Math 233b Lecture Notes
01/28/2010

References for this segment are Beilinson’s lecture 1 on homological algebra at http://www.math.uchicago.edu/~mitya/beilinson/ and Gelfand-Manin IV.1.

Let $\mathcal{A}$ be a category. We can consider the category of complexes $\text{Comp}(\mathcal{A})$ whose objects are complexes of objects of $\mathcal{A}$ and morphisms are morphisms of complexes.

**Definition (Shifts).** If $X^\bullet$ is complex, then $X^\bullet[k]$ is the "shifted complex" defined by

$$X^i[k] = X^{i+k}$$

$$d[k] = (-1)^k \cdot d$$

**Definition (Cones).** In $X^\bullet \rightarrow Y^\bullet$ is a map of complexes then the object $\text{Cone}(f)$, called the cone of $f$ is defined by as the total complex of the bicomplex with two columns - the top one being $Y$ and the bottom one being $X$. Namely,

$$\text{Cone}(f) = Y \oplus X[1]$$

i.e. $\text{Cone}(f)^n = Y^n \oplus X^{n+1}$ with differential

$$d(y, x) = (dy + f(x), -dx)$$

We immediately see that the cone of $f$ enters the exact sequences of complexes

$$X^\bullet \rightarrow Y^\bullet \rightarrow \text{Cone}(f)$$

and

$$0 \rightarrow Y^\bullet \rightarrow \text{Cone}(f) \rightarrow X[1] \rightarrow 0$$

Now suppose $Y^\bullet \rightarrow C^\bullet$ is a morphism of complexes which admits a splitting as a morphism of graded objects (i.e. the morphism does not have to commute with the differential):

$$Y^\bullet \xrightarrow{p} C^\bullet$$

Then defining to object $C^\bullet/Y^\bullet$ by $(C/Y)^n = C^n/Y^n$ with differential inherited from $Y$ (as $i$ commutes with the differential this is well-defined), we can define the map

$$d_Y \cdot p^n - p^{n+1} \cdot d_C: (C/Y)^n \rightarrow Y^{n+1}$$

**Claim.** If we denote $X^\bullet = (C/Y)^\bullet[-1]$ and get the map $f: X^\bullet \rightarrow Y^\bullet$ then $C^\bullet = \text{Cone}(f)$

Proof: $C^n \cong Y^n \oplus X^{n+1}$ via because $(C/Y)^n$ is the cokernel of the split embedding $Y^n \rightarrow C^n$. It remains to check that the differential is what is should be which is straightforward verification.

The idea is to apply this abstract theory when $\mathcal{A} = \text{QCoh}(X)$, take the category $\text{comp}(\mathcal{A})$ and look at the derived category, but this will be elaborated upon later. For now, it suffices to work in additive categories.

**Additive categories.**

In the following, let $\mathcal{C}$ be a category. We illustrate two approaches of making to viewing it as an additive category.

**Approach 1.** The structure of an additive category on $\mathcal{C}$ is defined as follows:
1) For any $X,Y \in \mathcal{C}$, $\text{Hom}(X,Y)$ is endowed with a structure of an abelian group such that the composition maps are bilinear:

$$\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$$

2) $\exists 0 \in C$ with $\text{Hom}(0, X) \cong 0, \text{Hom}(X, 0) \cong 0$

3) For any $X,Y \in \mathcal{C}$, there exists a direct sum object $X \oplus Y$ i.e. it is endowed with the following morphisms

$$X \xrightarrow{i} X \oplus Y \xrightarrow{q} Y$$

such that $pi = id_X, qj = id_Y, qi = pj = 0, ip + jq = id_{X \oplus Y}$

In particular, $X \oplus Y$ is both the product and coproduct of $X$ and $Y$.

**Approach 2.**

**Lemma.** If an abstract category $\mathcal{C}$ admits a structure of an additive category, then it’s unique. (i.e. being an additive category is a property).

Proof: The zero object is easy to find (up to isomorphism), so is the direct sum. Now we can use the direct sum to recover the addition law on $\text{Hom}(X,Y)$.

Now let $\mathcal{C}$ be an additive category.

**Definition.** 1) A graded structure on $\mathcal{C}$ is a grading on each morphism set $\text{Hom}(X,Y)$ such that the composition law $\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$ respects the grading.

2) A DG (differential grading) structure on $\mathcal{C}$ is, in addition to the grading structure, a choice of homomorphisms $d: \text{Hom}^i(X,Y) \to \text{Hom}^{i+1}(X,Y)$ for all $X,Y \in \mathcal{C}, i \in \mathbb{N}$, compatible under the composition law:

$$f \in \text{Hom}(X,Y)^i, g \in \text{Hom}(Y,Z)^j \implies d(gf) = (dg) \cdot f + (-1)^j g \cdot (dt)$$

An example of a DG category is the following construction:

Given $A$ an additive category, define the DG category $\text{Comp}_{DG}(A)$ as follows: its objects are complexes in $A$, and for $X,Y$ complexes, define the graded group structure on $\text{Hom}(X,Y)$ by $\text{Hom}(X,Y)^i = \prod_j \text{Hom}_A(X^j, Y^{i+j})$ (no compatibility with the differential is required).

Define the differential $\text{Hom}(X,Y)^i \to \text{Hom}(X,Y)^{i+1}$ by $(f_j) \to (df_j)$ where $df_j = df \cdot f_j + (-1)^j f_{j+1} \cdot dx$.

Other similar examples of DG categories are $\text{Comp}_{DG}^+(A)$ - consisting of complexes whose indices are bounded below (or equivalently, all chains with sufficiently long index are 0). Similarly, one can consider the category $\text{Comp}_{DG}^-(A)$ and $\text{Comp}_{DG}^{\text{bounded}}(A)$.

Let $\mathcal{C}$ be a DG category.

**Definition.** 1) Define $Z(\mathcal{C})$ be the category whose objects are the same as objects of $\mathcal{C}$, but morphisms are $\text{Hom}_{Z(\mathcal{C})}(X,Y) = Z^0 \text{Hom}(X,Y)$. In another words, these are morphisms of degree 0 that are killed by the differential - such morphisms are called closed. In the example of $\text{Comp}_{DG}(A)$ this is the same as maps (of degree zero) of complexes - that commute with the differential.

2) Define $Ho(\mathcal{C})$ - the homotopy category of $\mathcal{C}$ by $\text{Ob}(H^0(\mathcal{C})) = \text{Ob}(\mathcal{C}), \text{Hom}_{Ho(\mathcal{C})}(X,Y) = H^0(\text{Hom}_\mathcal{C}(X,Y))$.

In the case $\mathcal{C} = \text{Comp}_{DG}(A) = K(A)$ - the homotopy category of complexes. $\text{Hom}_{K(A)}$ - consists of morphisms (that commute with the differential) of complexes up to homotopy.

Let $\mathcal{C}$ be a DG category. Let $X \in \mathcal{C}$.

**Definition.** 1) (Shifts) For $k \in \mathbb{Z}$, define the object $X[k]$ (if it exists) via the universal property

$$\text{Hom}(Y, X[k])^* \cong \text{Hom}(X,Y)^*[k]$$
We say that the shift exists if there exists such an object.

2) (Cones) If \( X \xrightarrow{f} Y, f \in Z^0 \text{Hom}_C(X, Y) \) define \( \text{Cone}(f) \) by the universal property

\[
\text{Hom}(Z, \text{Cone}(f))^\bullet = \text{Cone}(\text{Hom}(X, Z)^\bullet \rightarrow \text{Hom}(Z, Y)^\bullet)
\]

For example if \( C \) is the category of complexes in an abelian category, shifts and cones exists.

**Lemma.** a) The shift can (if it exists) can be defined by the following (stronger)property:

- There is an isomorphism \( X \xrightarrow{\alpha} X[k] \) in \( C \) regarded as a plain category (i.e. forgetting about the grading and the differential structure), such that \( \alpha \in \text{Hom}_C(X, X[k]) \) and

  \[
  da = 0.
  \]

  If \( X[k] \) exists, then it’s unique up to a unique DG-isomorphism.

  Also, it satisfies the following ”mapping out” property:

  \[
  \text{Hom}_C(X[k], Y) \xrightarrow{\alpha} \text{Hom}_C(X, Y)
  \]

b) If \( f: X \rightarrow Y \) is a closed morphism of degree 0, then consider an object \( Z \) of \( C \) (if it exists) endowed with an isomorphism \( Z \cong X \oplus Y \) in \( C \) (regarded as a plain additive category) such that

(i) The morphism \( i: Y \rightarrow Z \) is closed of degree 0.

(ii) \( p: Z \rightarrow X \) is closed of degree 1.

(iii) The morphism \( j: X \rightarrow Z \) (which is automatically of degree \(-1\)) is such that \( dj \in \text{Hom}^0(X, Z) \) (which automatically is a cycle and belongs to \( \text{Hom}^0(X, Y) \)), equals \( f \).

Then \( Z \cong \text{Cone}(f) \).

If \( \text{Cone}(f) \) exists, then it’s unique up to a unique DG-isomorphism.

We also have:

\[
\text{Hom}^\bullet_C(W, Z) \cong \text{Cone}(\text{Hom}^\bullet_C(W, X) \rightarrow \text{Hom}^\bullet_C(W, Y))
\]

and

\[
\text{Hom}^\bullet_C(Z, W) \cong \text{Cone}(\text{Hom}^\bullet_C(Y, W) \rightarrow \text{Hom}^\bullet_C(X, W))
\]

Proof: The shifts and cones are defined up to isomorphism, by Yoneda’s lemma. In the following, we prove it directly according to the defining properties in the condition of the lemma.

a) Assume \( Y \) is another object satisfying the defining property of the condition: there exists a morphism \( \beta: X \rightarrow Y \) of degree \(-k\) which is an isomorphism in \( C \) as a plain category, and \( d\beta = 0 \). Then \( \beta\alpha^{-1} \) is an isomorphism between \( X[k] \) and \( Y \) - and it is of degree 0 since \( \alpha^{-1} \) is of degree \( k \): indeed if \( \alpha^{-1} = \oplus u_i \) then \( id = \alpha \circ \alpha^{-1} = \oplus u_i \) and as \( id \) has degree 0, \( \alpha \circ u_i = 0 \) for \( i \neq -k \) which implies \( u_i = 0 \) as \( \alpha \) is invertible. Moreover, \( d(\alpha \circ \alpha^{-1}) = d(id) = 0 \) hence \( da \circ \alpha^{-1} + (-1)^k \alpha(da^{-1}) = 0 \) and as \( da = 0 \) and \( \alpha \) is invertible we conclude \( da^{-1} = 0 \). Then we readily deduce \( d(\beta\alpha^{-1}) = 0 \) so this is closed. This implies the uniqueness clause.

The canonical isomorphism

\[
\text{Hom}^\bullet_C(Y, X[k]) \xrightarrow{\alpha^{-1}} \text{Hom}^\bullet_C(Y, X)[k]
\]

is induced by the natural isomorphism

\[
\text{Hom}_C(Y, X[k]) \xrightarrow{\alpha^{-1}} \text{Hom}_C(Y, X)
\]

(where the latter is regarded as an isomorphism in the plain category \( C \))

Since \( \alpha^{-1} \) has degree \( k \), the latter isomorphism increases the degree of homogeneous components by \( k \), hence it induces the desired isomorphism. (Clearly the inverse is obtained by composing with \( \alpha \)).

Similarly

\[
\text{Hom}_C(X[k], Y) \xrightarrow{\alpha} \text{Hom}_C(X, Y)
\]

3
is an isomorphism and since $\alpha$ has degree $-k$ it provides an isomorphism

$$\text{Hom}_{C}(X[k], Y) \cong \text{Hom}_{C}(X, Y)[-k]$$

b) Let $Z, Z'$ be two objects satisfying the given property, and let $\alpha: X \oplus Y \to Z, \beta: X \oplus Y \to Z$ be corresponding morphisms in the plain category. Then $\beta\alpha^{-1}$ provides an isomorphism $Z' \cong Z$, and we need to show it is closed of degree 0.

Let $i: Y \to X \oplus Y, q: X \oplus Y \to Y, j: X \to X \oplus Y, p: X \oplus Y \to X$ be the natural morphisms coming from the direct sum definition. Condition i) tells us that $\alpha \circ i, \beta \circ i$ are closed of degree 0, and ii) tells us that $p \circ \alpha^{-1}, p \circ \beta^{-1}$ are closed of degree 1. From here we deduce immediately that $\alpha \circ j, \beta \circ j$ have degree -1 (as their composition with $p \circ \alpha^{-1}$ and $p \circ \beta^{-1}$ are the identity), and $q \circ \alpha^{-1}, q \circ \beta^{-1}$ have degree 0.

Because the identity morphism $X \oplus Y \to X \oplus Y$ is the sum $jp + iq$, we deduce $\beta\alpha^{-1} = \beta(jp + iq)\alpha^{-1} = (\beta \circ j)(p \circ \alpha^{-1}) + (\beta \circ i)(q \circ \alpha^{-1})$, and therefore we immediately see that this is a sum of two maps that have degree 0, which completes the proof, provided we show that it’s closed. Indeed, $d(p \circ \alpha^{-1}) = 0$ by definition and $d(\beta \circ j) = \beta f$. Also $d(\beta \circ i) = 0$. It remains to compute $d(q \circ \alpha^{-1})$. Note that $(q \circ \alpha^{-1}) \circ (\alpha \circ i) = id$ and by differentiating this $d(q \circ \alpha^{-1}) \circ (\alpha \circ i) = 0$ (as $\alpha \circ i$ is closed). Further, by differentiating $(q \circ \alpha^{-1}) \circ (\alpha \circ j) = 0$ we deduce $d(q \circ \alpha^{-1}) \circ (\alpha \circ j) = -f$ hence $d(q \circ \alpha^{-1}) \circ (\alpha \circ j) = -f \circ (p \circ \alpha^{-1})$. As $jp + iq = id$ but $d(q \circ \alpha^{-1}) \circ (\alpha \circ i) = 0$ as shown before, we deduce that $d(q \circ \alpha^{-1}) = -f \circ (p \circ \alpha^{-1})$.

Hence the differential of our map will be $\beta if(p \circ \alpha^{-1}) - \beta if(p \circ \alpha^{-1}) = 0$, as desired.

Next, we prove that

$$\text{Hom}_{C}(W, Z) \cong \text{Cone} (\text{Hom}_{C}(W, X) \to \text{Hom}_{C}(W, Y))$$

(remark that the right hand-side is isomorphic in the plain category to $\text{Hom}_{C}(W, X[1]) \oplus \text{Hom}_{C}(W, Y)$ hence $Z \cong X[1] \oplus Y$).

This comes from the canonical isomorphism $\text{Hom}_{C}(W, Z) \xrightarrow{\text{proj}} \text{Hom}_{C}(W, X) \oplus \text{Hom}_{C}(W, Y)$ arising from the isomorphism in the plain category. Since $p$ has degree 1 and $q$ has degree 1, as graded modules, this provides the isomorphism respecting the grading between $\text{Hom}_{C}^{*}(W, Z)$ and $\text{Hom}_{C}^{*}(W, Y) \oplus \text{Hom}_{C}^{*}(W, X)[1]\text{Hom}_{C}^{*}(W, Y)$ and the latter is isomorphic to $\text{Cone} (\text{Hom}_{C}^{*}(Z, X) \to \text{Hom}_{C}^{*}(Z, Y))$ as graded complexes. To conclude, it remains to show that the differentials match. That is, if we have a map $h \in \text{Hom}_{C}^{*}(W, Z)$ we want $(q \circ dh, p \circ dh)$ to equal $d(q \circ h, p \circ h)$ - and recall that the latter is $(dq \circ h + f \circ p \circ h, -dp \circ h + p \circ dh)$. That can be rewritten as $(dq \circ h + q \circ dh + f \circ p \circ h, -dp \circ h + p \circ dh)$. Recall thought that we have proven above that $dq = -f \circ p$, and we also know $dp = 0$, from where we deduce $(dq \circ h + q \circ dh + f \circ p \circ h, -dp \circ h + p \circ dh) = (q \circ dh, p \circ dh)$, as desired.

The ”mapping out” property is similar. $\square$

**Definition.** Let $\mathcal{C}$ be a DG category. We say that $\mathcal{C}$ is strongly pre-triangulated if all shifts and cones exists.

**Definition.** A triangle is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

(we usually have $vu = uv = 0$)

Remark: one can create another triangle

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]}, Y[1]$$

then

$$Z \xrightarrow{w}, X[1] \xrightarrow{-u[1]}, Y[1] \xrightarrow{-u[1]}, Z[1]$$

and so on.
A morphism of triangles is a commutative diagram:

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow f & & \downarrow g \\
X' & \rightarrow & Y' \\
\end{array}
\sim
\begin{array}{ccc}
Y & \rightarrow & Z \\
\downarrow h & & \downarrow f[1] \\
Y' & \rightarrow & Z' \\
\end{array}
\rightarrow
\begin{array}{ccc}
Z & \rightarrow & X[1] \\
\downarrow \ & & \downarrow \ \\
Z' & \rightarrow & X'[1] \\
\end{array}
\]

**Definition.** A triangle

\[X \rightarrow Y \rightarrow Z \rightarrow X[1]\]

in \(Ho(C)\) is said to be distinguished if it is isomorphic to a triangle of a form

\[X \xrightarrow{\ell} Y \rightarrow \text{Cone}(f) \rightarrow X[1]\]

**Triangulated categories.**

Let \(D\) be an additive category endowed with an auto-equivalence \(X \rightarrow X[1]\) and a collection of triangles marked as distinguished, such that the following four axioms hold:

**Tr1.** a) A triangle isomorphic to a distinguished one is distinguished.

b) \(X \xrightarrow{id} X \rightarrow 0 \rightarrow X[1]\) is distinguished.

c) Any morphism \(X \rightarrow Y\) can be completed to a distinguished triangle

\[X \rightarrow Y \rightarrow \text{Cone}(f) \rightarrow X[1]\]

**Tr2.** \(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]\) is distinguished if and only if \(Y \rightarrow Z \rightarrow X[1] \xrightarrow{-u[1]} Y[1]\) is distinguished.

**Tr3.** If \(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]\) and \(X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]\) are distinguished triangles the any pair of morphisms \(f: X \rightarrow X', g: Y \rightarrow Y'\) such that \(gu = u'f\) can be completed with a morphism \(h: Z \rightarrow Z'\) such that the following diagram is a morphism of triangles:

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow f & & \downarrow g \\
X' & \xrightarrow{u'} & Y' \\
\end{array}
\sim
\begin{array}{ccc}
Y & \xrightarrow{v} & Z \\
\downarrow h & & \downarrow f[1] \\
Y' & \xrightarrow{v'} & Z' \\
\end{array}
\rightarrow
\begin{array}{ccc}
Z & \rightarrow & X[1] \\
\downarrow \ & & \downarrow \ \\
Z' & \rightarrow & X'[1] \\
\end{array}
\]

**Tr4.** This is an analogue of the second isomorphism theorem from algebra \(0 \rightarrow Y/X \rightarrow Z/X \rightarrow Z/Y \rightarrow 0\). We will choose notations to underline this similarity. Consider two distinguished triangles \(X \xrightarrow{\ell} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]\) and \(Y \xrightarrow{g} Z \xrightarrow{\delta} Z/Y \xrightarrow{d'} Y[1]\). Choose a distinguished triangle

\[(1)
\]

\[X \xrightarrow{g \circ f} Z \xrightarrow{\delta} X/Z \xrightarrow{d'} X[1]\]

Define \(\theta = u[1] \circ \delta': Z/Y \rightarrow X/Y[1]\). Choose maps \(\phi: Y/X \rightarrow Z/X, \psi: Z/X \rightarrow Z/Y\) such that the following two diagrams commute and form morphisms of distinguished triangles:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \id & & \downarrow g \\
X & \xrightarrow{g \circ f} & Z \\
\end{array}
\sim
\begin{array}{ccc}
Y/X & \xrightarrow{u} & Y \\
\downarrow \phi & & \downarrow \id \\
Y/X & \xrightarrow{\delta} & Z/X \\
\end{array}
\rightarrow
\begin{array}{ccc}
Y/X & \xrightarrow{d} & X[1] \\
\downarrow \ & & \downarrow \ \\
Z/X & \rightarrow & X[1] \\
\end{array}
\]

5
Consider the triangle

\[(2) \quad Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1] \]

The condition (Tr4) is that there exists a choice of the triangle (1), maps \(\phi, \psi\) such that (2) is a distinguished triangle. It turns out that this is equivalent to (2) being a distinguished triangle for any choice of triangle (1), \(\phi\) and \(\psi\).

It is also equivalent to what is called in other sources the "octahedron" axiom.

We can create the following "upper part" of the octahedron:

The starred triangles are distinguished and the other two marked \(\circ\) and \(\circ\) commute. Also the arrow \(\xrightarrow{[1]}\) means the morphism \(A \rightarrow B[1]\).

The conclusion of the "octahedron" axiom is that the "octahedron" can be completed with a "lower" part, namely a diagram of the form:

such that the two ways to get a morphism from \(Y\) to \(Z/X\) (via \(Y/X\) and \(Z\)) and from \(Z/X\) to \(Y[1]\) via \(X[1]\) and \(Z/Y\) coincide.

It is straightforward to check that this formulation is equivalent to the original one.

