.1 using the following lemma:

**Lemma 2.5.3.** Let \( \alpha : F \rightarrow E \) be a map of coherent sheaves on a locally Noetherian scheme \( X \), where \( E \) is locally free. Assume that for every \( x \in X \), the map \( F_x \rightarrow E_x \) is injective. Then: (i) the map \( \alpha \) is injective, (ii) \( F \) is locally free, (iii) \( \text{coker}(\alpha) \) is locally free. Moreover, the above condition is sufficient to check for closed points only.

\[ \square \]

Note the difference in the conditions of Proposition 2.5.1 and Theorem 2.5.2: for a closed point \( x \in X \), the former requires that we can generate\( \mathcal{I}_x := \ker(\mathcal{O}_{X',x} \rightarrow \mathcal{O}_{X,x}) \) by elements \( f_1, ..., f_{m-m'} \) such that their images \( \overline{f}_1, ..., \overline{f}_{m-m'} \) in \( m'/m'^2 \) are linearly independent. The latter requires that their further images \( df_i(x) \in (\Omega_{X'/x})_x \) be linearly independent.

### 2.6. Generic smoothness.

**Definition 2.6.1.** We say that \( X \) is generically smooth of dimension \( n \) over \( k \) if \( X \) contains a dense Zariski open \( \tilde{X} \), which is smooth of dimension \( n \) over \( k \).

**Lemma 2.6.2.** Let \( X \) be an irreducible scheme of finite type over \( k \).

1. The generic rank of \( \Omega_{X/k} \) is \( \geq \dim(X) \).
2. The equality in (1) holds if and only if \( X \) is generically smooth.

The proof follows from Lemma 2.2.2 by base change. \[ \square \]

**Definition 2.6.3.** A finitely generated field extension \( K/k \) is said to be separably generated if it can be written in the form \( K \subset K_0 = k(x_1, ..., x_m) \), where \( K/K_0 \) is a finite separable extension.

**Lemma 2.6.4.** Let \( K/k \) be a finitely generated field extension.

1. We have \( \dim_K(\Omega_{K/k}) \geq \text{tr.deg.}(K/k) \).
2. If \( K/k \) is separably generated, then the inequality in (1) is an equality.

NB: We’ll soon see that the assertion of (2) is in fact "if and only if".

**Proof.** To prove point (1) choose an integral scheme of finite type over \( X \) with field of fractions \( K \). Then the assertion follows from Lemma 2.6.2.

To prove point (2), consider the exact sequence:

\[ K \otimes_{K_0} \Omega_{K_0/k} \rightarrow \Omega_{K/k} \rightarrow \Omega_{K/K_0}, \]

and the assertion follows. \[ \square \]

**Lemma 2.6.5.** If \( k \) is perfect, then any finitely generated field extension is separably generated.

For the proof, see references in Hartshorne, Theorem 4.8A.

**Corollary 2.6.6.** Any integral scheme over a perfect field is generically smooth.
3. Smooth morphisms between schemes

3.1. Let \( f : X \rightarrow S \) be a morphism of schemes. Until the end of this write-up, we’ll assume that the base \( S \) is locally Noetherian.

Assume that \( f \) is of finite type (smoothness is only defined for morphisms of finite type).

**Definition 3.1.1.** We say that \( f \) is smooth of relative dimension \( n \) if the following conditions hold:

(i) \( X \) is flat over \( S \).

(ii) For every point \( s \in S \), the base change \( X \times_{S} \mathbb{k} \) is a smooth scheme of dimension \( n \) over the residue field \( \mathbb{k} \).

**Theorem 3.1.2.** Let \( f : X \rightarrow S \) be a morphism of finite type. The following conditions are equivalent:

(a) Condition (ii) in the definition of smoothness holds, and \( \Omega_{X/S} \) is locally free of rank \( n \).

(b) \( f \) is smooth of relative dimension \( n \).

(c) \( X \) is flat and locally on \( X \), we can find a closed embedding over \( X \hookrightarrow X' := \mathbb{A}^n_S \) (compatible with the projection to \( S \)), so that if we denote by \( \mathcal{I}_{X,X'} \) the corresponding sheaf of ideals on \( X' \), the sequence

\[
\mathcal{I}_{X,X'}/\mathcal{I}_{X,X'}^2 \rightarrow \Omega_{X'/S}|_X \rightarrow \Omega_{X/S} \rightarrow 0
\]

is a short exact sequence of vector bundles.

**Proof.** The fact that (a) implies (b) follows from Lemma 2.2.2. The implications (c) \( \Rightarrow \) (a) is tautological. So, it’s enough to show that (b) implies (c). We’ll use the following generalization of Lemma 2.5.3:

**Lemma 3.1.3.** Let \( \alpha : \mathcal{F} \rightarrow \mathcal{E} \) be a map of coherent sheaves on a locally Noetherian scheme \( X \), where \( \mathcal{E} \) is locally free. Assume that for every \( x \in X \), there exists a closed subscheme \( Z_x \subset X \) containing \( x \), such that the resulting map \( \alpha|_{Z_x} : \mathcal{F}|_{Z_x} \rightarrow \mathcal{E}|_{Z_x} \) is injective and the quotient is locally free on \( Z_x \). Then: (i) the map \( \alpha \) is injective, (ii) \( \mathcal{F} \) is locally free, (iii) \( \text{coker}(\alpha) \) is locally free. In fact, it is sufficient to check this condition for closed points only.

We apply this lemma to the map \( \mathcal{I}_{X,X'}/\mathcal{I}_{X,X'}^2 \rightarrow \Omega_{X'/S}|_X \). For every \( x \in X \) we take \( Z_x := X_{f(x)} \). We need to verify that the conditions of the lemma hold. First, since \( X \) is flat over \( Y \), for every \( y \in Y \), the natural map \( \mathcal{I}_{X_y} \rightarrow \mathcal{I}_{X_{f(y)}} \) is an isomorphism, and hence

\[
(\mathcal{I}_{X,X'}/\mathcal{I}_{X,X'}^2)|_{X_y} \simeq \mathcal{I}_{X_{y},X_{y'}}/\mathcal{I}_{X_{y},X_{y'}}^2.
\]

Now, the assertion follows from Theorem 2.5.2.

\[\square\]

NB: Note that from what we proved it follows that Theorem 3.1.2 can be rephrased as follows: \( X \rightarrow S \) is smooth if and only if it is flat, and for any closed embedding \( X \hookrightarrow X' \) with \( X' \) smooth over \( S \), the sequence in point (3) of the theorem is short exact. In one of the problems for this week you’ll show that the condition that \( X \rightarrow S \) is in fact automatic.
3.2. We’ll now discuss a differential criterion for smoothness of a map:

Let $X, Y$ be two schemes, smooth over a base $S$, and $f : X \to Y$ a map between them.

**Theorem 3.2.1.** The following conditions are equivalent:

(a) $f$ is smooth of relative dimension $\dim \text{rel}(X, S) - \dim \text{rel}(Y, S)$.

(b) $\Omega_{X/Y}$ is locally free of rank $\dim \text{rel}(X, S) - \dim \text{rel}(Y, S)$.

(c) The map $f^*(\Omega_Y/S) \to \Omega_{X/S}$ is injective, and the quotient is locally free.