Axiom (Tr4) is the most complicated of the four, and there are many important corollaries of the axioms (Tr1), (Tr2) and (Tr3) alone.

**Lemma.** Let \(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]\) be a distinguished triangle.

Then for any object \(W \in D\), the sequences

\[\ldots \rightarrow \text{Hom}_D(W, Z[-1]) \rightarrow \text{Hom}_D(W, X) \rightarrow \text{Hom}_D(W, Y) \rightarrow \text{Hom}_D(W, Z) \rightarrow \text{Hom}_D(W, X[1]) \rightarrow \text{Hom}_D(W, Y[1]) \rightarrow \ldots\]

\[\ldots \rightarrow \text{Hom}_D(Y[1], W) \rightarrow \text{Hom}_D(X[1], W) \rightarrow \text{Hom}_D(Z, W) \rightarrow \text{Hom}_D(Y, W) \rightarrow \text{Hom}_D(X, W) \rightarrow \text{Hom}_D(Z, W) \ldots\]
are long exact sequences of abelian groups. (In particular, the composition of any two consecutive morphisms in a triangle is zero).

Proof: Since distinguished triangles remain distinguished under ”shifts”, it is enough to show that $\text{Hom}_{D}(W, X) \rightarrow \text{Hom}_{D}(W, Y) \rightarrow \text{Hom}_{D}(W, Z)$ is exact. First, let’s show that the composition $X \rightarrow Y \rightarrow Z$ is zero. Consider the distinguished triangle $W \xrightarrow{id} W \rightarrow 0 \rightarrow W[1]$. For any morphism $\phi \in \text{Hom}(W, X)$ we can compose it with $X \rightarrow Y$ and form the commutative square

\[
x \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\psi[1]} X[1]
\]

that extends by (Tr3) to

\[
x \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\psi[1]} X[1]
\]

Since any morphism from 0 to 0 we deduce that the composite map in $\text{Hom}(W, Z)$ as desired.

Conversely if $\phi \in \text{Hom}(W, Y)$ and $v \circ \phi = 0$ then as the triangle $W \rightarrow 0 \rightarrow W[1] \xrightarrow{id} W[1]$ is distinguished (Tr1+Tr2) and we have the commutative square

\[
Y \xrightarrow{\phi} Z \xrightarrow{\psi[1]} W[1]
\]

that extends to

\[
Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]
\]

In particular $-\phi[1] = -u[1] \circ \psi$ so $\phi = u \circ \psi[-1]$ which completes the proof of the exactness.

The second (contravariant) sequence is done analogously by ”inverting arrows”.

**Corollary 1.** Say we have a map of distinguished triangles

\[
\begin{array}{c}
X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \\
\downarrow f \quad \downarrow g \quad \downarrow h \\
X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]
\end{array}
\]

If $f, g$ are isomorphisms, then so is $h$.

Proof: By Yoneda’s lemma it is enough to show that map $ho : \text{Hom}(W, Z') \rightarrow \text{Hom}(W, Z)$ is an isomorphism. This follows from the five lemma and the long exact sequence constructed in the lemma.

\[
\begin{array}{c}
\text{Hom}(W, X) \xrightarrow{\sim} \text{Hom}(W, Y) \xrightarrow{\sim} \text{Hom}(W, Z) \xrightarrow{\sim} \text{Hom}(W, X[1]) \xrightarrow{\sim} \text{Hom}(X, Y[1]) \\
\text{Hom}(W, X') \xrightarrow{\sim} \text{Hom}(W, Y') \xrightarrow{\sim} \text{Hom}(W, Z') \xrightarrow{\sim} \text{Hom}(W, X'[1]) \xrightarrow{\sim} \text{Hom}(X, Y'[1])
\end{array}
\]
**Corollary 2.** Let \( f: X \to Y \) be a morphism. Consider two extensions of \( f \) to a distinguished triangle:

\[
\begin{align*}
X &\xrightarrow{f} Y \to Z \to X[1] \\
X &\xrightarrow{f} Y \to Z' \to X[1]
\end{align*}
\]

Then there exists a (non-canonical) isomorphism \( Z \sim Z' \) that makes the following diagram into an isomorphism of distinguished triangles:

\[
\begin{CD}
X @>f>> Y @>>> Z @>>> X[1] \\
@V{id}VV @V{id}VV @V{\sim}VV @V{id}VV \\
X @>f>> Y @>>> Z' @>>> X[1]
\end{CD}
\]

Proof: the existence of a map \( Z \to Z' \) making the diagram commutative exists by (Tr3), and it is an isomorphism by the previous corollary.

**Corollary 3.** Let \( X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \) and \( X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1] \) be two distinguished triangles and \( g \in \text{Hom}(Y,Y') \). Then there exists \( f \in \text{Hom}(X,X') \) such that \( gu = u'f \) - or equivalently by (Tr3), \( g \) can be extended to a morphism of distinguished triangles, if and only if \( v'gu = 0 \). Moreover, in that case, \( f \) is non-unique up to an element of \( \text{Hom}(X,Z'[−1]) \) - more precisely, any two \( f, f_1 \) differ by \( f - f_1 = w'[-1] \circ j \) for some \( j \in \text{Hom}(X,Z'[−1]) \).

Proof: we are asking whether \( gu \) is in the image of \( \text{Hom}(X,X') \xrightarrow{w'} \text{Hom}(X,Y') \), which by the long exact sequence of the lemma is equivalent to \( v'gu = 0 \). Moreover, if \( f \) is any such morphism, then any other alternative \( f_1 \) may be recovered by \( u'(f - f_1) = 0 \) which by the long exact sequence of the lemma again, is equivalent to \( f - f_1 = w'[-1] \circ j \) for some \( j \in \text{Hom}(X,Z'[−1]) \).

**Proposition.** Define \( \mathcal{D} = K(\mathcal{A}) = Ho(\text{Comp}_{\text{DG}}(\mathcal{A})) \) where \( \mathcal{A} \) is an additive category. Then with the class of distinguished triangles specified earlier, \( \mathcal{D} \) is a triangulated category.

02/02/2010

Let \( \mathcal{C} \) be a DG-category (that is pre-triangulated).

**Proposition.** \( Ho(\mathcal{C}) \) is a triangulated category. Particularly, if \( \mathcal{A} \) is an abelian category, then \( \text{Comp}_{\text{DG}}^+(\mathcal{A}), \text{Comp}_{\text{DG}}^-_b(\mathcal{A}), \text{Comp}_{\text{DG}}^b(\mathcal{A}) \) are triangulated categories. They are denoted by \( K(\mathcal{A})^+, K(\mathcal{A})^- \) and \( K(\mathcal{A})^b \).

**Definition.** Let \( \mathcal{D}_1, \mathcal{D}_2 \) be two triangulated categories. A functor \( F: \mathcal{D}_1 \to \mathcal{D}_2 \) is called triangulated if it satisfies the following conditions:

a) \( F \circ [k]_{\mathcal{D}_1} \cong [1]_{\mathcal{D}_2} \circ F \) for all \( k \in \mathbb{Z} \) (it suffices to check it for \( k = 1 \))

b) \( F \) is additive (\( F(0) = 0 \), \( F \) commutes with direct sums, and the map \( F: \text{Hom}_{\mathcal{D}_1}(X,Y) \to \text{Hom}_{\mathcal{D}_2}(FX,FY) \) is a group homomorphism)

c) \( F \) takes distinguished triangles to distinguished triangles.

For example, here’s an easy proposition:

**Proposition.** If \( F: \mathcal{A} \to \mathcal{B} \) is an additive functor between abelian categories, then it induces a triangulated functor \( K(\mathcal{A}) \xrightarrow{F} K(\mathcal{B}) \).

**Proposition.** Let \( F: \mathcal{D}_1 \to \mathcal{D}_2 \) be a triangulated functor. If \( G \) is its adjoint (from any side), then \( G \) is also triangulated.

Proof: We will only do the case \( G \) being right-adjoint, since the other case is absolutely similar.
Define $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ to be quasi-distinguished if it satisfies the mapping-in long exact sequence of the lemma from lecture:

$$
\ldots \rightarrow \text{Hom}_D(W,Z[-1]) \rightarrow \text{Hom}_D(W,X) \rightarrow \text{Hom}_D(W,Y) \rightarrow \text{Hom}_D(W,Z) \rightarrow \text{Hom}_D(W,X[1]) \rightarrow \text{Hom}_D(W,Y[1]) \rightarrow \ldots
$$

Any distinguished triangle is quasi-distinguished.

Lemma: consider a map of quasi-distinguished triangles

$$
\begin{array}{ccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1]
\end{array}
$$

If $f, g$ are isomorphisms, then so is $h$.

Proof: the lemma was proven before for distinguished triangles, but we only used the mapping-in long exact sequence in process (the proof works with the mapping out sequence as well, in the case of the left adjoint).

Now let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a distinguished triangle. We want to prove that $GX \xrightarrow{Gu} GY \xrightarrow{Gv} GZ \xrightarrow{Gw} GX[1]$ is distinguished. A priori, it is quasi-distinguished, as easily seen from the adjointness condition.

Consider a distinguished triangle $GX \xrightarrow{Gu} GY \xrightarrow{Gv'} K \xrightarrow{w'} GX[1]$. We will show that it is isomorphic to $GX \xrightarrow{Gu} GY \xrightarrow{Gv} GZ \xrightarrow{Gw} GX[1]$.

$FGX \xrightarrow{FGu} FGY \xrightarrow{Fv'} FK \xrightarrow{Fw'} FGX[1]$ is distinguished, and it has a map to $X, Y, Z$ by the axiom (Tr3):

$$
\begin{array}{ccc}
FGX & \xrightarrow{FGu} & FGY & \xrightarrow{Fv'} & FK & \xrightarrow{Fw'} & FGX[1] \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}
$$

where the unlabeled vertical maps are the adjunction maps.

By applying the functor $G$ to this diagram, and using the adjunction map $\cdot \rightarrow GF\cdot$, we obtain the following diagram:

$$
\begin{array}{ccc}
GX & \xrightarrow{Gu} & GY & \xrightarrow{v'} & K & \xrightarrow{w'} & GX[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
GFGX & \xrightarrow{GFu} & GFGY & \xrightarrow{GFv'} & GFK & \xrightarrow{GFw'} & GFGX[1] \\
\downarrow & & \downarrow & & \downarrow G\circ h & & \downarrow \\
GX & \xrightarrow{Gu} & GY & \xrightarrow{Gv} & GZ & \xrightarrow{Gw} & GX[1]
\end{array}
$$

The composites of the unlabeled vertical arrows are the identity, and the upper diagram commutes - i believe both of these facts have been proven a couple of times in the problem sets of last semester using the natural-ness of the adjoint property. Therefore we get a map between $GX \xrightarrow{Gu} GY \xrightarrow{v'} K \xrightarrow{w'} GX[1]$ and $GX \xrightarrow{Gu} GY \xrightarrow{Gv} GZ \xrightarrow{Gw} GX[1]$ which is an isomorphism from the lemma. □

Given a triangulated category $\mathcal{D}$ and $\mathcal{D}' \subset \mathcal{D}$ a full "triangulated" subcategory (i.e. the inclusion map is a triangulated functor/ or for a weaker assumption distinguished triangles in $\mathcal{D}'$ are distinguished in $\mathcal{D}$) one would like to define the quotient category $\mathcal{D}/\mathcal{D}'$ define by the universal property

$$
\{\text{Funct}^{tr}(\mathcal{D}, \hat{\mathcal{D}}) \mid F(\mathcal{D}') = 0\} \cong \{\text{Funct}^{tr}(\mathcal{D}/\mathcal{D}', \hat{\mathcal{D}})\}
$$
A fundamental example would be $\mathcal{D} = K(A), \mathcal{D}' = K(A)_{acyc}^\ast$ consisting of acyclic complexes. The quotient $D(A) = K(A)/K(A)_{acyc}^\ast$ is called the derived category of $A$. One can similarly define $D^+(A) = K(A)^+/K(A)^+ \cap K(A)_{acyc}^\ast$ and so on.

**Construction.**

We say that $f: X \to Y$ is a $\mathcal{D}'$-quasi-isomorphism if $\text{Cone}(f) \in \mathcal{D}'$ (here and later on we will always consider subcategories that are closed under isomorphism).

We will define $\mathcal{D}/\mathcal{D}'$ as the localization (see last semester) of $\mathcal{D}$ with respect to $\mathcal{D}'$-quasi-isomorphisms:

Morphisms between $X$ and $Y$ are represented by upper or lower huts:

![Hut Diagram](image)

where the morphisms labeled by $\sim$ are $\mathcal{D}'$-isomorphisms.

Two huts are equivalent if they can be "lifted" to a bigger diagram i.e. a commutative diagram like those two diagrams:

![Larger Diagrams](image)

To compose two huts we $X_1 \to Y \leftarrow X_2 \to Z \leftarrow X_3$ respectively $X_1 \leftarrow Y \to X_2$ and $X_2 \leftarrow Z \to X_3$ we also complete a diagram to combine them into a larger single hut:

![Combined Hut](image)

We need to prove that any two such huts can be enlarged to a hut, to prove that this definition makes sense - note that the category can always be constructed as generated by compositions of such huts, but we want to prove that every morphism is represented by one single hut.

If $X \in \mathcal{D}$, denote by $\mathcal{Q}_i/X$ the category of quasi-isomorphisms into $X$, i.e. consisting of morphisms $\tilde{X} \xrightarrow{f} X$ with $\text{Cone}(f) \in \mathcal{D}'$, and morphisms consisting of maps that make the corresponding diagrams commute:

![Quasi-Morphisms Diagram](image)

We define the category $X\setminus\mathcal{Q}_i$ in a similar way, consisting of quasi-morphisms coming out of $X$.

**Proposition.** 1) $\text{Hom}_{\mathcal{D}/\mathcal{D}'}(X,Y) \cong \lim_{(\tilde{X},f) \in \mathcal{Q}_i/X} \text{Hom}_{\mathcal{D}}(\tilde{X},Y)$

2) The category $\mathcal{Q}_i/X$ is filtered
3) $\text{Hom}_{\mathcal{D}/\mathcal{D}'}(X,Y) \cong \lim_{(\tilde{Y},f) \in Q_i \setminus Q_i} \text{Hom}_\mathcal{D}(X,\tilde{Y})$

4) $Y\setminus Q_i$ is filtered.

Note that the proposition finishes the construction of the quotient category.

**Digression on direct limits and filtered categories.** Let $I$ be a category and let $A: I \to \text{Ab}$ be a functor from $I$ to the category of abelian groups (more generally we can consider any abelian category as the target).

We then define the direct limit of $A$, characterized by the functorial universal property:

$$
\text{Hom}(\lim_{i \in I} M, M) = \left\{ \begin{array}{ll}
\forall i, A_i \xrightarrow{f_i} M \\
\forall i,j \in I, i \xrightarrow{f} j,
\end{array} \right.
\text{if } i \xrightarrow{f} j
$$

**Definition** $I$ is filtered if for any $i_1, i_2 \in I$ there exists $j \in I$ which receives maps from both $i_1$ and $i_2$, and for any two $i$ and $j$ with two morphisms from $i$ to $j$, there exists a morphism $j \to k$ such that these two morphisms become equal when composed with it: $i \Rightarrow j \Rightarrow k$.

**Lemma (2 out of 3 property):** Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$. If two of $f, g, f \circ g$ is a quasi-isomorphism then the third is, too.

**Proof:** By axiom Tr4 we have the distinguished triangle $\text{Cone}(Y/X) \to \text{Cone}(Z/X) \to \text{Cone}(Z/Y) \to \text{Cone}(Y/X)[1]$. If two of its elements are in $\mathcal{D}'$ then so is the third. □

**Proof of proposition:** Let’s first prove 4), 2) is analogous. Assume we have two quasi-isomorphisms $f_1: Y \to \tilde{Y}_1, f_2: Y \to \tilde{Y}_2$. We will take $Z = \text{Cone}(Y \xrightarrow{\text{antidiag}} \tilde{Y}_1 \oplus \tilde{Y}_2)$ (the map is the anti-diagonal map i.e. $f_1 - f_2$) - note that it receives canonical maps from $Y, \tilde{Y}_1, \tilde{Y}_2$.

First we have the distinguished triangle $\tilde{Y}_1 \to \tilde{Y}_1 \oplus \tilde{Y}_2 \to \tilde{Y}_2 \to \tilde{Y}_1[1]$ and from there using (Tr4) we deduce the distinguished triangle $\tilde{Y}_1 \to \text{Cone}(Y \to \tilde{Y}_1 \oplus \tilde{Y}_2) \to \text{Cone}(Y \to \tilde{Y}_2) \to \tilde{Y}_1[1]$ and since $\text{Cone}(Y \to \tilde{Y}_2) \in \mathcal{D}'$ we deduce that $\tilde{Y}_1 \to \text{Cone}(Y \to \tilde{Y}_1 \oplus \tilde{Y}_2)$ is a quasi-isomorphism i.e. $\tilde{Y}_1 \to Z$ is a quasi-isomorphism. Now using the 2 out of 3 lemma we can conclude that $Y \to Z$ is a quasi-isomorphism which shows the first part of the definition of being filtered.

For the second part, assume one has two maps $f, g: \tilde{Y}_1 \to \tilde{Y}_2$ such that $f \circ f_1 = g \circ f_1 = f_2$. Let $Z \in \mathcal{D}'$ be the cone of $f_1$. Because $(f - g) \circ f_1 = 0$ the map $f - g$ factors through a map $Z \to \tilde{Y}_2$ whose cone we denote by $W$ - then as $Z, Z[1] \in \mathcal{D}'$ the map $\tilde{Y}_2 \to W$ is a quasi-isomorphism and it is clear that the map $\tilde{Y}_2 \to W$ verifies the second property of filtered categories.

For proving 1) or 3), as shown last semester, we need to prove that for any lower hut there is an upper hut that is equivalent to it (i.e. the obtained rhombus diagram commutes) or vice versa.

Assume, for example that we have a hut $X \xleftarrow{f} \tilde{X} \to Y$. We then set $\tilde{Y} = \text{Cone}(\tilde{X} \to X \oplus Y)$ - again we take the anti-diagonal map. It is immediate (from the fact that the composition of two maps in a distinguished triangle is 0) that the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{X} \\
\downarrow{g} & & \downarrow{g} \\
Y & & Y
\end{array}
$$

Therefore the proposition will follow once we show that $Y \xrightarrow{g} \tilde{Y}$ is a quasi-isomorphism. In fact, we prove something stronger: $\text{Cone}(\tilde{X} \to X) \cong \text{Cone}(Y \to \tilde{Y})$.

Indeed, considering the distinguished triangle $Y \to X \oplus Y \to X \to X[1]$ and applying the axiom (Tr4) we obtain the distinguished triangle $Y \to \text{Cone}(\tilde{X} \to X \oplus Y) \to \text{Cone}(\tilde{X} \to X) \to Y[1]$ and from here the result follows.
**Proposition.** a) The projection functor $\pi: \mathcal{D} \to \mathcal{D}/\mathcal{D}'$ is additive.

b) If we define the distinguished triangles in $\mathcal{D}/\mathcal{D}'$ to be the (isomorphic) projections of the distinguished triangles in $\mathcal{D}$ then $\mathcal{D}/\mathcal{D}'$ is triangulated and the projection functor $\pi$ is a triangulated functor.

c) $\mathcal{D}/\mathcal{D}'$ satisfies the universal property mentioned before.

**Proof:** Part a) is trivial. Part b) is also relatively easy, once we notice that quasi-isomorphisms in $\mathcal{D}$ become isomorphisms in $\mathcal{D}/\mathcal{D}'$ so every morphism in $\mathcal{D}/\mathcal{D}'$ up to isomorphism comes from an actual morphism in $\mathcal{D}$ - therefore every morphism in $\mathcal{D}/\mathcal{D}'$ can be extended to a distinguished triangle - which shows (Tr1). All the other axioms follow immediately from the fact that every distinguished triangle can be isomorphically mapped to a triangle which is the projection of something in $\mathcal{D}$ and applying the axioms for $\mathcal{D}$.

Now let’s show part c). On one hand any functor that factors through $\mathcal{D}/\mathcal{D}'$ must send $\mathcal{D}'$ to 0 because every object in $\mathcal{D}'$ is isomorphic to 0 in $\mathcal{D}/\mathcal{D}'$. On the other hand, if it does send $\mathcal{D}'$ to 0 then any quasi-isomorphism will get mapped to an isomorphism because it will be sent to a morphism whose cone is isomorphic to 0 (it is in the image of $\mathcal{D}'$). By the standard Serre quotient stuff from the first semester, this shows that it factors through $\mathcal{D}/\mathcal{D}'$.

**Lemma.** If $X \in \text{ker}(\pi)$ then $X$ is a direct summand of an object of $\mathcal{D}'$.

**Proof:** Assume $\pi(X) = 0$. This implies that there is an isomorphism in $\mathcal{D}/\mathcal{D}'$ to 0. This isomorphism is represented by a hut $X \leftarrow X' \rightarrow 0$ where $X' \rightarrow X$ is a quasi-isomorphism. It will have an inverse - which is unique and is represented by the map $0 \rightarrow X$ (since 0 is the zero object in $\mathcal{D}/\mathcal{D}'$). By applying the composition law we will get the hut $X \leftarrow X' \rightarrow X$ which must represent the identity morphism. Now it’s an easy exercise to show that such a hut represents the identity morphism if and only if there is some $Z$ and a quasi-isomorphism $Z \rightarrow X'$ such that the two possible composite maps $Z \rightarrow X' \rightarrow X$ are equal. But one of them is the 0 map and the other is a quasi-isomorphism. Therefore $Z \rightarrow X$ is both the zero map and a quasi-isomorphism, in which case its cone is both (isomorphic to) $X \oplus Z[-1]$ and is in $\mathcal{D}'$ which shows the forward direction. For the other direction, if $X \oplus Z[1] \in \mathcal{D}'$ the hut $X \leftarrow Z \rightarrow X$ (where both maps $Z \rightarrow X$ are 0) represents the identity morphism (since both it and the identity hut trivially extend to a larger hut) and so we conclude that the hut $X \leftarrow Z \rightarrow 0$ and the morphism $0 \rightarrow X$ are mutually inverse.

**Definition.** Let $i: \mathcal{D}' \hookrightarrow \mathcal{D}, j: \mathcal{D}'' \hookrightarrow \mathcal{D}$ be embeddings of full triangulated subcategories. We say that $\mathcal{D}' \subset \mathcal{D} \supset \mathcal{D}''$ is an admissible triple, and call $\mathcal{D}', \mathcal{D}''$ left admissible respectively right admissible if the following conditions are satisfied:

i) $\text{Hom}_\mathcal{D}(X', X'') = 0$ for any $X' \in \mathcal{D}', X'' \in \mathcal{D}''$

ii) For any $X$ an object of $\mathcal{D}$, there exists a distinguished triangle $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$ where $X' \in \mathcal{D}', X'' \in \mathcal{D}''$.

**Lemma.** If the above condition i) is satisfied, then triangles as in ii), if exist, are functorial (hence unique).

**Proof:** Assume that $X \rightarrow Y$ is a map, and $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$ and $Y' \rightarrow Y \rightarrow Y'' \rightarrow Y'[1]$ are distinguished triangles.

Because of condition i), the composite map $X' \rightarrow X \rightarrow Y \rightarrow Y''$ is 0 and hence the map extends to a map of distinguished triangles. Moreover the map must be unique: a map $X' \rightarrow Y'$ must be unique up to a map $X' \rightarrow Y''[-1]$ but there are no such maps according to assumption i). Similarly for $X'' \rightarrow Y''$. □

It is an easy application of i) and ii) that $\mathcal{D}'' = (\mathcal{D}')^\perp = \{X'' \in \mathcal{D} \mid \text{Hom}_\mathcal{D}(X', X'') = 0, \forall X' \in \mathcal{D}'\}$: indeed, $\mathcal{D}'' \subset (\mathcal{D}')^\perp$ follows from i), and if $X \in (\mathcal{D}')^\perp$ then inserting it into a distinguished triangle $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$ since the map $X \rightarrow X''$ must be zero, we conclude that $X' = X \oplus X''[-1]$ and since non-zero mappings from $X$ to $X''[-1]$ must be impossible we must have $X''[-1] = 0$ so $X'' = 0$ thus $X = X' \in \mathcal{D}'$.

**Proposition.** Let $\mathcal{D}' \subset \mathcal{D}$ be a full triangulated subcategory and $i: \mathcal{D}' \hookrightarrow \mathcal{D}$ be the inclusion map.
The following are equivalent:

a) \( D' \) is left admissible i.e. together with \( D'' = (D')^\perp \) it forms an admissible pair of subcategories.

b) \( i \) admits a right adjoint.

c) \( D/D' \) exists and the projection \( D \to D/D' \) admits a right adjoint.

Moreover under these conditions the composite functor \( D'' \leftarrow D \to D/D' \) is an equivalence of categories.

Let \( D'' \subset D \) be a full triangulated subcategory and \( j : D'' \leftarrow D \) be the inclusion map.

The following are equivalent:

a) \( D'' \) is right admissible.

b) \( j \) admits a left adjoint.

c) \( D/D'' \) exists and the projection \( D \to D/D'' \) admits a left adjoint.

Moreover, under these conditions \( (D'')^\perp = D' \leftarrow D \to D/D'' \) is an equivalence.

**Proof:** we will only prove the first of the two symmetric parts.

a)\( \Rightarrow \) b): since taking distinguished triangles \( X' \to X \to X'' \to X'[1] \) is functorial in \( X \), we can take \( q(x) = X' \) and this will be a functor. To show that it’s right adjoint, we need to show \( \text{Hom}(X'_1, X) \cong \text{Hom}(X'_1, X') \) which is easy using the long exact sequence of a distinguished triangle together with \( \text{Hom}(X'_1, X'') = 0 \).

b)\( \Rightarrow \) a): Let \( q \) be the right adjoint to \( i \), and set \( D'' = (D')^\perp \).

Consider the distinguished triangle \( iqX \xrightarrow{\text{adj}} X \to \text{Cone}(iqX \to X) \to iqX[1] \)

We need to check that \( C = \text{Cone}(iqX \to X) \) is indeed in \( D'' \). Indeed, for any \( X'_1 \in D' \), adjunction induces an isomorphism between \( \text{Hom}(iX'_1, X) \cong \text{Hom}(X'_1, iqX) \) and \( \text{Hom}(iX'_1, X) = \text{Hom}(X'_1, X) \) hence the long exact sequence of a distinguished triangle implies the claim.

\[ a) \Rightarrow \mathclap{c): } \text{Define } J : D \rightarrow D \text{ by } JX = X'' \text{ where } X'' \text{ comes from the familiar distinguished triangle. Because it sends } D' \text{ to } 0, \text{ it factors through } D''/D'' . \text{ We need to show } \text{Hom}_{D/D''}(X, Y) \cong \text{Hom}_D(X, Y''). \text{ Indeed, any morphism between } X \text{ and } Y \text{ in } D''/D'' \text{ is a hut } X \to Z \leftarrow Y. Y \to Z \text{ a quasi-isomorphism implies that } Y'' = Z''. \text{ Indeed, the map induces a map of distinguished triangles } Y' \to Y \to Y'' \to Y'[1] \text{ and } Z' \to Z \to Z'' \to Z'[1] \text{ and by (Tr4) a distinguished triangle } \text{Cone}(Y \to Z) \to \text{Cone}(Y \to Z'') \to Z'[1] \to \text{Cone}(Y \to Z)[1] \text{ and since } \text{Cone}(Y \to Z), Z'[1] \in D' \text{ we conclude } \text{Cone}(Y \to Z'') \in D' \text{ and the equality } Y'' = Z'' \text{ follows from the uniqueness of the corresponding distinguished triangle.} \]

Therefore the map \( Y \to Z \) produces a map \( Y'' \to Z'' \) i.e. a map \( X'' \to Y'' \) which by composition yields a map \( X \to Z'' \).

Conversely, a map from \( X \to Y'' \) must become \( 0 \) when composed with \( X' \to X \) hence it must be induced from a map \( X'' \to Y'' \) and so it produces the hut \( X \to Y'' \leftarrow Y \). Checking that these operations are mutually inverse is another diagram chase.

c) \( \Rightarrow \) a): Assume \( p \) has a right adjoint \( j \). Since anything in \( D' \) becomes \( 0 \) in the quotient category, the adjointness property immediately implies that the essential image of \( j \) is in \( D'' \). Consider the distinguished triangle \( \text{Cone}(X \to jpX)[\sim] \to X \to jpX \to \text{Cone}(X \to jpX) \). Applying \( p \) to it gives an isomorphism \( jX \to pjpX \) thus \( \text{Cone}(X \to jpX) \) must lie in the kernel of \( p \) which by a previous lemma implies that it is a direct summand of something in \( D' \). In particular if \( X \in D'' \) it implies that the cone is \( 0 \) and \( X \equiv jpX \). To finish, we need to show that a direct summand of an object in \( D' \) is actually in \( D' \). [????]

Finally, it remains to show that \( D'' \rightarrow D/D' \) is an equivalence. We have just seen that \( j \) provides a right inverse, which must also be a left inverse. Indeed, we want \( p(j) = id \) which is true since \( p \) is essentially surjective so any element in \( D/D' \) is \( pX \) for some \( X \) and \( pjpX = pX \) is a standard property of adjoint functors.

02/04/2010

Consider the category \( D = K(A) \) where \( A \) is an abelian category, and define \( D' = K(A)^{acyc} \) the triangulated subcategory consisting of acyclic objects (i.e. exact complexes). We then define \( D(A) = D/D' \) - the derived category of \( A \). We define \( D^+(A), D^-(A) \) and \( D^b(A) \) to be the subcategories consisting of objects whose cohomologies are 0 in
degree $\ll 0, \gg 0$ or both. These categories can be defined in another way: for example, $\mathcal{D}^+(A)$ can be taken as the quotient of $\mathcal{K}^+(A)$ modulo $\mathcal{K}^+(A) \cap \mathcal{K}(A)^{acycl}$. The following proposition shows that these two definitions coincide.

**Proposition.** The natural map $\mathcal{K}^+(A)/\mathcal{K}^+(A) \cap \mathcal{K}(A)^{acycl} \to \mathcal{D}^+(A)$ (inherited from the map to $\mathcal{D}(A)$ which clearly factors through $\mathcal{D}^+(A)$) is an equivalence of categories. The same holds for $\mathcal{D}^−$ and $\mathcal{D}^h$.

Proof: I will only do the $+$ case, because the method clearly adapts to the other cases as well.

**Lemma.** A map of complexes which induces an isomorphism of the cohomologies is a quasi-isomorphism.

To see this, note that problem 3a), b) from PS 1 guarantee that every cone sequence is homotopy equivalent to a short exact sequence $0 \to A \to B \to C$ of complexes. Any such short exact sequence induces a long exact sequence of cohomologies, and the cone is acyclic if and only if the cohomologies of $C$ are 0, which is equivalent to the cohomologies of $A$ and $B$ being isomorphic. The lemma is proven.

It’s quite clear that the projection is indeed a well-defined functor, just because the category we’re modding out on the left is a subcategory of the category we mod out on the right.

Now we show $F$ is essentially surjective. That is, we need to show that every object with zero homology in low degrees is equivalent to an object with zero objects in low degree. According to the lemma a map is an equivalence (modulo acyclics) if it induces an isomorphism on cohomology. So assume $\mathcal{A}$ is a complex with negative cohomologies. Take the complex $A′$ defined by $A′[k] = 0, 0 < k, A′[0] = A[0]/\text{Im}(A[−1]), A′[k] = A[k] for k > 0$, with obvious differentials.

The map map from $A$ to $A′$ is zero in negative degrees, identity in positive degrees and $A[0] → A[0]/\text{Im}(A[−1])$. It’s clear that these maps are isomorphism on complexes.

Also they induce an isomorphism on cohomologies. This is clear in all degrees except the zero degree. But indeed, $\text{ker}(A′[0] → A′[1])$ is the projection of the kernel of $A[0] → A[1]$ as the map is induced, so it equals $\text{ker}(A[0] → A[1])/\text{Im}(A[−1] → A[0])$ which is the zeroth cohomology. As $\text{Im}(A[−1] → A[0]) = 0$, we deduce the result.

Finally we prove that $\text{Hom}(X, Y) → \text{Hom}(FX, FY)$ is an isomorphism -i.e. that $F$ is fully faithful. We need to construct the inverse. Consider a map from $X$ to $Y$, regarded as complexes in $\mathcal{K}(A)$, modulo $\mathcal{K}(A)^{acycl}$. It is a hut $X → W → Y$ with $W → X$ being the identity on cohomologies. We can change $W$ by isomorphism so we can assume $W$ is $FZ$ as well. Then clearly the hut $FX → FZ → FY$ induces a hut $X → Z → Y$ - simply because the objects are actually the same. It is clear that this map is indeed the inverse to $F$, so we are done.

Now recall the concept of an admissible triple. Assume that $\mathcal{D}' \subset \mathcal{D} \supset \mathcal{D}^\circ$ is an admissible triple, and call the inclusions $\mathcal{D}' → \mathcal{D}, \mathcal{D}^\circ → \mathcal{D}$ be $i_*$ and $j^*$.

**Proposition.** 1) $i_*$ admits a right adjoint $i'_*$

2) $j^*$ admits a left adjoint $j^*_*$

3) $j^*_* i_* = 0$ therefore it induces a map $\mathcal{D}/\mathcal{D}' → \mathcal{D}^\circ$. This map is an equivalence.

4) $i'_* j_* = 0$ and therefore it induces a map $\mathcal{D}/\mathcal{D}^\circ → \mathcal{D}'$. This is also an equivalence.

Proof: Parts 1) and 2) have been proved in the previous lecture. So were the first part of 3) and the first part of 4).

For the second part of 3): Consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{j^*} & \mathcal{D}^\circ \\
\downarrow p & & \downarrow j^*_* \\
\mathcal{D}/\mathcal{D}' & \xrightarrow{j^*} & \mathcal{D}/\mathcal{D}^\circ
\end{array}
$$

We claim that the inverse to $j^*_*$ is $pj_*$

Recall that $j^*, j_*$ are adjoint to each other.

Now let’s show that $pj_*, j^*_*$ is isomorphic to the identity.
As shown before, $X$ can be inserted into the distinguished triangle $X' \to X \to X'' \to X'[1]$ where $X' = i_*i'X, X'' = j_*j^*X$. In particular $X'' = pX$ gets mapped under $j^*$ to $j^*X$ and then under $j_*$ it gets mapped to $j_*j^*X = X''$.

For the converse, $j^*pj_* = j^*j_* \cong id$ as $j^*$ is essentially surjective. □

Remark: we can also show $D \xrightarrow{j^*} D''$ satisfies the universal property of mapping onto.

Now let $A$ be an abelian category, and $D = K^-(A), D'' = K^-(A)_{acycl}$. We claim that $D''$ is right admissible, with $D'$ consisting of complexes whose hom. into $D''$ is acyclic.

**Proposition.** Suppose $A$ has enough projectives. Then

$$K^-(A^{proj}) \subset K^-(A) \subset K^-(A)_{acycl}$$

is an admissible triple.

**Corollary.** $K^-(A^{proj}) \hookrightarrow K^-(A) \to D'(A)$ is an equivalence.

The proposition (and its corollary) admits a dual:

**Proposition.** If $A$ has enough injectives then

$$K^+(A)_{acycl} \subset K^+(A) \subset K^+(A^{inj})$$

is admissible.

**Corollary.** $K^+(A^{inj}) \hookrightarrow K^+(A) \to D^+(A)$ is an equivalence.

Proof of the first proposition:

Say $P^\bullet$ is a complex of projective bounded from above, and $X^\bullet$ is acyclic. Then $Hom(P^\bullet, X^\bullet)$ is also acyclic. We construct the bicomplex $Hom(P^i, X^j)$ and note that each row $(Hom(P^i, X^j))_j$ is exact as $X^\bullet$ is acyclic and $P^i$ is projective: if $\partial^j f = 0$ then $f$ factors through $Ker(\partial^j) = Im(\partial^{j-1})$ and since $P^i$ is projective, $f$ lifts to a function $f': P^i \to X^{j-1}$ such that $\partial^{j-1} \circ f' = f$ which implies that the row is exact. Hence as rows are exact, the total complex is also exact but it represents the complex $Hom(P^\bullet, X^\bullet)$.

This shows that the two subcategories have no morphisms between each other.

Also, since every object has a projective resolution, this translates into every complex being inserted into a distinguished triangle involving a complex of projectives and an acyclic complex, which finishes the proof of the proposition. □

**Definition.** $P^\bullet \in K(A)$ is called $K$-projective if it is in $(K(A)^{acycl})^\perp$ i.e. $Hom(P^\bullet, X^\bullet)$ is acyclic for every $X^\bullet$ acyclic.

We have show that any $P^\bullet \in Ho^- (A)$ that consists of projective objects is $K$-projective. This is not true for unbounded complexes however. For example, let $A = Mod_{\mathbb{C}[t]/t^2}$ and consider the complex

$$\mathbb{C}[t]/t^2 \xrightarrow{t} \mathbb{C}[t]/t^2 \xrightarrow{t} \ldots$$

It consists of projectives but is not $K$-projective: mapping from it to the complex $\ldots \xrightarrow{t} \mathbb{C}[t]/t^2 \xrightarrow{t} \mathbb{C}[t]/t^2 \xrightarrow{t} \ldots$ we can easily prove that the morphisms of degree 0 up to homotopy, are isomorphic to $\mathbb{C}$.

**Theorem.** If $A$ has enough projectives, then for any $X^\bullet \in Comp(A)$ there is a quasi-isomorphism $P^\bullet \to X^\bullet$ from a $K$-projective object.

The following explicit lemma is instrumental in the proof of this theorem:

**Lemma.** Let $A$ be an abelian category, and let $P^\bullet$ be a complex of objects in $A$ with the following property:
• As a complex
\[ P^* \cong \lim_{\leftarrow} P_i^* \]

• The maps \( P_i^* \to P_{i+1}^* \) are term-wise injective
• Each quotient complex \( R_i^* = P_{i+1}^*/P_i^* \) has zero differential and consists of projective objects of \( A \).

Then \( P^* \) is projective.

Proof of lemma: Consider a complex map \( f: P^* \to Q^* \) where \( Q^* \) is an acyclic complex. We want to show that \( f = d_Q g + g d_P \) where \( g \in \text{Hom}_{DG}^{-1}(P^*, Q^*) \). Notice that \( f \) is equivalent to a system of compatible maps \( f_i: P_i^* \to Q_i^* \), and therefore it is enough to construct a system of compatible maps \( g_i: P_i^* \to Q_i^* \) with \( f_i = d_Q g_i + g_i d_P \).

We will construct them by induction on \( i \).

Base case: \( i = 1 \). The map \( f_1: P_1^* \to Q_1^* \) must satisfy \( d_Q f_1 = 0 \), because \( P_1^* \) has zero differential. Because \( Q^* \) is acyclic, it has to factor through the image of \( d_Q \), i.e. \( f_1 = i f'_1 \) where \( i[k]: \text{Im}(d_Q[k-1]) \to Q[k] \). As \( P^* \) is projective, and \( d_Q: Q[k-1] \to \text{Im}(d_Q[k-1]) \) is surjective (\( d_Q \) also factors through the image), there is a map \( g[k]: P[k] \to Q[k-1] \) such that \( f[k] = d_Q \circ g[k-1] \) - and this map satisfies the conclusion.

Before we proceed to the induction step, let’s first prove that all objects \( P_i^* \) are \( K \)-projective. First we have the following proposition:

If \( 0 \to A \to B \to C \) is a short exact sequence and \( A, C \) are projective, then \( B \) is projective too, and the short exact sequence splits. Indeed, as \( B \to C \) is surjective the map \( C \to C \) gives rise to a splitting map \( C \to B \) - that implies that the short exact sequence splits and \( B \cong A \oplus C \) via identifying \( A \) with the cokernel of the map \( C \to B \) (this is a standard result). Subsequently \( B \) is projective as a direct sum of two projectives.

This proposition readily implies that all the \( P_i^* \) are projective by induction, and that there is a splitting map \( \beta_i: P_{i+1}^* \to P_i^* \) of degree 0 (but not necessarily commuting with the differential). Let \( \alpha_i: P_i^* \to P_{i+1}^* \) be the injection map.

We are now ready to prove the induction step.

Assume we have constructed a map \( g_i: P_i^* \to Q_i^* \) such that \( f_i \circ \alpha_i = f_i = d_Q g_i + g_i d_P \). Define \( g_i' + 1 = g_i \circ \beta_i \), and let \( h_i = f_i + 1 - d_Q g_i' + 1 - g_i' d_P \). Then \( h_i \circ \alpha_i = f_i + 1 \circ \alpha_i - d_Q g_i' + 1 \circ \beta_i \circ \alpha_i - g_i \circ \beta_i \circ d_P \circ \alpha_i \). Because \( \alpha_i \) commutes with the differential, \( d_P \circ \alpha_i = \alpha_i \circ d_P \) and as \( \beta_i \circ \alpha_i = id \) we deduce \( h_i \circ \alpha_i = 0 \) hence \( h_i \) factors through a map \( h_i': P_i^* \to Q_i^* \). We now apply the base case to produce a map \( g_{i+1}' \in \text{Hom}^{-1}(R_{i+1}^*, Q_i^*) \) with \( d_Q g_i + 1 - g_{i+1}' d_P \). This map induces a map \( g_i + 1 \in \text{Hom}^{-1}(P_i^*, Q_i^*) \) (by abuse of notation) with \( d_Q g_i + 1 - g_i + 1 d_P = h_i + 1 \), and hence we can set \( g_i = g_i' + 1 \). \( \square \)

Proper of theorem: Consider \( P_0^* \) be the zero complex mapping to \( X^* \). Now assume a map \( f_i: P_i^* \to X^* \) has been constructed. Let \( Q_i^* = \text{Cone}(P_i^* \to X_i^*)[-1] \). We claim that there exists \( R_i^* \) a complex with projective components and zero differential that maps to \( Q_i^* \) and induces a surjection on the cohomology - just choose \( R_i[1] \) to be a projective surjecting onto \( \ker(d) \subset Q_i[k] \), then it will also surject on the cohomology - and the corresponding maps will induce a map of complexes as the composite of a vertical map and the differential will always be zero.

Because \( Q_i^* \) maps to \( P_i^* \) from the cone property, we have a composite map \( r_i: R_i^* \to P_i^* \), and consider \( P_{i+1}^* \) be the cone of this map. Note that \( P_{i+1}^* \) has a map back to \( P_i^* \) therefore it maps into \( X^* \).

We claim that the sequence \( P_i^* \) satisfies the hypothesis of part a). The injectivity is true since \( P_i^* \) is the cone of a map to \( P_i^* \). Note that the limit \( P^* \) exists because as shown in the proof of the lemma the sequence \( P_i^* \to P_{i+1}^* \) splits so we can build the objects of the complex as being infinite direct sums.

Now \( Q_i^* = X^*[-1] \oplus P_i^* \) with differential \( d(x, p) = (d_X x + f_i(p), -d_P p) \). Further \( P_{i+1}^* = P_i^* \oplus R_i[1] \) with differential \( d(p, r) = (d_P p + r_i(r), 0) \) - from here we immediately see that \( R_i[1] \) can be identified as the quotient of \( P_{i+1}^* \). Whence we can apply the previous part to conclude that \( P^* = \lim_{i \in \mathbb{N}} P_i^* \) is quasi-projective, and it maps to \( X^* \).

It remains to show that the map is a quasi-isomorphism.
Consider the following diagram:

\[
\begin{array}{ccc}
R_i^* & \longrightarrow & P_i^* \longrightarrow P_{i+1}^* \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X^* \longrightarrow X^* \\
\downarrow & & \downarrow \\
R_i^*[1] & \longrightarrow & Q_i^*[1] \longrightarrow Q_{i+1}[1]
\end{array}
\]

The upper two rows are cone sequences, so are all the three columns, therefore the bottom row is also a cone sequence according to the nine lemma, since cone sequences are homotopy equivalent to exact sequences.

It follows that \(R_i^*[1] \rightarrow Q_i^*[1] \rightarrow Q_{i+1}[1]\) is a cone sequence, hence it induces a long exact sequence on cohomology, and because we know that \(R_i^* \rightarrow Q_i^*\) is surjective on the cohomologies, we deduce that the map \(Q_i^*[1] \rightarrow Q_{i+1}[1]\) is homotopy equivalent to zero.

Now let \(Q^*[1]\) be the cone of the map \(P^* \rightarrow X^*\). We claim that \(Q^*[1] = \lim_{n} Q_i^*[1]\). This can be proved using the universal property, but in fact in this case it is best checked in a straightforward manner: just use the fact that \(Q_i^*[1] = X^* \oplus P^*[1]\) and \(Q_i^*[1] = X^* \oplus P_i^*[1]\), and that \(P^* = \lim_{n} P_i^*\) and the differentials also clearly commute. Also from this fact it is clear that \(Q_i^*[1]\) injects into \(Q_{i+1}[1]\). Our result now follows from the following lemma:

Lemma: Let \(Q^* = \lim_{n} Q_i^*\) and assume that \(\alpha_i: Q_i^* \rightarrow Q_{i+1}^*\) is injective and homotopic to 0. Then \(Q^*\) is acyclic.

Proof: I can only prove it for modules, so using the Mitchell embedding theorem let’s assume all objects are modules. Then it’s well-known from commutative algebra that every element of \(Q[k]\) equals \(\pi_i(a)\) for some \(a \in Q_i[k]\) (where \(\pi_i: Q_i[k] \rightarrow Q[k]\) are the induced maps, which are readily seen to be injective). For the latter injectivity, we use the well-known lemma that \(\pi_i(b) = 0\) if and only if \(b\) projects to 0 in some higher \(Q_j[k]\), and now we are done using injectivity (both these facts are well-known and for example found in Atiyah-MacDonald).

So now assume \(b = \pi_i(s)\) satisfies \(db = 0\). As \(db = \pi_i(d_i a)\) we deduce \(d_i a = 0\) - in particular \(\alpha_i(d_i a) = d_{i+1}(\alpha_i a) = 0\). But \(\alpha_i\) is homotopic to the identity so \(\alpha_i a = d_{i+1} q_i a + q_{i+1} d_i a = 0\) and as \(d_i a = 0\) we deduce \(\alpha_i a = d_{i+1} q_i a\) hence \(b = \pi_{i+1}(\alpha_i a) = \pi_{i+1}(d_{i+1} q_i a) = d\pi_i(q_i a)\) and this finishes the proof. □

**Corollary.** \(K \rightarrow \text{proj} \subset K(A) \supset K(A)^{acyc}\) is an admissible triple.

**Derived functors**

Assume that the functor \(F: \mathcal{D} \rightarrow \mathcal{D}'\) does not factor through \(\mathcal{D}/\mathcal{D}'\). We would still like to produce a functor \(G: \mathcal{D}/\mathcal{D}' \rightarrow \mathcal{D}\)

**Definition.** A functor \(F^\#: \mathcal{D}'' \rightarrow \mathcal{D}'\) is called a Left Kahn Extension (LKE) of \(F, F' = LKE(F)\) is satisfies the universal property

\[
\text{Hom}_{\mathcal{D}/\mathcal{D}'}(F^\#, G) \cong \text{Hom}_{\mathcal{D}, \mathcal{D}'}(F, G \circ \pi)
\]

for every \(G \in \text{Funct}(\mathcal{D}'', \mathcal{D})\).

In other words, LKE is the left adjoint to the functor \(\circ \pi\) from \(\text{Funct}(\mathcal{D}'', \mathcal{D})\) to \(\text{Funct}(\mathcal{D}, \mathcal{D})\) (although it is not defined for all \(F\)).

In particular, there exists an adjunction natural transformation \(F \rightarrow F'' \circ \pi\) such that if \(F'' \rightarrow G\) is a morphism of functors, the composite morphism \(F \rightarrow F'' \circ \pi \rightarrow G \circ \pi\) is the one corresponding to it by adjunction.

We can also define the Right Kahn Extension in the same way: the right adjoint to composition with \(\pi\), i.e. to \(F\) one associates the functor \(F''\) such that

\[
\text{Hom}_{\mathcal{D}/\mathcal{D}'}(G, F'') = \text{Hom}_{\mathcal{D}, \mathcal{D}'}(G \circ \pi, F)
\]

17
Definition. If $F$ is a functor from $\mathcal{D} \to \mathcal{D}'$ and $\mathcal{D}'$ is a triangulated subcategory of $\mathcal{D}$, then the \textit{right derived functor} of $F$ (if it exists) is

$$R_{\mathcal{D}'}(F) = LKE(F)$$

and similarly the \textit{left derived functor} of $F$ is

$$L_{\mathcal{D}'}(F) = RKE(F)$$

Proposition. a) Let $\mathcal{D}$ be a triangulated category, and $\mathcal{D}' \subset \mathcal{D}$ a full triangulated subcategory. Then the right (left) adjoint to the projection $\mathcal{D} \to \mathcal{D}'$ (if it exists) is the right (resp. left) derived functor of the identity functor $\mathcal{D} \to \mathcal{D}$.

b) Let $F: \mathcal{D}_1 \to \mathcal{D}$ be an adjoint pair of triangulated functors. Let $\mathcal{D}'_1 \subset \mathcal{D}_1$ and $\mathcal{D}' \subset \mathcal{D}$ be full triangulated subcategories. Assume that the projection $\pi: \mathcal{D} \to \mathcal{D}'$ admits a right adjoint and the projection $\pi_1: \mathcal{D}_1 \to \mathcal{D}_1'$ admits a left adjoint. Then the $\mathcal{D}'$-right derived functor of $\pi_1 \circ G: \mathcal{D} \to \mathcal{D}_1/\mathcal{D}_1'$ and the $\mathcal{D}_1'$-left derived functor of $\pi \circ F: \mathcal{D}_1 \to \mathcal{D}'$ are mutually adjoint.

Proof: a) Let $\pi$ be the projection functor, and $j$ its right adjoint (right-adjointness is done in the same way).

We want a

$$\text{Hom}_{\text{Funct}(\mathcal{D}', \mathcal{D})}(j, G) = \text{Hom}_{\text{Funct}(\mathcal{D}, \mathcal{D})}(id, G \circ \pi)$$

for any functor $G$ from $\mathcal{D}'$ to $G$, where $\pi$ is the projection $\mathcal{D} \to \mathcal{D}/\mathcal{D}'$.

Here is a way to get the identification:

Assume we have a natural transformation $\alpha: j \to G$. Then $\alpha \circ \pi$ is a natural transformation from $j\pi$ to $G$ and composing it with the adjoint natural map $adj_1: id \to j\pi$ we get a map $id \to G$.

Conversely, assume we have a natural transformation $\beta: id \to G\pi$. Then $\beta \circ j$ is a natural transformation from $j$ to $G \circ \pi \to j$ which then maps to $G$ via $G$ composed with the adjointness map $adj_2: \pi \circ j$.

We need to prove that these operations are mutually inverse to each other.

Let’s start from $\beta: id \to G \circ \pi$. We produce $\alpha = (G \cdot adj_2) \circ (b \circ j)$ and then $\beta' = (G \cdot adj_2 \circ \pi) \circ (b \circ j \circ \pi) \circ adj_1$ and we want $\beta = \beta'$.

In terms of object we want the following diagram to commute for any $X$:

$$\begin{array}{ccc}
X & \xrightarrow{adj_1(X)} & j\pi X \\
\downarrow{\beta(X)} & & \downarrow{\beta(j\pi X)} \\
G\pi(X) & \xleftarrow{G\cdot adj_2(\pi X)} & G\pi j\pi(X)
\end{array}$$

But recall that $adj_2 \circ \pi: \pi j\pi \to \pi$ has a quasi-inverse $\pi \circ adj_1$, whence applying $ti$ we transform our diagram into another one

$$\begin{array}{ccc}
X & \xrightarrow{adj_1(X)} & j\pi X \\
\downarrow{\beta(X)} & & \downarrow{\beta(j\pi X)} \\
G\pi(X) & \xleftarrow{G\pi \cdot adj_1(X)} & G\pi j\pi(X)
\end{array}$$

and this is just the defining diagram for a natural transformation.

For the other composition to be the identity, the trick is absolutely the same. This time we’ll have to show that a composition $j \to j\pi j \to G\pi j \to G$ equals $\alpha$, and it’s done in the same way by using the quasi-inverse $j\pi j \to j$ to make it into a similar diagram.

b) $G: \mathcal{D} \to \mathcal{D}_1$ is right adjoint, $F: \mathcal{D}_1 \to \mathcal{D}$ is left adjoint.
Let \( j \) be right adjoint to \( \pi \) and \( j_1 \) be left adjoint to \( \pi_1 \). I claim the two derived functors are \( \pi_1Gj \) and \( \pi Fj_1 \) - which are clearly adjoint to each other (right respectively left) as compositions of adjoint functors: \( \text{Hom}(X, \pi_1GjY) = \text{Hom}(\pi Fj_1X, jY) = \text{Hom}(\pi Fj_1X, Y) \).

The proof of the claim is based on the very same idea as in a) (in fact a) is a special case of this for \( G = id \) respectively \( F = id \), so i will not repeat the diagrams of “naturality” here - but see the next problem. I will only write the identifications, for \( R \) the right adjoint to \( G \circ \pi \), for example:

\[
R \text{ is defined by } \text{Hom}_{\text{Funct}(D/D', D_1/D'_1)}(R, H) = \text{Hom}_{\text{Funct}(D, D_1/D'_1)}(\pi_1 \circ G, H \circ \pi). \]

If we take a natural transformation from \( \pi_1Gj \) to \( H \), we compose it with \( \pi \) to get \( \pi_1Gj\pi \rightarrow H\pi \), and as \( id \) maps to \( j\pi \) we get the composite natural transformation \( \pi_1G \rightarrow \pi_1Gj\pi \rightarrow H\pi \). Conversely if we have a natural transformation \( \pi_1G \rightarrow H\pi \) we get a natural transformation \( \pi_1Gj \rightarrow H\pi j \rightarrow H \). \( \square \)

Proposition. Let \( f : D \rightarrow A \) be a cohomological functor, where \( A \) is an abelian category. Let \( D' \subset D \) be a full triangulated subcategory.

a) Assume that \( D' \) is left admissible. Then \( Rf \) exists and is isomorphic to \( f \circ j_* \), where \( j_* \) is the right adjoint to the projection \( D \rightarrow D/D' \).

b) If, in addition to the assumptions of a), \( f \) is of the form \( H^0 \circ F \), where \( F \) is a triangulated functor \( D \rightarrow D(A) \), then \( Rf = H^0 \circ RF \).

c) If \( A = Ab \) then \( Rf \) always exists and is given by

\[
X \rightarrow \lim_{(X', \alpha) \in X \setminus D' - qi} f(X').
\]

d) If \( A = Ab \) and \( f = \text{Hom}_D(X, -) \) then \( Rf(Y) = \text{Hom}_{D/D'}(X, Y) \). This is called the Ext functor.

Proof: a) This (as well as the previous proposition) is a special case of the following lemma (which admits a dual version):

Lemma: Let \( D' \) be a full triangulated subcategory of \( D \) and assume the projection \( \pi : D \rightarrow D/D' \) admits a right adjoint \( j \). Then for any functor \( F : D \rightarrow \mathbb{D} \), the right adjoint to \( F \) is simply \( F \circ j \).

Proof: we need to show the following identification

\[
\text{Hom}_{\text{Funct}(D/D'), \mathbb{D}}(F \circ j, G) \cong \text{Hom}_{\text{Funct}(D, \mathbb{D})}(F, G \circ \pi)
\]

for any functor \( G : D/D' \rightarrow \mathbb{D} \).

Indeed, consider a natural transformation \( \alpha : F \circ j \rightarrow G \). We produce \( \alpha \pi \circ j \circ \pi \rightarrow G \circ \pi \). Because there exists an adjoint natural transformation \( a_1 : id \circ j \pi \rightarrow \alpha \pi \) with \( Fa_1 \) to get \( \beta = \alpha \pi \circ Fj_1 \rightarrow F \circ G \circ \pi \).

Conversely, consider a natural transformation \( \beta : F \rightarrow G \circ \pi \). We get \( \beta j : F \circ j \rightarrow G \circ \pi j \) and using the adjunction \( a_2 : id \rightarrow \pi j \) we get \( Ga_2 \circ \beta j : F \circ j \rightarrow G \).

Now we show these operations are mutually inverse. Suppose we start with \( \alpha : F \circ j \rightarrow G \). Then we get \( \beta = \alpha \pi \circ Fa_1 : F \rightarrow G \circ \pi \) and back \( \alpha' = Ga_2 \circ \beta j = Ga_2 \circ \alpha j \circ Fa_1 j \) and we need to show \( \alpha' = \alpha \). So we have to show that the following composite functor is \( \alpha \):

\[
F \circ j \xrightarrow{F \circ a_1 j} F \circ j \pi \xrightarrow{\alpha \pi j} G \circ \pi j \xrightarrow{G a_2} G
\]

But recall that \( ja_2 \) is quasi-inverse to \( a_1 j \) (\( a_1 j \circ ja_2 = id : j \rightarrow j \)) whence \( F ja_2 \) is quasi-inverse to \( Fa_1 j \). Hence it suffices to prove that \( Ga_2 \circ \alpha j = \alpha \circ Fja_2 \) (because then composing with \( Fa_1 j \) to the right will yield \( \alpha' = \alpha \)).

So we need to show that the following diagram commutes:

\[
\begin{array}{ccc}
F \circ j \pi j & \xrightarrow{\alpha \pi j} & G \circ \pi j \\
F j \alpha_2 \downarrow & & \downarrow G a_2 \\
F \circ j & \xrightarrow{\alpha} & G \\
\end{array}
\]

19
This is simply the diagram of \( \alpha \) being a natural transformation. Namely, if we choose an object \( X \in \mathcal{D}/\mathcal{D}' \) (which, by abuse of notation, we also consider to be in \( \mathcal{D} \)), the aforementioned diagram transforms to

\[
\begin{array}{ccc}
Fj\pi j(X) \overset{\alpha_{\pi j(X)\pi j}}{\longrightarrow} G\pi j(X) \\
Fj(\alpha_2)X & \downarrow & G(\alpha_2)X \\
Fj(X) \overset{\alpha_X}{\longrightarrow} G(X)
\end{array}
\]

which is the condition of \( \alpha \) being a natural transformation with respect to the adjunction map \( \pi jX \overset{(\alpha_2)X}{\longrightarrow} X \)

The fact that the other composite is the identity is done in the same way. \( \square \)

b) Follows immediately from the previous part and the lemma from it.

c) Clearly the object \((Rf)(X)\lim_{(X',\alpha)\in X/\mathcal{D}'-qi} f(X')\) exists as limits exist in \( Ab \). Unfortunately, I don’t know how to show it’s functorial.

We need to show \( \text{Hom}_{\text{Funct}(\mathcal{D}/\mathcal{D}',Ab)}(Rf,G) = \text{Hom}_{\text{Funct}(\mathcal{D}),Ab}(f,G \circ \pi) \).

Assume now that we have a map of functors from \((Rf)\) to \( G \). This means that we get a map \((Rf)(X) \to G(X)\) for any maps i.e. compatible maps \( f(X') \to G(X) \). In particular, this induces a map \( f(X) \to G(X) \) and the compatibility clause says that \( X' \to X \) is a quasi-isomorphism then the diagram

\[
\begin{array}{ccc}
& f(X') \downarrow & \\
& \downarrow & \\
& f(X) & \downarrow
\end{array}
\]

commutes. In particular, this tells that the map factors through \( \mathcal{D}/\mathcal{D}' \) i.e. induces a map \( f \to G \circ \pi \). Conversely, if we have such a map then as \( X \cong X' \) in \( \mathcal{D}/\mathcal{D}' \) we get the map \( f(X') \to G(X) \) and hence we get a map of limits, which finishes the claim.

Here is a more detailed proof of this part, due to Jonathan:

Let \( f : \mathcal{D} \to Ab \) be a cohomological functor, \( \mathcal{D}' \subset \mathcal{D} \) a full triangulated subcategory, and \( \pi : \mathcal{D} \to \mathcal{D}/\mathcal{D}' \) the projection. Direct limits exist in \( Ab \). We define functor \( f' : \mathcal{D}/\mathcal{D}' \to Ab \) as follows. The objects of \( \mathcal{D} \) and \( \mathcal{D}/\mathcal{D}' \) are the same, so for \( \pi X \in \mathcal{D}/\mathcal{D}' \), where \( X \in \mathcal{D} \), we let

\[
f''(\pi X) = \lim_{(X,\alpha)\in X/\mathcal{D}'-qi} f(\tilde{X})
\]

Now take another object \( \pi Y \in \mathcal{D}/\mathcal{D}' \). We define \( f' : \text{Hom}_{\mathcal{D}/\mathcal{D}'}(\pi X, \pi Y) \to \text{Hom}_{Ab}(f''(\pi X), f''(\pi Y)) \) as follows: take a hut \( X \to Z \to Y \) and for \( (\tilde{X},\alpha) \in X/\mathcal{D}'-qi \), form a lower hut as in the diagram:

\[
\begin{array}{ccc}
\tilde{X} & \overset{\alpha}{\longrightarrow} & X \\
\downarrow \beta & & \downarrow \\
Y & \overset{u}{\longrightarrow} & Z
\end{array}
\]

where \( (\tilde{Y},\beta) \in Y/\mathcal{D}'-qi \). Now compose \( f(\tilde{X}) \circ f(\tilde{Y}) \to f''(\pi Y) \). Observe that if we have \( (\tilde{X}_2,\alpha_2) \) with \( v : \tilde{X}_2 \to \tilde{X} \) compatible with \( \alpha,\alpha_2 \), then by commutativity we see that the same procedure gives map \( u \circ v : \tilde{X}_2 \to \tilde{Y} \). Therefore by universal property of direct limit, we get a map \( f''(\pi X) \to f''(\pi Y) \). From the direct limits we have \( f'' \) is well-defined on \( \text{Homs} \). Clearly \( f''(id) = id \). Lastly we check composition: let \( \pi X, \pi Y, \pi Z \in \mathcal{D}/\mathcal{D}' \) and take huts that represent morphisms \( \pi X \to \pi Y \) and \( \pi Y \to \pi Z \). Since qi are localizing set of morphisms, we have the following commutative
diagram, which implies \( f'' \) respects composition of maps.

We conclude that \( f'' : \mathcal{D}/\mathcal{D}' \to \text{Ab} \) is a valid functor.

Now let \( G \in \text{Funct}(\mathcal{D}/\mathcal{D}', \text{Ab}) \). We define a map \( \Phi : \text{Hom}_{\text{Funct}(\mathcal{D}/\mathcal{D}', \text{Ab})}(f, G\pi) \to \text{Hom}_{\text{Funct}(\mathcal{D}/\mathcal{D}', \text{Ab})}(f'', G) \). Take a natural transformation \( \phi : f \to G\pi \). Now take an object \( \pi X \in \mathcal{D}/\mathcal{D}' \). For \((\tilde{X}, \alpha) \in X \setminus \mathcal{D}' - qi\), since \( \alpha \) is qi, \( G\pi(\alpha) \) is isomorphism, and we have a map

\[
f\tilde{X} \overset{\phi_{\tilde{X}}}{\longrightarrow} G\pi\tilde{X} \overset{G\pi(\alpha)^{-1}}{\longrightarrow} G\pi X
\]

Given \( u : (\tilde{X}, \alpha) \to (\tilde{X}_2, \alpha_2) \), using naturality of \( \phi \) we have following commutative diagram:

\[
\begin{array}{ccc}
f(\tilde{X}) & \overset{\phi_{\tilde{X}}}{\longrightarrow} & G\pi\tilde{X} \\
\downarrow{u} & & \downarrow{G\pi u} \\
f(\tilde{X}_2) & \overset{\phi_{\tilde{X}_2}}{\longrightarrow} & G\pi\tilde{X}_2
\end{array}
\]

Therefore by universal property of direct limit, the collection of \( G\pi(\alpha)^{-1}\phi_{\tilde{X}} \) induce a map \( f''(\pi X) \to G(\pi X) \) which we define as \( \Phi(\phi)_{\pi X} \). From our definitions, it can be checked that \( \Phi(\phi) \) is a natural transformation and \( \Phi \) is natural in \( G \).

Next we define the reverse map \( \Psi : \text{Hom}_{\text{Funct}(\mathcal{D}/\mathcal{D}', \text{Ab})}(f'', G) \to \text{Hom}_{\text{Funct}(\mathcal{D}/\mathcal{D}', \text{Ab})}(f, G\pi) \). Take natural transformation \( \psi : f'' \to G \) and an object \( X \in \mathcal{D} \). Since \((X, \text{id}_X) \in X \setminus \mathcal{D}' - qi\), we have \( f(X) \to f''(\pi X) \), so define \( \Psi(\psi)_X \) to be the composition \( f(X) \hookrightarrow f''(\pi X) \overset{\psi_{\pi X}}{\longrightarrow} G(\pi X) \). Again it can be checked that \( \Psi(\psi) \) is a natural transformation and \( \Psi \) is natural in \( G \).

Now we check that \( \Phi, \Psi \) are inverses. First consider \( \Psi\Phi(\phi) \), which is equal to \( f(X) \to f''(\pi X) \overset{\Phi(\phi)_{\pi X}}{\longrightarrow} G(\pi X) \), which by definition of \( \Phi \) is \( G\pi(\text{id}_X)^{-1}\phi_X = \phi_X \). Therefore \( \Psi\Phi = \text{id} \). Next consider \( \Phi\Psi(\psi) \). For any \((\tilde{X}, \alpha) \in X \setminus \mathcal{D}' - qi\), the map \( f(\tilde{X}) \to f''(\pi X) \overset{\Phi(\psi)_{\pi X}}{\longrightarrow} G(\pi X) \) is defined to be \( G\pi(\alpha)^{-1}\Psi(\psi)_{\tilde{X}} \). Since \( \psi \) is a natural transformation, we have the commutative diagram

\[
\begin{array}{ccc}
f''(\pi X) & \overset{\psi_{\pi X}}{\longrightarrow} & G(\pi X) \\
\downarrow{f''(\pi(\alpha))} & & \downarrow{G\pi(\alpha)} \\
f(\tilde{X}) & \overset{\phi_{\tilde{X}}}{\longrightarrow} & G\pi\tilde{X} \overset{G\pi(\alpha)^{-1}}{\longrightarrow} G(\pi X)
\end{array}
\]

where the composition of the bottom row equals \( G\pi(\alpha)^{-1}\Psi(\psi)_{\tilde{X}} \). Since \( \alpha : X \to \tilde{X} \) is a qi, \( f''(\pi(\alpha)) \) is isomorphism, and by definition of the functor \( f'' \), we see that \( f''(\pi(\alpha)^{-1} \) is the obvious inclusion \( f''(\pi\tilde{X}) \to f''(\pi X) \). Thus \( f(\tilde{X}) \to f''(\pi X) \overset{\Phi(\psi)_{\pi X}}{\longrightarrow} G(\pi X) \) is equal to \( f(\tilde{X}) \to f''(\pi X) \overset{\psi_{\pi X}}{\longrightarrow} G(\pi X) \). We deduce that \( \Phi\Psi = \text{id} \). Therefore \( \Phi, \Psi \) are natural isomorphisms. Hence \( f'' \) is the LKE of \( f \), i.e., \( Rf = f'' \). d) Follows from the previous part. We only need to show that \( \lim_{(Y', \alpha) \in Y \setminus \mathcal{D}' - qi} \text{Hom}(X, Y') \) which is the "lower hut" definition of morphisms in \( \mathcal{D}/\mathcal{D}' \).

**Example.** Let \( F : \mathcal{A} \to \mathcal{B} \) be an additive functor between abelian categories. It induces the functor \( K(F) : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B}) \), and \( F \) is exact precisely when \( K(F) \) factors through \( D(\mathcal{A}) \).
We convene to denote by $RF : D(A) \rightarrow D(B)$ the right derived functor

$$R_{K(A)\rightarrow K(B)}(\pi_B \circ K(F))$$

Now recall that $\pi_A$ has a right adjoint if there exist enough injectives: namely choosing for each object in $D(A)$ represented by $\pi_A(X^\bullet)$ a projective resolution of $X^\bullet$ (there is a unique one up to quasi-isomorphism).

As a corollary of the previous propositions, we have the following observation:

In the case of $D(A)$, the right derived functor $RF$ is calculated as follows:

Take an object $X^\bullet \in D(A)$ represented by $X^\bullet \in K(A)$. Find a $K$-injective resolution $X^\bullet \rightarrow I^\bullet$, and take $F(I^\bullet)$. The projection of $F(I^\bullet)$ to $B$ is the right derived functor.

We similarly construct $LF$ via projective resolutions, in case there are enough projectives.

Remark: we don’t actually use the universal property of being LKE/RKE. This universal property is useful in at least one situation: in case one wants to show two functors are isomorphic - and one of them is expressed in terms of an LKE/RKE, the universal property may be used to construct a morphism from one to another, and then one can show it is an isomorphism. This allows us to get for free that the morphism is a natural transformation, as it may not be at all obvious to prove directly.

Observation: if $F$ is left exact, then as it sends $K^+(A)$ to $K^+(B)$ and for any single object $M \in A$,

$$H^i(RF(M)) = \begin{cases} 0 & i < 0 \\ F(M) & i = 0 \end{cases}$$

This is obtained by taking an injective resolution $M \rightarrow I_0 \rightarrow I_1 \rightarrow \ldots$ and noticing that the zero-th cohomology of $F(I_0) \rightarrow F(I_1) \rightarrow \ldots$ is $Ker(F(I_0) \rightarrow F(I_1)) = F(Ker(I_0 \rightarrow I_1)) = F(M)$ since $F$ is right exact.

Similarly, if $F$ is right exact then the left derived functor when applied to a single object has $i$-th cohomology equal to 0 for $i > 0$ and its 0-th cohomology equals $F$.

Also, if $f : D \rightarrow A$ is a functor, we use the notation

$$f^i(X) = f(X[i])$$

**Definition.** Let $D$ be a triangulated category, and let $f : D \rightarrow A$ be a functor into an abelian category. We say that $f$ is cohomological if $f$ takes distinguished triangles to long exact sequences in the following way:

if $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ is a distinguished triangle, then

$$f(X) \rightarrow f(Y) \rightarrow f(Z) \rightarrow f^1(X) \rightarrow \ldots$$

is a long exact sequence.

**Examples:** 1) if $X \in D$ then the functor $Hom_D(X, -) : D \rightarrow Ab$ is cohomological.

2) The functor $D(A) \xrightarrow{H^0} A$ is cohomological.

Remark: If we have a functor $D \xrightarrow{f} \tilde{D}$ we can try to take LKE/RKE of $F$ in the appropriate world - Delta functors, cohomological functors, etc. The lemma for adjoints still holds: $Rf = f \circ j_*$.
Recall also the results proved in a previous proposition:

- If \( F = H^0 \circ f \) then \( RF = H^0 \circ Rf \)

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{f} & \mathcal{D}(A) \\
\downarrow{Rf} & & \downarrow{H^0} \\
\mathcal{D}/\mathcal{D}' & \xrightarrow{Rf} & A
\end{array}
\]

- If \( f \) is a functor from \( \mathcal{D} \) to \( \text{Ab} \) then \( Rf \) always exists and can be expressed as

\[
Rf = \varinjlim_{(X', \alpha) \in X \setminus \mathcal{D}' - qi} f(X')
\]

- If \( X \in \mathcal{D} \), \( f = \text{Hom}_\mathcal{D}(X, -) \) then \( Rf = \text{Hom}_{\mathcal{D}/\mathcal{D}'}(X, -) \)

Now let \( X \) be a topological space and \( \mathcal{R}_X \) a sheaf of rings on \( X \). Let \( A = \text{Sh}_{\mathcal{R}_X} - \text{mod}(X) \).

If \( X f \rightarrow Y \) is a map of topological spaces, and we have a sheaf of rings \( \mathcal{R}_Y \) on \( Y \) with a map to \( f_* \mathcal{R}_X \), we can consider the functor

\[
f_* : \text{Sh}_{\mathcal{R}_X} - \text{mod}(X) \rightarrow \text{Sh}_{\mathcal{R}_Y} - \text{mod}(X)
\]

This functor is left exact, and it makes sense to construct the right derived functor to it. For this functor to be of any use, we need the following result:

**Lemma.** The category \( \text{Sh}_{\mathcal{R}_X} - \text{mod} \) has enough injectives.

**Proof:** Take a point \( x \in X \) and consider the inclusion \( i_x : \{x\} \rightarrow X \). If \( M_x \) is an injective module over \( (\mathcal{R}_X)_x \) then \( i_x^* M_x \) is an injective sheaf of \( \mathcal{R}_X \)-rings: this easily follows by adjunction since \( i_x^* \) is right adjoint to an exact functor.

Therefore for any sheaf \( \mathcal{F} \) we can embed it into \( \prod_{x \in X} (i_x^*)(\mathcal{F}_x) \) and since the category of modules over a ring has enough injectives, each \( \mathcal{F}_x \) can be embedded into an injective module \( M_x \) producing the composite embedding \( \mathcal{F} \hookrightarrow \prod_{x \in X} (i_x^*)(M_x) \). \( \square \)

Now let \( \{U_\alpha\} \) be a covering of \( X \) and take \( X \in \text{Comp}^+(X) - \) a complex of sheaves of rings over \( X \).

Take \( \check{C}(X, \{U_\alpha\}, \mathcal{F}^\bullet) \) be the complex associated to the bicomplex

\[
\begin{array}{ccc}
\prod_{\alpha, \beta} \Gamma(U_\alpha \cap U_\beta, \mathcal{F}^0) & \rightarrow & \prod_{\alpha, \beta} \Gamma(U_\alpha \cap U_\beta, \mathcal{F}^1) \\
\downarrow & & \downarrow \\
\prod_{\alpha} \Gamma(U_\alpha, \mathcal{F}^0) & \rightarrow & \prod_{\alpha} \Gamma(U_\alpha, \mathcal{F}^1) \\
\end{array}
\]

Remark: we can order the the entries and then take the product over \( \alpha < \beta \) etc. which yields an alternative Cech complex.

Now let \( \{U_\alpha\}_{\alpha \in A} \) be a covering, and \( \{U'_\beta\}_{\beta \in B} \) be another covering.

**Definition.** We say that \( \{U'_\beta\} \) is a refinement of \( \{U_\alpha\} \) if for every \( \beta \in B \), there exists \( \alpha \in A \) such that \( U_\beta \subset U_\alpha \).

**Lemma.** There exists a map of complexes \( \check{C}(X, \{U_\alpha\}, \mathcal{F}^\bullet) \rightarrow \check{C}(C, \{U'_\beta\}, \mathcal{F}^\bullet) \) well-defined up to homotopy.

For every \( \beta \) choose an \( \alpha \) such that \( U'_\beta \subset U_\alpha \), thus producing a map \( \phi : B \rightarrow A \).

02/09/2010
Continuation of the proof of the previous lemma:

We can now produce the map of complexes: starting an object \( O \) in the Čech complex of \( \{ U_\alpha \} \), we can produce a complex \( O' \) in \( \{ U_\beta \} \) by letting \((O')_{\beta_1,\ldots,\beta_k} = \text{Res}_{\cap U_{\beta_1}}^{U_{\beta_k}} O_{\alpha_1,\ldots,\alpha_k}\).

Now we have to show that the map is unique up to homotopy, i.e. if \( \phi, \psi \) are two maps from \( B \) to \( A \) with the target containing the image, then they define the same map of Čech complexes up to homotopy.

The homotopy making the difference: a map in \( \text{Hom}^{-1}(\check{C}(X,\{U_\alpha\},\mathcal{F}^\bullet),\check{C}(C,\{U'_\alpha\},\mathcal{F}^\bullet)) \) is defined by
\[
(O')_{\beta_1,\ldots,\beta_k} = \text{Res}_{\cap U_{\beta_1}}^{U_{\beta_k}} O_{\beta_1,\ldots,\beta_k} - \text{Res}_{\cap U_{\beta_1}}^{U_{\beta_k}} O_{\beta_1,\ldots,\beta_k} + \ldots
\]
Verifying that this works is standard. \( \square \)

**Definition.** The Čech Cohomology of a sheaf or a complex of sheaves is defined by
\[
\hat{H}^i(X,\mathcal{F}^\bullet) = \lim\sup_{\longrightarrow} \check{H}^i(X,\{U_\alpha\},\mathcal{F}^\bullet)
\]
the limit being taken over the category of open covers with morphisms defined by refinements.

It can be easy to show that \( \hat{H}^0(X,\cdot) \) is cohomological - this is the old long exact sequence of Čech cohomology from last semester.

**Lemma:** \( \hat{H}^0(X,\cdot) \) factors through \( D^+(Sh(X)) \). In other words, it satisfies the following conditions (which are equivalent because the functor is cohomological):

i) Quasi-isomorphisms go to isomorphisms

ii) Acyclic complexes go to zero.

Proof: Assume that \( \mathcal{F}^\bullet = \mathcal{F}_0 \to \mathcal{F}_1 \to \ldots \to \mathcal{F}_n \) is bounded below and acyclic. We want to show \( \hat{H}^n(X,\mathcal{F}) = 0 \).

Consider a cocycle \( f \), represented by a cover \( (U_\alpha) \) and elements \( f^n \in \prod_\alpha \Gamma(U_\alpha,\mathcal{F}^n) \), \( f^{n-1} \in \prod_{\alpha_1,\alpha_2} \Gamma(U_{\alpha_1} \cap U_{\alpha_2},\mathcal{F}^{n-1}) \), \ldots, \( f^0 \in \prod_{\alpha_1,\ldots,\alpha_n} \Gamma(U_{\alpha_1} \cap \ldots \cap U_{\alpha_n},\mathcal{F}^0) \).

We will work with direct sums instead of direct products, so that \( f^i \) are represented by finitely many non-zero sections (this works if the cover is finite, for example). The theorem is true in general.

Then \( f^n \) dies under \( \mathcal{F}^n \to \mathcal{F}^{n+1} \). Then \( f^n \) comes from \( \mathcal{F}^{n-1} \) locally (on stalks) so that by refining the cover we can assume \( f^n \) comes from something \( g^{n-1} \) in \( \mathcal{F}^{n-1} \). By subtracting the coboundary \( \partial g^{n-1} \) from \( f \) we can assume that \( f^n = 0 \). We similarly dispose of \( f^{n-1} \) and so on, subtracting coboundaries until we get \( f = 0 \) so the original \( f \) was a coboundary. \( \square \)

**Theorem.** On \( D^+(Sh(X)) \),
\[
\hat{H}^0(X,\cdot) = H^0(\Gamma(X,\cdot))
\]
(the latter is known as \( H^0(X,\cdot) \)).

Unless \( X \) is nice enough, this is no longer true if we replace \( D^+ \) by the other "\( D^\ast \)-categories.

Proof: By the universal property it is possible to map \( H^0(X,\cdot) \) to \( \hat{H}^0(X,\cdot) \).

\[
\begin{array}{c}
K^+(SH) \\
\pi \\
\downarrow \Gamma \\
D^+(Sh) \longrightarrow \text{Ab}
\end{array}
\]

It is the same as a natural transformation \( \Gamma \to \hat{H}^0 \circ \pi \) i.e. a map \( H^0(\Gamma(X,\mathcal{F}^\bullet)) \to \hat{H}^0(X,\mathcal{F}^\bullet) \). The map is coming from mapping every global section \( \Gamma(X,\mathcal{F}^\bullet) \) to the first member in \( i\)-th column via restriction.

After constructing the map, we claim it is an isomorphism.

It is enough to prove it on injectives. Let \( \mathcal{F}^\bullet \in \mathcal{K}(Sh^{1\text{inj}}) \).
We need to show that the above map induces an isomorphism between complexes of abelian groups

\[ H^0(X, \mathcal{F}^\bullet) \to \tilde{H}^0(X, \mathcal{F}^\bullet) \]

Consider the following augmentation of the Čech bicomplex:

\[
\cdots \to \prod_{\alpha} \Gamma(U_\alpha, \mathcal{F}^0) \to \prod_{\alpha} \Gamma(U_\alpha, \mathcal{F}^1) \to \cdots \\
\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \cdots
\]

We claim that all columns are exact, and that this (somehow by a known homological algebra manipulation) implies that the map from the bottom row to the total complex of the Čech bicomplex is a quasi-isomorphism, for each cover.

For showing exactness of columns, we more generally show that if \( \mathcal{F} \) is an injective sheaf of \( R \)-modules (where \( R \) is a sheaf of rings), then the following complex of abelian groups is exact.

\[
0 \to \Gamma(X, \mathcal{F}) \to \prod_{\alpha} \Gamma(U_\alpha, \mathcal{F}) \to \prod_{\alpha_1, \alpha_2} \Gamma(U_{\alpha_1} \cap U_{\alpha_2}, \mathcal{F}) \to \cdots
\]

Recall that if \( U_j \hookrightarrow X \) is an open embedding, then \( j_{U_0} \) is right adjoint to \( j_\ast \), left adjoint to \((j_U)_0\) (extension by 0).

Take the complex

\[
\cdots \Rightarrow \bigoplus_{\alpha_1, \alpha_2} (j_{U_{\alpha_1}, \cap \alpha_2}) \ast \mathbb{Z} \to \bigoplus_{\alpha} (j_{U_\alpha}) \ast \mathbb{Z} \to \mathbb{Z}
\]

Call this complex \( S^\bullet \).

We claim

\[ \tilde{C}(X, \{U_\alpha\}, \mathcal{F}) = \text{Hom}(S^\bullet, \mathcal{F}) \]

as \( \text{Hom}((j_U)\ast \mathbb{Z}, \mathcal{F}) = \text{Hom}(\mathbb{Z}, (j_U)_0(\mathcal{F})) = \Gamma(U, \mathcal{F}) \).

Now we claim \( S^\bullet \) is acyclic - which will then imply the exactness of the Čech complex - because the sheaf is injective, mapping into it is exact.

To prove that \( S^\bullet \) is acyclic is the same as showing that for each \( \alpha \) \((j_{U_\alpha})^\ast \) is acyclic, which is reducing to the situation when one of the elements of the cover contains \( X \). In that case that can be explicitly shown.

Indeed, assume we have \( f \) such that for any \( i_1, \ldots, i_{k+1} \) we have

\[ f_{\alpha_1, \ldots, \alpha_k} - f_{\alpha_1, \ldots, \alpha_{i_1}} + \ldots + (-1)^{k+1} f_{\alpha_1, \ldots, \alpha_{i_{k+1}}} = 0 \]

regarded as a section of \( U_{\alpha_1 \cap \alpha_2 \cap \ldots \cap \alpha_{i_{k+1}}} \).

Now take \( U_{\alpha_0} = X \). We can define \( g \) by \( g_{\alpha_1, \ldots, \alpha_k} = f_{\alpha_0, \alpha_1, \ldots, \alpha_k} \) and it’s obvious that \( g = \partial f \). □

Let \( \mathcal{F}^\bullet \in K^+(\text{Sh}) \). There is a map \( \tilde{H}^i(X, \{U_\alpha\}, \mathcal{F}^\bullet) \to H^i(X, \mathcal{F}^\bullet) \). There is the following diagram, which is generally non-commutative:

\[
\begin{array}{ccc}
K^+(\text{Sh}) & \xrightarrow{\tilde{C}(X, \{U_\alpha\}, -)} & K^+(\text{Ab}) \\
\pi \downarrow & & \downarrow H^i \\
D^+(\text{Sh}) & \xrightarrow{H^i(X, -)} & \text{Ab}
\end{array}
\]

However, the following is true:
Proposition. Let $\mathcal{F} \in K^b(Sh)$. Assume $\mathcal{F}^i$ are such that $H^j(U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}, \mathcal{F}^i) = 0$, $\forall \alpha_0, \ldots, \alpha_k, k \geq 0, j > 0$. Then the map

$$\tilde{H}^i(X, \{U_{\alpha}\}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$$

is an isomorphism.

Proof: First assume $\mathcal{F}$ is a single sheaf. Choose an injective resolution $\mathcal{F} \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots$ of $\mathcal{F}$. Consider the bicomplex

$$\prod_{\alpha_0, \alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{F}) \rightarrow \prod_{\alpha_0, \alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, I^0) \rightarrow \prod_{\alpha_0, \alpha_1} \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, I^1)$$

The cohomologies of the columns (except first column) are given by $\tilde{H}^k(X, \{U_{\alpha}\}, I^j)$, which we showed equals 0 earlier for $k > 0$, since $I^j$ is injective. The cohomologies of the rows are just $\prod_{\alpha_0, \ldots, \alpha_k} H^j(U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}, \mathcal{F}) = 0$ by assumption. Therefore using the total complex again, we see that the first column is quasi-isomorphic to the $-1$ row (not shown). This exactly says $\tilde{H}^i(X, \{U_{\alpha}\}, \mathcal{F}) \simeq H^i(X, \mathcal{F})$.

For the general case of $\mathcal{F} \in K^b(Sh)$, we use the fact that $\tilde{H}^i(X, \{U_{\alpha}\}, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ is a natural transformation of cohomological functors. Now we consider the “stupid truncation”: for any $i \in \mathbb{Z}$, we consider the complex $\sigma^{\geq i}(\mathcal{F}^\bullet)$ defined as $\mathcal{F}^i \rightarrow \mathcal{F}^{i+1} \rightarrow \ldots$. There is an obvious map $\sigma^{\geq i}(\mathcal{F}^\bullet) \rightarrow \sigma^{\geq i-1}(\mathcal{F}^\bullet)$ which gives distinguished triangle

$$\sigma^{\geq i}(\mathcal{F}^\bullet) \rightarrow \sigma^{\geq i-1}(\mathcal{F}^\bullet) \rightarrow \mathcal{F}^{i-1}[i+1] \rightarrow [1]$$

Now the 5-lemma implies that if we know the result holds for the two ends, it holds for the middle. By induction we get the claim. □

02/11/2010

Remark: we can define $R\Gamma$ in two different ways

Let $X, Y$ be locally ringed spaces and $\Phi: X \rightarrow Y$ a map of topological spaces. We have the functor $\Phi_*: Sh(X) \rightarrow Sh(Y)$ and the right derived functor $R\Phi_*: D^+(Sh(X)) \rightarrow D(Sh(Y))$.

Underlying $\Gamma_Y \circ \Phi_* \cong \Gamma_X$, we have:

**Theorem (Leray)** The following diagram is commutative:

$$D^+(Sh(X)) \longrightarrow D^+(Sh(Y))$$

More generally, if $\Psi$ is the map from $X$ to $Spec(\mathbb{Z})$ then $\Gamma$ is just $\Psi_*$. Leray’s theorem has the following generalization:

if $X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$, then

$$R((\Psi \circ \Phi)_*) \cong R\Psi_* \circ R\Phi_*$$

Leray’s theorem is part d) of the following exercise:

**Proposition.** Let $\mathcal{A}$ be an abelian category with enough injectives. Let $F: \mathcal{A} \rightarrow Ab$ be a left-exact functor, and let $RF: D^+(\mathcal{A}) \rightarrow D^+(Ab)$ be its derived functor. We say that an object $X \in \mathcal{A}$ is $f$-acyclic if $R^iF(X) := RF(X[i]) = 0$ for $i \neq 0$.

a) Let $X^\bullet \in K^+(\mathcal{A})$ be a complex consisting of $F$-acyclic objects. Then the natural morphisms $F(X^\bullet) \rightarrow RF(X^\bullet)$ is an isomorphism.
b) Let $\mathcal{A}$ be $Sh(X)$. A sheaf $\mathcal{F}$ is called "flabby" or "flasque" if for any pair of open subsets $U \subset V$, the map $\Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective.

b) Injective sheaves are flasque.

bii) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves with $\mathcal{F}_1$ flasque, then the map $\Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3)$ is surjective.

biii) If in a short exact sequence as above $\mathcal{F}_1$ and $\mathcal{F}_2$ are flasque, then so is $\mathcal{F}_3$.

biv) Flasque sheaves are acyclic for the functor $H^0(X, -)$ and more generally for any direct image functor $\Phi_*$, where $\Phi : X \to Y$ is a map of topological spaces.

c) Let $\Phi : X \to Y$ be a morphism between topological spaces. Then $\Phi_*$ sends flasque sheaves on $X$ to flasque sheaves on $Y$.

d) $R\Gamma_Y \circ R\Phi_* \cong R\Gamma_Y$

Proof: a) Say the complex is $X_1 \to X_2 \to \ldots$ (all lower terms are zero). Now let $X_1 \to I_{i,1} \to I_{i,2} \to \ldots$ be injective resolutions for $X_i$.

We consider the bicomplex

$$
\begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
I_{2,1} & I_{2,2} & \cdots & \\
I_{1,1} & I_{1,2} & \cdots & \\
X_1 & X_2 & \cdots & \\
\end{array}
$$

Its rows are exact therefore the first row is homotopy equivalent to the total complex, which gives an injective resolution of $X_1 \to X_2 \to \ldots$.

Now apply $F$ to the bicomplex - except the first row is deleted. Rows are still exact (except at the first term) by assumption, and the same assumption show that the first homology of the rows are isomorphic to the homologies of the first complex - but just gives our desired isomorphism.

b) Take $i$ to be the restriction to $V$ and $j$ the restriction to $U$. Consider the sheaf $i!*\mathbb{Z}_V$. Note that $i!*\mathbb{Z}_V$ is the left adjoint to the inverse image $i^*$ functor $V(\mathbb{Z}_V)$ is the constant sheaf on $V$.

Since $i!$ is left adjoint we conclude $\text{Hom}(i!*\mathbb{Z}_V, \mathcal{F}) = \text{Hom}(\mathbb{Z}_V, i^*\mathcal{F}) = \Gamma(i^*\mathcal{F}, V) = \Gamma(\mathcal{F}, V)$. Similarly we get $\Gamma(\mathcal{F}, U) = \text{Hom}(j!*\mathbb{Z}_U, \mathcal{F})$. It remains to show that $j!*\mathbb{Z}_U$ injects into $i!*\mathbb{Z}_V$ as then the injectivity of $\mathcal{F}$ will do. But the first is actually a subsheaf of the other since $\mathbb{Z}_U$ is naturally a subsheaf of $\mathbb{Z}_V$ (and extension by zero is left exact as proven in the past semester).

More concretely, recall that sections of $j!*\mathbb{Z}_U$ over $W$ are given by pairs of $W_1 \subset W \cap U, W_2 \subset W, W_1 \cup W_2 = W$ and $s \in \Gamma(\mathbb{Z}_U, W)$ such that $s|_{W_1 \cap W_2} = 0$. Of course such a section immediately gives a section in $i!*\mathbb{Z}_V$ by the same procedure since $W_1 \subset W \cap V$, and the latter is zero if and only if $s$ is if and only if the former is zero.

bii) Say $f$ is the map $\mathcal{F}_2 \to \mathcal{F}_3$. Let $u \in \Gamma(X, \mathcal{F}_3)$. Then there is a cover $(U_\alpha)_{\alpha \in I}$ and sections $v_\alpha \in \Gamma(U_\alpha, \mathcal{F}_2)$ such that $f(v_\alpha) = u|_{U_\alpha}$.

Consider the partially ordered set consisting of subsets $J \subset I$ and sections $v_J \in \Gamma(U, \mathcal{F}_2)$ where $U = \bigcup J U_\alpha$ such that $f(v_J) = u|_U$ (the ordering is given by inclusion and restriction). It is clear from the sheaf axiom that it satisfies the ascending chain principle therefore there is a maximal such subset by Zorn’s lemma. We claim it is everything. Indeed, say $\alpha \notin J$. Then $v_J|_{U \cap U_\alpha} = u|_{U \cap U_\alpha}$ must project to 0 therefore it must be in $\Gamma(U \cap U_\alpha, \mathcal{F}_1)$. It will therefore be the restriction of a section $s_\alpha$ in $\Gamma(U_\alpha, \mathcal{F}_1)$ since $\mathcal{F}_3$ is flasque. Subsequently $v_\alpha - s_\alpha$ coincides with $u$ on $U \cap U_\alpha$, and therefore by the gluing axiom $U$ and $v_\alpha - s_\alpha$ define a section on $U \cup U_\alpha$ contradicting maximality.

The same method works for any open subset instead of just $X$.

biii) Take $U \subset V \subset X$ be open. Take the following diagram:
\[ \Gamma(\mathcal{F}_2, V) \longrightarrow \Gamma(\mathcal{F}_3, V) \]

\[ \Gamma(\mathcal{F}_2, U) \longrightarrow \Gamma(\mathcal{F}_3, U) \]

The left vertical map is surjective because \( \mathcal{F}_2 \) is flasque, and the two horizontal maps are surjective from the previous problem. So the composite map \( \Gamma(\mathcal{F}_2, V) \to \Gamma(\mathcal{F}_3, U) \) is surjective. Since it factors through \( \Gamma(\mathcal{F}_3, V) \), we conclude that \( \Gamma(\mathcal{F}_3, V) \to \Gamma(\mathcal{F}_3, U) \) is surjective.

Again, it works the same not just for global sections, but for sections over all opens.

biv) The part bii) tells us that \( \Gamma \) is exact on short exact sequences of flasque sheaves. Now that \( \mathcal{F} \) a flasque sheaf, and \( \mathcal{F} \to I_1 \to I_2 \to \ldots \) be an injective resolution.

According to the previous part, we have the following short exact sequences of flasques: \( 0 \to \mathcal{F} \to I_1 \to I_1' \to 0, 0 \to I_1' \to I_2 \to I_2' \to 0, \ldots \) where \( I_k' = I_k/\text{Im}(I_{k-1}) \). So \( \Gamma \) turns them into short exact sequences as well, and patching up we deduce that

\[ 0 \to \Gamma(X, F) \to \Gamma(X, I_1) \to \ldots \]

is exact. In particular all but the zero-th cohomology of

\[ \Gamma(X, I_1) \to \Gamma(X, I_2) \to \ldots \]

are zero, which shows the claim.

Being acyclic for \( \Phi_* \) is similar, once we prove that \( \Phi_* \) also turns short exact sequences of flasques into flasques.

Indeed, \( \Phi_* \) is left exact so we only need to show that if \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) is a short exact sequence of flasques, then \( \Phi_* \mathcal{F}_2 \to \Phi_* \mathcal{F}_3 \) is surjective, i.e. surjective on stalks.

Take a stalk in \((\Phi_*, \mathcal{F}_3)_x\) represented by \( \Gamma(\Phi_*, \mathcal{F}_3, U) = \Gamma(\mathcal{F}_3, \Phi^{-1}(U)) \). But because of part biii), it is in the image of some section of \( \Gamma(\mathcal{F}_3, \Phi^{-1}(U)) = \Gamma(\Phi_*, \mathcal{F}_3, U) \), and this proves surjectivity.

c) Let \( U \subset V \subset Y \) are opens, then we have the following diagram:

\[ \Gamma(\mathcal{F}, \Phi^{-1}(V)) \longrightarrow \Gamma(\mathcal{F}, \Phi^{-1}(U)) \]

\[ \Gamma(\Phi_* \mathcal{F}, V) \longrightarrow \Gamma(\Phi_* \mathcal{F}, U) \]

If \( \mathcal{F} \) is flasque then the upper horizontal map is surjective, hence so is the lower horizontal map.

d) We can compute the right derived functor of \( \Gamma \) using injective resolutions. In fact, we claim it can be computed using flasque resolutions as well. This follows from a) and biv), because if \( 0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots \) is a flasque resolution then as \( \mathcal{F}_1 \) are \( \Gamma \)-acyclic we can apply a) to \( \mathcal{F}_1 \to \mathcal{F}_2 \to \ldots \) to deduce that the homologies of \( \Gamma \mathcal{F}_1 \to \Gamma \mathcal{F}_2 \to \ldots \) are isomorphic to \( R\Gamma(\mathcal{F}_1 \to \mathcal{F}_2 \to \ldots) \cong R\Gamma(\Gamma) \), as desired. The same method works for complexes instead of single objects.

Now let’s come back to the statement. It can be done for complexes in the same way, but for simplicity let’s assume \( \mathcal{F} \) is a sheaf.

Consider an injective (and flasque) resolution \( \mathcal{F} \to I_1 \to I_2 \to \ldots \). According to the previous parts, \( \Phi_* \mathcal{F} \to \Phi_* I_1 \to \Phi_* I_2 \to \ldots \) is a flasque resolution of \( \Phi_* \mathcal{F} \), hence \( R\Gamma_Y(R\Phi_* \mathcal{F}) \) is computed to be the cohomology of the sequence \( \Gamma(\Phi_* I_1, Y) \to \Gamma(\Phi_* I_2, Y) \to \ldots \). But that sequence is the same as \( \Gamma(I_1, X) \to \Gamma(I_2, X) \to \ldots \) so it’s cohomology is the same as \( \Gamma(\mathcal{F}, X) \).

**Proposition.** Let \( X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z \) be maps of topological spaces. There is a natural isomorphism

\[ R(\Psi \circ \Phi)_* \rightarrow R\Psi_* \circ R\Phi_* \]
Proof: we use $\text{R}(\pi \circ f)$ instead of the old definition of $Rf$, to make the functor from $D^+$ to $D^+$. For this proof, alter the notation in order to compute the explicit maps:

$$\text{Hom}(\text{R}(\pi_Z \circ \Psi_* \circ \Phi_*), \text{R}(\pi_Z \Psi_* R(\pi_Y \Phi_*)))$$

is the same as

$$\text{Hom}(\pi_Z \circ \Psi_* \circ \Phi_* \circ \pi_Y, R(\pi_Y \Phi_*) \circ \pi_X)$$

Now observe that we have the adjunction map

$$\pi_Z \Psi_* \rightarrow R(\pi_Z \Psi_*) \circ \pi_Y$$

and therefore we produce the map

$$\pi_Z \circ \Psi_* \circ \Phi_* \rightarrow R(\pi_Z \Psi_*) \circ \pi_Y \circ \Phi_*$$

Similarly, we have the adjunction map

$$\pi_Y \Phi_* \rightarrow R(\pi_Y \Phi_*) \circ \pi_X$$

and hence we get the map

$$R(\pi_Z \Psi_*) \circ \pi_Y \circ \Phi_* \rightarrow R(\pi_Z \Psi_*) \circ R(\pi_Y \Phi_*) \circ \pi_X$$

The composite map

$$\pi_Z \circ \Psi_* \circ \Phi_* \rightarrow R(\pi_Z \Psi_*) \circ R(\pi_Y \Phi_*) \circ \pi_X$$

is what we seek.

Now to show that is provides an isomorphism, we will work with the explicit objects.

Recall how the adjunction property of the right derived functors is explicitly written out in this context.

Say $f: A \rightarrow B$ is a functor, and we call $f$ by abuse of notation the induced map $K^+(A) \rightarrow K^+(B)$. We then have $Rf = R(\pi_B \circ f): D^+(A) \rightarrow D^+(B)$, and we have a natural transformation

$$\text{Hom}(Rf, G) \cong \text{Hom}(\pi_B \circ f, G \circ \pi_A)$$

Here is how it is computed (*):

Assume we have an object $X^* \in D(A)$, represented by an injective resolution $I^*$. We then map $Rf(X^*)$ i.e. $f(I^*)$ to $G(I^*)$.

We need to map $\pi_B(f(X^*))$ to $G(\pi_A(X))$. But $f(X^*)$ maps to $f(I^*)$ which in turn maps to $G(I^*) = G(\pi_A(I^*))$ and this is equal to $G(\pi_A X^*)$.

Conversely, assuming we have a map from $\pi_B(f(X^*))$ to $G(\pi_A(X))$ we proceed backwards, by sending $Rf(X^*) = f(I^*)$ to $G(I^*)$ which is the same as $G(I^*)$ as $I^*$ and $X^*$ are the same in $D^+(A)$ - he domain of $G$.

Now take an injective resolution $I^*$ of $\mathcal{F}^*$. Then $R(\pi_Z \circ \Psi_* \circ \Phi_*) = R(\pi_Z \circ (\Psi \circ \Phi)_*)$ which is equivalent to $(\Psi \circ \Phi)_*(I^*)$.

Similarly, $R(\pi_Z \Psi_*) R(\pi_Y \Phi_*)$ can be computed by first taking an injective resolution $I^*$ of $X$ to produce $\Phi_*(I^*)$, to which $R(\pi_Z \circ \Psi_*)$ is applied. We know $\Phi_*(I^*)$ is not injective, but it consists of flasque objects, and we know the right derived functors of $\Psi_*$ can be computed using flasque resolutions. Hence we get to the same $\Psi_* \circ \Phi_*(I^*)$, so for every objects the cohomologies are isomorphic.

It remains to see that the map above induces this isomorphism. This is a tautological (but long) application of the principle (*). □

There is another version of Cech cohomology (for example the one used in Hartshorne):

If we introduce a partial ordering on the elements $\alpha$ we can define the $n$-th term of the Cech complex by

$$\prod_{\alpha_1 < \alpha_2 < \ldots < \alpha_n} \Gamma(\cap U_{\alpha_i}, \mathcal{F})$$
We call this Čech complex $\tilde{C}_{ord}$.

There is a natural embedding $\tilde{C}_{ord}(X, \{U_\alpha\}, \mathcal{F}) \hookrightarrow \tilde{C}(X, \{U_\alpha\}, \mathcal{F})$ obtained by $f \to g$ where $(g)_{\alpha_1, \ldots, \alpha_n} = sgn(\pi) f_{\pi(\alpha_1), \ldots, \pi(\alpha_n)}$ where $\pi$ is the permutation that orders the elements in increasing order (if two elements are equal the map on that component is 0). There is also a quasi-inverse $\tilde{C}(X, \{U_\alpha\}, \mathcal{F}) \rightarrow \tilde{C}_{ord}(X, \{U_\alpha\}, \mathcal{F})$ obtained by restriction.

It can be proved that the two versions of Čech cohomology are the same. One way is using delta functors - see my project for the fall term. Another way, for nice spaces, is to show that both equal the "normal" cohomology, i.e. the right derived functor of $\Gamma$ - that happens when the space is sufficiently nice (for example see the notes on the website), because essentially the same reasoning works for the alternative Čech cohomology. It may be also possible to exhibit an explicit homotopy to show that the two above maps are quasi-inverse to each other.

In what follows, we will always assume that the Čech cohomology equals the right derived functor of $\Gamma$.

**Definition.** $X$ is said to have (covering) dimension $n$ if any cover can be refined such that no more than $n+1$ of the subsets intersect.

For example $\mathbb{R}^n$ has covering dimension $n$.

**Proposition.** If $\mathcal{F}$ is a sheaf and $X$ has covering dimension $n$, then $H^i(X, \mathcal{F}) = 0$ for $i > n$.

Proof: the Čech complex terminates after $n$ steps.

Recall that a topological space is Noetherian if and only if every descending chain of closed subsets terminates.

**Definition.** A Noetherian topological space $X$ has dimension $\leq n$ if any chain of irreducible closed subsets has length $n+1$.

For example $\mathbb{A}^n$ has dimension $n$, as we know from commutative algebra.

**Lemma.** A Noetherian space of dimension $\leq n$ is of covering dimension $\leq n$.

**Corollary.** $H^i(X, \mathcal{F}) = 0, \forall i > n$ for every $\mathcal{F} \in Sh(X)$ if $X$ is Noetherian of dimension $\leq n$.

Let $X$ be an affine scheme, $\mathcal{F} \in QCoh(X)$

**Theorem.** $H^i(X, \mathcal{F}) = 0, \forall i > 0$ if $X$ is affine

Proof: it is enough to prove that all higher Čech cohomologies vanish (because then they always equal the usual cohomologies). That follows from Serre’s lemma - or alternatively because for an affine scheme $\Gamma$ is exact.

Recall that given a partially ordered set $I$, a subset $J \subset I$ is called cofinal if for every $i \in I$ there exists $j \in J$ so that $j > i$. If $J$ is cofinal, it is easy to prove that canonically

$$\lim_{j \in J} A_j \cong \lim_{i \in I} A_i$$

For example if $X$ is compact the finite covers are cofinal.

**Corollary.** Let $X$ be a separated scheme and $U_\alpha$ be any affine cover. Then if $\mathcal{F}^\bullet$ is a complex of quasi-coherent sheaves, then the natural map

$$\tilde{C}(X, \{U_\alpha\}, \mathcal{F}^\bullet) \rightarrow H^i(X, \mathcal{F}^\bullet)$$

is an isomorphism.

Proof: by Serre’s theorem, the left hand side can compute cohomology - as the higher cohomologies are 0 since intersections of $U_\alpha$ are affine. □

$$QCoh(X) \hookrightarrow Sh_{\mathcal{O}_X-mod}(X)$$. It was proven last semester that it’s a fully faithful embedding closed under extensions.
Also if \( \mathcal{F}_1, \mathcal{F}_2 \in QCoh(X) \),
\[
\text{Ext}^i QCoh(\mathcal{F}_1, \mathcal{F}_2) \xrightarrow{\sim} \text{Ext}^i_{Sh(X)}(\mathcal{F}_1, \mathcal{F}_2)
\]

From now on, assume that \( X \) is separated and quasi-compact.

**Claim:** \( QCoh(X) \) has enough injectives.

Proof: if \( U \) is an affine then \( QCoh(U) \) has enough injectives. This is equivalent to the category of \( R = \Gamma(U, \mathcal{O}_U) \)-modules having enough injectives. Indeed, \( Z \) does have enough injectives (it can be easily shown that \( \mathbb{R}/\mathbb{Z} \) is injective and then we embed any module into a product of copies of \( \mathbb{R}/\mathbb{Z} \) indexed by elements of \( M \) in order to make every element not map to zero), and if \( Q \) is \( Z \)-injective then \( \text{Hom}_Z(R, Q) \) is injective since \( \text{Hom}_R(M, \text{Hom}_Z(M, Q)) \cong \text{Hom}_Z(M, Q) \) and being injective is equivalent to \( \text{Hom}(M, -) \) being exact. If \( M \) injects into \( Q \) as abelian groups (\( Z \)-modules) then \( M \) injects into \( \text{Hom}_Z(R, Q) \) as \( R \)-modules. □

Therefore, \( \Gamma \) over the category \( D^+(QCoh(X)) \) has a right derived functor that can be computed using injective resolutions of quasi-coherent sheaves. On the other hand, we can consider it as a functor on sheaves over the ringed space \( \mathcal{O}_X \) and then we can take its right derived functor with respect to this category. It turns out these two versions are the same.

**02/16/2010**

Let \( \mathcal{A} \) be an abelian category, and let \( D(\mathcal{A}) \) be its derived category. For each \( n \in \mathbb{Z} \), consider the full subcategory \( D^{\leq n} \) consisting of all objects \( X^\bullet \) such that \( H^i(X^\bullet) \) for \( i > n \) and the full subcategory \( D^{\geq n} \) consisting of objects \( X^\bullet \) with \( H^i(X^\bullet) = 0, \forall i < n \). Note that \( D^{-}(\mathcal{A}) = \cup D^{\leq n} \) and \( D^{+}(\mathcal{A}) = \cup D^{\geq n} \).

**Proposition.** For any \( n \in \mathbb{Z} \), \( D(\mathcal{A})^{\leq n} \subset D(\mathcal{A}) \subset D(\mathcal{A})^{\geq n+1} \) is an admissible triple.

Proof: First we show that \( D(\mathcal{A})^{\leq n} \) and \( D(\mathcal{A})^{\geq n+1} \) are "orthogonal". Say \( X \) is represented by a complex \( X^\bullet \). Consider the complex \( X^\bullet \) defined by \( X^n = X^n \) for \( i < n \), \( X^n = \text{Ker}(X^n \rightarrow X^{n+1}) \) and \( X^n = 0 \) for \( i > n \). This complex maps to \( X^\bullet \), and the map readily induces an isomorphism on cohomology, so \( X^\bullet \) represents the same object as \( X \). Similarly for \( Y \) we can create a map \( Y \rightarrow Y_1 \) that is an isomorphism and such that \( Y_1 = 0 \) in degrees \( \leq n \).

Now a h.t \( X \rightarrow X' \rightarrow Y \) is equivalent to a h.t \( X \rightarrow X'_1 \rightarrow Y \). Because \( X_1 \rightarrow X'_1 \) and \( Y \rightarrow Y_1 \) are quasi-isomorphisms, it suffices to show that \( X'_1 \rightarrow Y_1 \) is 0 in \( D(\mathcal{A}) \). However it is zero already in \( K(\mathcal{A}) \), since for every index, one of the corresponding components of the two complexes is 0.

Next, we have to show that every \( X \) can be placed into a distinguished triangle \( X^{\leq n} \rightarrow X \rightarrow X^{\geq n+1} \rightarrow X^{\leq n}[1] \) and that it is functorial - that is every morphism \( X \rightarrow Y \) can be completed to a unique morphism of distinguished triangles

\[
\begin{array}{cccc}
X^{\leq n} & \rightarrow & X & \rightarrow & X^{\geq n+1} & \rightarrow & X^{\geq n}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Y^{\leq n} & \rightarrow & Y & \rightarrow & Y^{\geq n+1} & \rightarrow & Y^{\geq n}[1]
\end{array}
\]

Assuming these distinguished triangles exist, we show the existence and uniqueness of the extension. According to the lemma in lecture 1, there exists a map \( X^{\leq n} \rightarrow Y^{\leq n} \) that makes the corresponding square commute of and only if the composition \( X^{\leq n} \rightarrow X \rightarrow Y \rightarrow Y^{\geq n+1} \) is 0, which is the case from what was proven before. Moreover, the map is unique up to something in \( \text{Hom}_{D(\mathcal{A})}(X^{\leq n}, Y^{\geq n+1}[-1]) \) which is again 0. By axiom Tr3, or by applying the same reasoning to the other pair, we construct the (unique) map of triangles.

Particularly, if we have two decompositions \( T_1 : X^{\leq n} \rightarrow X \rightarrow X^{\geq n+1} \rightarrow X^{\neq n}[1] \) and \( T_2 : X^{\leq n} \rightarrow X \rightarrow X^{\geq n+1} \rightarrow X^{\neq n}[1] \), then by the above argument we have maps \( T_1 \rightarrow T_2 \) and \( T_2 \rightarrow T_1 \) whose composites \( T_1 \rightarrow T_1 \) and \( T_2 \rightarrow T_2 \) are equal to the identity by the uniqueness clause, so the two decompositions are isomorphic.
Now we show the existence of such a triangle. Recall the complex \( X_{\leq n} \) defined by \((X_{\leq n})^i = X^i \) for \( i < n \), \((X_{\leq n})^n = \ker(X^n \to X^{n+1})\) and the quotient complex \( X_{\geq n+1} \) defined by \((X_{\geq n+1})^i = X^i \) for \( i > n \), \((X_{\geq n+1})^n = X^n/\ker(X^n \to X^{n+1})\) and \((X_{\geq n+1})^i = 0 \) for \( i < n \). The two complexes satisfy the condition of the problem. □

The two functors \( \tau_{\leq n} : D(A) \to D_{\leq n}(A) \), \( X \to X_{\leq n} \) and \( \tau_{\geq n} : D(A) \to D_{\geq n}(A) \), \( X \to X_{\geq n} \) obtained in this way are the left respectively right adjoint to the embeddings \( D_{\leq n}(A) \to D(A) \) and \( D_{\geq n}(A) \). They are called the truncation functors.

The next proposition is a useful tool when reducing theorems from the general case to the case of bounded complexes.

Let \( D \) be a triangulated category that contains countable direct sums. Let \( X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \ldots \) be an inductive system of objects. We define a new object of \( D \) called a homotopy direct limit of \( X_i \), denoted \( \text{holim} X_i \) to be

\[
\text{Cone}(\oplus X_i \xrightarrow{f} \oplus X_i)
\]

where the map \( f \) sends \( X_i \) to \( X_i \oplus X_{i+1} \) by means of \((id, -f_i)\).

**Proposition.** Assume that \( D = D(A) \) for an abelian category \( A \) which has countable direct sums and in which the functor of direct limit over \( \mathbb{N} \) is exact (e.g. \( Ab, R - \text{mod}, QCoh(X), Sh(X) \))

a) There are natural isomorphisms

\[
H^n(\text{holim} X_i) \cong \lim_{\leftarrow} H^n(X_i)
\]

b) Assume that the maps \( X_i \to X_{i+1} \) come from actual maps of complexes \( X_i^\bullet \to X_{i+1}^\bullet \) and let \( \lim X_i^\bullet \) be their inductive limit. Then there is an isomorphism in \( D(A) \) between \( \lim X_i^\bullet \) and \( \text{holim} X_i \).

c) In general, \( \text{holim} X_i \) is not isomorphic to the categorical direct limit of the \( X_i \)'s taken in \( D(A) \)

d) For an object \( X \in D(A) \) set \( X_i = \tau_{\leq i}(X) \). Then \( X \) is isomorphic to \( \text{holim} X_i \).

Proof: a) We may assume that the maps come from actual maps of complexes: if they are huts \( X_i \to X_{i+1} \xrightarrow{f} X_{i+1} \) then replace \( X_{i+1} \) by \( X_{i+1}' \) and continue by induction. This will not change the cohomologies.

We claim that \( \oplus X_i \xrightarrow{f} \oplus X_i \) is an injective map with quotient \( \text{lim} X_i \), which will provide the required assertion as the cohomologies of the cone will be the same as the cohomologies of the quotient.

First we prove injectivity - component wise. We will use the Mitchell embedding theorem (although we could do the same thing with Yoneda). So assume that \( X_i \) are modules and that \( f(a_0, a_1, \ldots, a_k, 0, \ldots, 0) = 0 \). This yields \((a_0, a_1 - f_0(a_0), a_2 - f_1(a_2), \ldots) = (0, 0, \ldots)\) so \( a_0 = 0, a_1 = f(a_0) = 0 \) and so on, proving injectivity.

We also have that map \( \oplus X_i \to \lim X_i \) by sending \( X_i \) to \( \lim X_i \), and let's prove that this makes \( 0 \to \oplus X_i \xrightarrow{f} \oplus X_i \to \lim X_i \to 0 \) into a short exact sequence. We have already proven that the first map is injective. The surjectivity of the last map can be done by the Mitchell embedding theorem again, but here's how to do it with Yoneda: if \( g \in \text{Hom}(\lim X_i, M) \) such that the composite map in \( \text{Hom}(X_i, M) \) is 0 then the (compatible) maps \( X_i \to M \) that determine \( g \) must be 0 (as they also determine the composite map). Also, the composite of the two maps is 0, because \( X_i \to X_i \to X_i \oplus X_{i+1} \to \lim X_i \) is clearly 0 as the maps \( X_i \to \lim X_i \) and \( X_i \xrightarrow{f} X_{i+1} \to \lim X_i \) are equal.

Finally, we must show that the kernel of the second map is contained in the image of the first. Again, we assume that \( X_i \) are modules, and say \( \sum a_i \) projects to 0, where \( a_i \in X_i \). Because the system if filtered we deduce that \( \sum a_i \) projects to 0 in some \( X_k \) i.e. \( \sum a_i f_k = 0 \). But then \( \sum a_i f_{i+1} = \sum_{i=1}^n a_i f_{i+1} \) and in general, \( x - f_{i+1} f_{i+2} \ldots f_{i+1} f_{i+2} \ldots \) is in the image of \( f \), as it can be written as \( x - f_{i+1} f_{i+2} \ldots f_{i+1} f_{i+2} \ldots \).

b) This follows immediately from what we did in a), combined with the fact that the cone of an injection is quasi-isomorphic to the quotient.

c) Let \( X_i = k[t] \) with maps given by multiplication by \( t \). As shown before, \( \text{holim} X_i \) consists of the limit \( \lim k[t] \).

That latter limit equals \( \oplus k[t] \epsilon_i \) modulo the submodule generated by \( \epsilon_i - t\epsilon_{i+1} \) - and by sending \( \epsilon^i \) to \( t^{-i} \) we see it's
the same as \(k[t,t^{-1}]\). If, on the other hand, it were the categorical inductive limit of the \(X_i\) in \(\mathcal{D}(A)\), we would have 
\[
\text{Hom}_D(k[t,t^{-1}], k[t,y]) = \lim \text{Hom}_D(k[t], k[t,y]) \quad \text{or by problem 1a), } \text{Ext}^1(k[t,t^{-1}], k[t,y]) = \lim \text{Ext}^1(k[t], k[t,y]) = 0 
\]
(We regard \(k[t,y]\) here as a \(k[t]\)-module). Indeed, if \(0 \to k[t,y] \to A \to k[t] \to 0\) is a short exact sequence, then \(A = k[t,y] \oplus e k[t]\) where \(e\) is any preimage of 1 in \(A\), so the sequence splits.

However \(\text{Ext}^1(k[t,t^{-1}], k[t,y]) \neq 0\). Here is an example of a non-split extension.

Consider the injective map \(k[t,y] \to k[t,y]\) given by multiplication by \(ty - 1\). It’s quotient is \(k[t,t^{-1}]\) vie identifying \(y\) with \(t^{-1}\). If, on the other hand, it had a splitting map \(g: k[t,t^{-1}] \to k[t,y]\) then \(g(t^{-1})\) would have to be \(ty^i + p_i(y)(ty - 1)\) and since \(t \cdot t^{-1} = t^{1-i}\) we would have \(ty^i + tp_i(t,y)(ty - 1) = y^{i-1} + p_{i-1}(t)(ty - 1)\) i.e. \(tp_i + y^{i-1} = p_{i-1}\). By iterating, we get \(t^k p_i + y^{i-1}k^{-1} + y^{i-2}k^{-2} + \ldots + y^{i-k} = p_{i-k}\). In particular, if \(i = k\) we get that \(p_0\) must be congruent to \(y^{k-1}k^{-1} + \ldots + 1\) modulo \(t^k\). But of course, there is no such polynomial for arbitrarily big \(k\).

d) Follows directly from b), as both are equal to the limits of the complexes \(\tau^{\leq i}(X)\) (which stabilize in every degree).

As shown in the online notes, the map \(\hat{H}^i(X, \mathcal{F}) \to \hat{H}^i(X, \mathcal{F}^\bullet)\) is an isomorphism if \(X\) is paracompact and Hausdorff (which is pretty useless since schemes are not Hausdorff).

**Lemma.** If \(\mathcal{F}\) is such that there is a basis of the topology got which \(\hat{H}^i(V, \mathcal{F}) = 0\) for all \(V\) in the basis then 
\[
\hat{H}^i(U, \mathcal{F}) = 0
\]
for all open sets \(U\).

More generally:

**Theorem.** If \(\Phi: X \to Y\) is an affine map of schemes, \(\mathcal{F}\) a quasi-coherent sheaf over \(X\), then \(\mathcal{R}^i\Phi_*(\mathcal{F}) = 0\).

**Proof:**

**Lemma:** if \(\Phi: X \to Y\) is a map of topological spaces then \(\mathcal{R}^i\Phi_*(\mathcal{F}^\bullet)\) is isomorphic to the sheaf associated to the presheaf \(U \to \hat{H}^i(\Phi^{-1}(U), \mathcal{F}^\bullet)\)

**Proof of lemma:** Let \(I^\bullet\) be an injective resolution for \(\mathcal{F}^\bullet\). Then \(\mathcal{R}^i\Phi_*(\mathcal{F}) = \ker(\Phi_*(I_i) \to \Phi_*(I_{i+1}))/\text{Im}(\Phi_*(I_{i-1}) \to \Phi_*(I_i))\). As we know, the cokernel of a map is the sheaf associated to the sheaf whose sections are the cokernels of the corresponding sections, which is computed to be \(U \to \ker(\Gamma(\Phi^{-1}(U), I_i) \to \Gamma(\Phi^{-1}(U), I_{i+1}))/\text{Im}(\Gamma(\Phi^{-1}(U), I_{i-1}) \to \Gamma(\Phi^{-1}(U), I_i))\).

However we know that \((I_i)_U\) (meaning restricted to \(U\)) are injective on \(U\) (or flasque, for that matter, as right derived functors can be computed from flasque resolutions, according to a previous proposition), so they provide an injective resolution for \(\mathcal{F}^\bullet_U\). Then, \(\ker(\Gamma(\Phi^{-1}(U), I_i) \to \Gamma(\Phi^{-1}(U), I_{i+1}))/\text{Im}(\Gamma(\Phi^{-1}(U), I_{i-1}) \to \Gamma(\Phi^{-1}(U), I_i))\) will be \(\hat{H}^i(\Phi^{-1}(U), \mathcal{F}^\bullet)\) which finishes the proof of the lemma.

Now we return to the proof of the theorem. To show the higher cohomologies are 0 it suffices to show that their stalks are 0. Now from the previous problem, the higher cohomologies are sheaves associated to the presheaves \(U \to \hat{H}^i(\Phi^{-1}(U), \mathcal{F}^\bullet)\) so it suffices to prove these sections are 0 over a basis. Naturally, we choose a basis of affines in which case the preimage will also be affine. So it suffices to consider \(Y = \text{Spec}(B)\) affine, in which case \(X = \text{Spec}(A)\) will also be affine, and we’ll have a map \(B \overset{f}{\leftarrow} A\). Then \(\mathcal{F} = \text{loc}_B(M)\) for some module \(M\). Choose \(0 \to \mathcal{F} \to I_1 \to \ldots\) be an injective resolution in \(\text{Qcoh}(\mathcal{F})\). The sheaves \(I_i\) will not be injective as sheaves of abelian groups, but they will be flasque - that’s because they are flasque as quasi-coherent sheaves and this is proved in the very same way as in bi) of a previous proposition, except using the structure sheaf instead of \(\mathbb{Z}\) (alternatively, see Hartshorne III.2.4.), and of course being flasque as a quasi-coherent sheaf means being flasque as a sheaf of abelian groups. Therefore, as we have show in part d) of a previous proposition, they can be used to compute the cohomology. However, because the functor \(\Phi_*\) is obviously exact in this setting (over quasi-cherent sheaves, and the two schemes are affine), the sequence \(\Phi_*(I_1) \to \Phi_*(I_2) \to \ldots\) is exact which implies the conclusion. \(\square\)
**Theorem.** Let $X$ be a quasi-compact, quasi-separated scheme. The following are equivalent:
(i) $X$ is affine
(ii) $H^i(X, \mathcal{F}) = 0$ for all $i > 0, \forall \mathcal{F} \in QCoh(X)$
(iii) $\Gamma$ is exact on short exact sequences of quasi-coherent sheaves
(iv) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of sheaves with $\mathcal{F}_1 \subset O_X$ a sheaf of ideals, then $\Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3)$ is surjective.

Proof: (i)$\Rightarrow$ (ii) follows from Serre’s theorem.
(ii)$\Rightarrow$ (iii) follows immediately from the long exact sequence of cohomology, and (iii)$\Rightarrow$ (iv) is clear.
The hardest part is (iv)$\Rightarrow$(i).

**Lemma.** Let $X$ be a scheme, $f_1, f_2, \ldots, f_n \in A \in \Gamma(X, O_X)$. Consider $X_{f_i}$ be the locus of non-vanishing (invertibility) of $f_i$. Suppose that $X_{f_i}$ are affine, $\cap X_{f_i} = X$ (i.e. $X$ is generated by the global sections $f_1, \ldots, f_n$), $\cup Spec A_{f_i} = Spec A$ (i.e. $f_1, \ldots, f_n$ generate the unit ideal in $A$), and finally that $X$ is quasi-separated.

Then $X \xrightarrow{\sim} Spec A$.

The proof of the lemma follows from the following sub-lemma:

**Sub-lemma:** if $X$ is quasi-compact and quasi-separable and $f \in \Gamma(X, O_X)$ then
$$\Gamma(X, O_X) = \Gamma(X_{f}, O_{X_f})$$
where the map is given by restriction plus inverting $f$ ($f$ is invertible on $X_f$ as it is invertible locally - the inverses easily glue).

Let’s show the sub-lemma. First the map is injective: assume that $\frac{a}{f^m} \in \Gamma(X, O_X)$ gets sent to 0. Since $f$ is invertible on both sides, it is enough to consider $k = 0$ so $a$ restricts to 0 on $X_f$. Choose a finite affine cover of $X$ by $Spec(B_i)$ to that $X_f \cap Spec(B_i)$ is isomorphic to $Spec((B_i)_f)$ where we regard $f$ as an element of $B_i = \Gamma(Spec(B_i), O_X)$ by restriction. Since a restrict to 0 on $Spec((B_i)_f)$ we deduce that $af^m$ restricts to 0 on $Spec(B_i)$. Choosing a large enough $m$ to work for all $i$ implies that $af^m = 0$ on $X_f$ so that $a$ is zero on the right-hand side as well. Now we show surjectivity. Choose the same affine cover, and assume we have compatible elements $\frac{b_i}{f^m}$ over $Spec(B_i)$. Again by multiplying by a suitable power of $f$, we may assume $m_i = 0$. By multiplying by a suitable power of $f$ again, we may make $b_i$ and $b_j$ actually agree. Indeed, $Spec(B_i) \cap Spec(B_j)$ is a union of finitely many affines and on each affine $b_i$ and $b_j$ become equal when multiplied by a certain power of $f$. In that case, $b_i$ and $b_j$ glue to a global section. By dividing back by the power of $f$ we multiplied by, we get surjectivity. □

Now let’s see how the sub-lemma implies the lemma. According to it, we can immediately deduce $X_{f_i} \xrightarrow{\sim} Spec(A_{f_i})$, the maps $X_{f_i} \to Spec(A_{f_i})$ being natural. In particular they glue and then we immediately deduce the claim as compatible maps define a map to $Spec(A)$ which is covered by $Spec(A_{f_i})$ and the map is an isomorphism because being an isomorphism is local. □

Now let’s go back to the proof of the theorem. Choose $P$ be a closed point of $X$ and $U$ be an affine open subscheme containing $P$. Let $Y$ be a closed subscheme structure on $X - P$ then $\{P\} \cap U$ gets a closed subscheme structure.

An easy general lemma tells that if $Y_1, Y_2$ are two disjoint closed subschemes the $O_{Y_1 \cup Y_2} = O_{Y_1} \oplus O_{Y_2}$ (for a suitable subscheme structure on $Y_1 \cup Y_2$) - this is easily done by picking the ideal sheaf $I_{Y_1 \cup Y_2}$ via circumscribing a neighborhood of every point $P$ disjoint from either $Y_1$ or $Y_2$ and then making it equal to either $I_{Y_2}$ or $I_{Y_1}$ on that subset (or everything if $Y_1, Y_2$ are both disjoint from that neighborhood).

We then also have the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \rightarrow & I_{Y_1 \cup Y_2} & \rightarrow & O_X & \rightarrow & O_{Y_1 \cup Y_2} & \rightarrow & 0 \\
& & \downarrow & & & \downarrow & & & \\
0 & \rightarrow & I_{Y_1} & \rightarrow & O_X & \rightarrow & O_{Y_1} & \rightarrow & 0
\end{array}
$$

34
where the left vertical map is injective and the right vertical map is surjective with cokernel $\mathcal{O}_{Y_2}$ according to the general lemma. The middle vertical map is the identity so now the snake lemma means that $\mathcal{O}_{Y_2}$ is isomorphic to the cokernel of the injection $I_{Y_1 \cup Y_2} \hookrightarrow I_{Y_1}$ hence we produce the short exact sequence

$$0 \to I_{Y_1 \cup Y_2} \to I_{Y_1} \to \mathcal{O}_{Y_2}$$

In particular for our case of $\{P\}$ and $Y$, we get the short exact sequence

$$0 \to I_{Y_1 \cup \{P\}} \to I_Y \to \mathcal{O}_{\{P\}} \to 0$$

which in particular gives a surjection (by property (iv) and the long exact sequence of cohomology) $\Gamma(X, I_Y) \to k_P$ so we can choose $f_P \in \Gamma(I_Y)$ project to 1.

Let $X_{f_P}$ be the corresponding open. Then it contains $\{P\}$ and is disjoint from $Y$ meaning it is contained in the affine $U$ and then it is affine because it is the locus of non-vanishing of a global section of an affine scheme, which is just a basic open affine.

We now choose such sections for all closed point $P$. In a quasi-compact scheme, closed points form a dense subset. This is obtained from the auxiliary proposition that every quasi-compact scheme contains a closed point, applied to the closure of every point that is quasi-compact. The auxiliary proposition follows from the following observation: with respect to the partial order induced by $x < y$ is $y$ is in the closure of $x$, a point is closed if and only if it is maximal. It remains to observe that every affine contains points maximal within that affine - this is the theorem about a ring having a maximal ideals, and now the proposition follows from the assertion that if $X = \bigcup_{i=1}^{n} X_i$ is such that $X_i$ all have a maximal element, then $X$ has a maximal element too. Essentially pick a maximal element in $X_1$, if it’s not maximal in $X$, there is a larger element in some other set $X_2$ and we go on like that, noting that we can not return to a previously visited $X_i$ so we must stop.

Therefore the subsets $X_{f_P}$ cover $X$ and we can choose a finite subcover producing the sections $f_1, \ldots, f_n$ that clearly satisfy all requirements of the lemma - except we are left to prove that $f_1, \ldots, f_n$ generate $\Gamma(X, \mathcal{O}_X)$.

That is equivalent to saying that the induced map $\mathcal{O}_X^\oplus n \to \mathcal{O}_X$ induces a surjection on the global sections. First, it is a surjection of sheaves as $f_i$ induces a surjection of stalks over $X_{f_i}$. Let $T$ be the kernel.

We now can filter $\mathcal{O}_X^\oplus n$ by $\mathcal{F}^i \cong \mathcal{O}_X$. We can then produce filtrations $0 \to T^i \to \mathcal{F}_i \to \mathcal{G}_i \to 0$ which yield

$$0 \to T^i/T^{i-1} \to \mathcal{F}^i/\mathcal{F}^{i-1} \to \mathcal{G}^i/\mathcal{G}^{i-1} \to 0$$

and $\mathcal{F}^i/\mathcal{F}^{i-1} \cong \mathcal{O}_X$ whence $T^i/T^{i-1}$ is a sheaf of ideals. According to (iv), we get $\Gamma(X, \mathcal{F}^i/\mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{G}^i/\mathcal{G}^{i-1})$.

From here we can show by induction on $i$ that $\Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{G}^i)$ using the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
\Gamma(X, \mathcal{F}^i/\mathcal{F}^{i-1}) \longrightarrow \Gamma(X, \mathcal{G}^i/\mathcal{G}^{i-1}) \\
\downarrow \\
\Gamma(X, \mathcal{F}^i) \longrightarrow \Gamma(X, \mathcal{G}^i) \\
\downarrow \\
\Gamma(X, \mathcal{F}^{i-1}) \longrightarrow \Gamma(X, \mathcal{G}^{i-1}) \\
\downarrow \\
0 \quad 0
\end{array}
\]

in which the first row is a split short exact sequence which finishes the proof.
Next, we return to the problem of taking cohomologies in two ways - namely as sheaves of modules versus quasi-coherent sheaves. We have the following commutative diagram:

\[
\begin{array}{ccc}
K^+(Qcoh(X)) & \xrightarrow{\text{fully faithful}} & R(\mathcal{Sh}_{\mathcal{O}_X-mod}(X)) \\
\xrightarrow{\pi_{Qcoh}} & & \xrightarrow{\pi_{Sh}} \\
D^+(Qcoh(X)) & \xrightarrow{\text{emb}_D} & D^+(\mathcal{Sh}_{\mathcal{O}_X-mod}(X))
\end{array}
\]

Note that \(\text{emb}_D\) takes acyclics to acyclics hence it factors through \(\text{emb}_D\).

**Theorem.** If \(X\) is a quasi-coherent and separated scheme then

\[R\Gamma_{Qcoh} \sim \to R\Gamma_{Sh} \circ \text{emb}_D\]

Proof: The map from the left hand-side to the right hand side is easily given by the universal property. Again, it is easy to define it in terms of resolutions, too. Since cohomology can be computed by acyclics, it is easy to see that statement can be reformulated as "any \(I \in Qcoh(X)^{\text{noj}}\) is acyclic for \(R\Gamma_{Sh}\) (in fact, for \(X\) noetherian, injective quasi-coherent sheaves are flasque - see Hartshorne ch. III).

It’s enough to show that any \(F \in Qcoh(X)\) can be embedded into an injective quasi-coherent sheaf \(I\) which is acyclic for \(\Gamma_{Sh}\).

Choose \(X = U_i\) where \(U_i\) are finitely many affines, and let \(j_i: U_i \to X\) be the open embedding which is affine as \(X\) is separated.

Then \(\mathcal{F} \mapsto \oplus (j_i)_*(\mathcal{F})\)

Now embed \(j_i^*: \mathcal{F}\) into \(I_i\) an injective quasi-coherent sheaf on \(U_i\) so that \(\mathcal{F}\) embeds into \(\oplus (j_i)_* I_i\).

We need to show that \((j_i)_*(I_i)\) is injective in \(Qcoh(X)\) and \(H^n(X,(j_i)_*(I_i)) = 0\) for \(n > 0\).

It is injective because \((j_i)_*\) is right adjoint to the exact functor \(j_i^*\), as for the second part, it follows immediately from Leray’s spectral sequence as \(R(j_i)_* = (j_i)_*\) since the map is affine (done before). □

**Theorem.**

\(\text{emb}_D: D^+(Qcoh(X)) \to D^+(\mathcal{Sh}_{\mathcal{O}_X-mod}(X))\)

is fully faithful.

**Corollary.** if \(\mathcal{F}_1, \mathcal{F}_2 \in Qcoh(X)\) then

\[\text{Ext}_{Qcoh}^n(\mathcal{F}_1, \mathcal{F}_2) \sim \to \text{Ext}_{\mathcal{O}_X-mod}^n(\mathcal{F}_1, \mathcal{F}_2)\]

(both are \(\text{Hom}\) in the derived category)

Proof: We have to show that \(\text{Hom}_{D(Qcoh(X))}(\mathcal{F}_1, \mathcal{F}_2) \sim \to \text{Hom}_{D(\mathcal{O}_X-mod)}(\text{emb}(\mathcal{F}_1), \text{emb}(\mathcal{F}_2))\)

First we can assume \(\mathcal{F}_1\) is in \(D^b\) - using the "holim" construction: \(\mathcal{F}_1 \cong \text{Cone}(\oplus \mathcal{F}_1^{\leq i} \to \oplus \mathcal{F}_1^{\leq i+1})\) and by long-exact sequence of mapping out, it suffices to consider \(\mathcal{F}_1^{\leq i}\).

Next, we can also consider \(\mathcal{F}_2 \in D^b\): if \(\mathcal{F}_1\) is trivial after \(n\)-th term then we have the distinguished triangle \(\mathcal{F}_2^{\leq n} \to \mathcal{F}_2 \to \mathcal{F}_2^{\leq n+1} \to \mathcal{F}_2^{\leq n}[1]\) and use the long exact sequence of mapping in to get \(\text{Hom}(\mathcal{F}_1, \mathcal{F}_2^{\leq n}) \cong \text{Hom}(\mathcal{F}_1, \mathcal{F}_2)\)

Further, we assume that \(\mathcal{F}_1\) lives cohomological degree - again using the short exact sequence of truncation and the long exact sequence of mapping out. In the same way, we can assume \(\mathcal{F}_2\) lives in one cohomological degree, so that we can assume \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are sheaves. That’s because any object that lives in one cohomological degree is homotopy equivalent to an object consisting of a single sheaf - namely a complex \(\mathcal{F}_1 \to \mathcal{F}_2 \to \ldots\) is equivalent to the sheaf \(\text{Ker}(\mathcal{F}_1 \to \mathcal{F}_2)\).

Next, we claim we can reduce to \(X\) affine.

Indeed, assume that we know the statement for affine sheaves.
It’s sufficient to show that any $\mathcal{F}_2 \in QCoh(X)$ can be embedded into some $\mathcal{F}'_2$ such that $\mathcal{F}'_2$ is injective as an object of $QCoh(X)$ and $Ext^i_{Sh(O_X-\text{mod})}(\mathcal{F}, \mathcal{F}_2') = 0$ for $i > 0$ (as then $\mathcal{F}'_2 \to \ldots \to$) will be used to compute both Ext functors, and will give the same result.

Cover $X$ by affines, $j_i: U_i \to X$ then $\mathcal{F}_2 \simeq (j_i)_* j_i^*(\mathcal{F}_2)$ so injecting $j_!^*(\mathcal{F}_2)$ into an injective object $I_i$, we produce $\mathcal{F} \simeq (j_i)_* I_i = \mathcal{F}'_2$ and $\mathcal{F}'_2$ is injective as before.

Next, observe that in $D(Sheaf(O_X-\text{mod})$, $j^*$ is left adjoint to $Rj_*$ - since to compute $Hom(j^* \mathcal{F}_1, \mathcal{F}_2)$ we take an injective resolution of $\mathcal{F}_2$ and map $j^* \mathcal{F}_1$ into it - but this is the same as mapping $\mathcal{F}_1$ into $j_*$ of the resolution, which is $Rj_*(\mathcal{F}_2)$ as $j_*$ sends injectives to injectives.

Thus, $Ext^n(\mathcal{F}_1, \mathcal{F}_2') = \oplus RHom(\mathcal{F}_1, R(j_i)_*(I_i)_n) = \oplus RHom(\mathcal{F}_1, (j_i)_*(I_i)_n) = \oplus RHom(j^* \mathcal{F}_1, I_i) = 0$ as $I_i$ are injective.

Finally, we do the affine case $X = Spec(A)$. As above, it suffices to map $\mathcal{F}_2$ into an injective modules with zero higher ext into it. In fact, every injective module satisfies the property.

Indeed, choose $\mathcal{F}$ a complex of $A$-modules and let $I$ be an injective $A$-module. We want $Ext^3_{Sh}(O_X-\text{mod})(\mathcal{F}, I) = 0$.

We know this when $\mathcal{F} = A$ because $RHom(A, -)$ is $R\Gamma(-)$ and we have shown $R\Gamma^n(I) = 0$ for $n > 0$ for affine schemes - as this is just $R\Phi_\ast$ where $\Phi$ is the map to the point which is affine if the scheme itself is affine.

Now choose a left resolution $P^\bullet$ of $\mathcal{F}$ by free $A$-modules, and an injective resolution $Q^\bullet$ of $I$ by injective sheaves of $A$-modules. Take the bicomplex $Hom(P^n, Q^j)$ and note that each column is exact except for the 0th cohomology being $Hom(\mathcal{F}, Q_1)$ hence the total cohomology of the bicomplex is $Ext^0(\mathcal{F}, I)$. But in the other hand rows are also exact except for the 0th cohomology being $Hom(P_0, I)$ - but these are acyclic as $I$ is injective - hence the total complex is acyclic and we get the result. □

The Ext functors also parametrizes short exact sequences in some sense, as shown by the next proposition.

**Proposition.** Let $\mathcal{A}$ be an abelian category.

a) Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence. Consider the ”hut”

$$Z \leftarrow (X \to Y) \to X[1]$$

where $X \to Y$ is considered as a complex in degree -1 and 0. This construction establishes a bijection between isomorphism classes of extensions as above and $Ext^1(Z, Y) = Hom_{D(\mathcal{A})}(Z, X[1])$

b) Let $X \to Y$ be a morphism in $\mathcal{A}$, let $K, I, J$ denote its kernel, image and cokernel respectively. We obtain the short exact sequences

$$0 \to K \to X \to I \to 0 \text{ and } 0 \to I \to Y \to J \to 0$$

along with the corresponding elements in

$$Ext^1(J, I) \cong Hom_{D(\mathcal{A})}(J, I[1]) \text{ and } Ext^1(I, K) \cong Hom_{D(\mathcal{A})}(I, K[1])$$

Their composition (a.k.a. cup product) gives an element in $Ext^2(J, K) = Hom_{D(\mathcal{A})}(J, K[2])$. This element represents the same element as the class of

$$J \leftarrow (K \to X \to Y) \to K[2]$$

c) In the situation of b) the resulting element of $Hom_{D(\mathcal{A})}(J, K[2])$ vanishes if and only if there exists an object $Z \in \mathcal{A}$ endowed with maps $K \to Z, Z \to J$, and isomorphisms $Z/K \cong Y$ compatible with the projection to $J$ and $X \cong Ker(Z \to J)$ compatible with the embedding of $K$.

Proof: Assume we have a hut $Z \leftarrow W \to X[1]$. Take $C[1]$ the cone of $W \to X$, then $X \to C \to W \to X[1]$ is a cone sequence. Now taking the long exact sequence of homology we deduce that $C$ has one non-zero homology in degree 1, equal to $Y$, and we have a short exact sequence $0 \to X \to Y \to Z \to 0$. 37
It remains to show that these operations are mutually equivalent.

First, we show that two equivalent huts give equivalent extensions. Indeed, if \( W' \to W \) is a quasi-isomorphism then we can complete to a map of triangles

\[
\begin{array}{ccc}
C' & \to & W' \to X[1] \\
\downarrow & & \downarrow id \\
C_1 & \to & W \to X[1]
\end{array}
\]

And by taking cohomologies and using the five lemma we deduce that the extensions are isomorphic.

Now take an extension \( 0 \to X \to Y \to Z \to 0 \) and produce the hut \( Z \sim (X \to Y) \to X[1] \). Since we’ve proved equivalent huts give equivalent extensions, it suffices that when we go back we work with the same hut. Then \( C \) will obviously be \( Y \), and we deduce the conclusion.

Conversely, start with a hut \( Z \sim W \to X[1] \). Since the category \( \mathcal{Q}/X \) is filtered, we may assume that \( W \to Z \) factors through \( W \sim (X \to Y) \to Z \). Then the diagram

\[
\begin{array}{ccc}
W & \to & X[1] \\
\sim & & \downarrow id \\
(X \to Y) & \to & X[1]
\end{array}
\]

will be commutative because the maps are the same on cohomologies and the cohomology of \( X[1] \) is \( X \) (so no quotient to worry about). It follows that the diagram extends to a map of triangles which means the two huts are the same, and we finish the proof. b) Consider the following diagram:

\[
\begin{array}{ccc}
\sim & & \sim \\
Y \oplus I[1] & \sim & K[2] \\
J & \to & I[1] & \to & K[2]
\end{array}
\]

The two smaller huts represent the morphisms in \( \text{Hom}_{\mathcal{D}(A)}(J, I[1]) \) and in \( \text{Hom}_{\mathcal{D}(A)}(I, K[1]) \) shifted by 1. The bigger hut represents the composition of these morphisms - where the map \( Y \oplus X[1] \oplus K[2] \to Y \oplus I[1] \) arises from the map \( X \oplus K[1] \to I \) which is a quasi-isomorphism and \( X \oplus I[1] \) is the cone of \( K \to X \) and \( I \) is the cokernel of the map, and the map \( Y \oplus X[1] \oplus K[2] \to X[1] \oplus K[2] \) is coming from the representation.

Now we introduce the differentials in \( Y \oplus X[1] \oplus K[2] \) to be such that the complex is the same as \( K \to X \to Y \). This is consistent with the maps described above - as the differential in the cone \( X \oplus K[1] \) is exactly the map \( K \to X \), and the differential in \( Y \oplus I[1] \) is the map \( I \to Y \).

Therefore the big hut represents the composite morphism \( J \to K[2] \) given by \( J \leftarrow (K \to X \to Y) \to K[2] \).

C) Assume there is such an object \( Z \). The composite map \( X \to Z \to Y \) then equals the map \( X \to Y \). Now consider the complex \( (0 \to X \to Z) \). The following diagram is commutative:

\[
\begin{array}{ccc}
0 & \to & X \to Z \\
\downarrow & & \downarrow id \\
K & \to & X \to Y
\end{array}
\]

Because the first two maps in both rows are injective, and \( Y/I \cong Z/X \cong J \), this induces an isomorphism on cohomologies so it is a quasi-isomorphism. But the map \( (0 \to X \to Z) \to K[2] \) is clearly zero, therefore the map
equals the map

$$J \leftarrow (0 \rightarrow X \rightarrow Z) \rightarrow K[2]$$

and the latter map is zero.

This method does not give a way to show the converse. Here is a more detailed proof of c), due to Jonathan:

In the problem statement, add the condition that the composition $X \rightarrow Z \rightarrow Y$ coincides with the original map $X \rightarrow Y$.

First we prove that short exact sequences of complexes in $K(\mathcal{A})$ can completed to distinguished triangles in $D(\mathcal{A})$.

**Lemma.** Let $0 \rightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \rightarrow 0$ be a short exact sequence of complexes of $\mathcal{A}$. Then we have the following commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & X^\bullet & \longrightarrow Y^\bullet & \longrightarrow \text{Cone}(f) \simeq Y^\bullet \oplus X[1]^\bullet \\
\downarrow{id} & & \downarrow{id} & & \downarrow{\pi} \\
0 & \longrightarrow & X^\bullet & \longrightarrow Y^\bullet & \longrightarrow Z^\bullet
\end{array}
\]

where $\pi = (g,0)$ is the canonical projection. We easily see that $\pi$ is a map of complexes (cf. PS 1), and the diagram is clearly commutative. We have that $\pi$ is a quasi-isomorphism.

**Proof:** We see that $\ker(\pi) \simeq \ker(g) \oplus X[1]^\bullet \simeq X^\bullet \oplus X[1]^\bullet$. We show that this complex $X^\bullet \oplus X[1]^\bullet$ is homotopy equivalent to the 0 complex. For this it is enough to show the identity map is nullhomotopic. Define $h^i : X^i \oplus X^{i+1} \rightarrow X^{i-1} \oplus X^i$ to be

$$h^i = \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix}$$

Now we just check that

$$d \circ h + h \circ d = \begin{pmatrix} d_X & \text{id}_X \\ 0 & -d_X \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix} \begin{pmatrix} d_X & \text{id}_X \\ 0 & -d_X \end{pmatrix} = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}$$

Therefore $X^\bullet \oplus X[1]^\bullet$ is homotopy equivalent to 0 complex, and is consequently acyclic. Clearly $\pi$ is surjective. Therefore we can apply LES of cohomology to the SES

$$0 \rightarrow X^\bullet \oplus X[1]^\bullet \rightarrow \text{Cone}(f) \xrightarrow{\pi} Z^\bullet \rightarrow 0$$

which shows cohomology of $\text{Cone}(f)$ coincides with that of $Z^\bullet$. □

In particular, in $D(\mathcal{A})$ can now take the hut $Z^\bullet \leftarrow \text{Cone}(f) \rightarrow X[1]^\bullet$ as the connecting morphism to get a distinguished triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $D(\mathcal{A})$.

Now consider the SES $0 \rightarrow K \rightarrow X \rightarrow I \rightarrow 0$. By Lemma 1, we have $K \rightarrow X \rightarrow I \rightarrow K[1]$ is distinguished, where $I \rightarrow K[1]$ is given by the hut $I \leftarrow \text{Cone}(K \rightarrow X) \rightarrow K[1]$. It is easy to see that $\text{Cone}(K \rightarrow X) = (K \rightarrow X)$, so the hut is the one described in part (a). Now taking LES of Hom, we get

$$\text{Ext}^1(J,X) \rightarrow \text{Ext}^1(J,I) \rightarrow \text{Ext}^2(J,K)$$

If we take $J \leftarrow (I \rightarrow Y) \rightarrow I[1]$ in $\text{Ext}^1(J,I)$, the image of composition in $\text{Ext}^2(J,K)$ is $J \leftarrow (K \rightarrow X \rightarrow Y) \rightarrow K[2]$ by part (b). If this hut is zero, by exactness we know there is element in $\text{Ext}^1(J,X)$ that maps to the extension $I \rightarrow Y \rightarrow J$. By (a), this means there exists extension $0 \rightarrow X \rightarrow Z \rightarrow J \rightarrow 0$, which is represented by the hut $J \leftarrow (X \rightarrow Z) \rightarrow X[1]$. Composing with $X \rightarrow I$, we are supposed to get the extension $0 \rightarrow I \rightarrow Y \rightarrow J \rightarrow 0$. From the proof of part (a), this means that

$$0 \rightarrow I \rightarrow H^{-1}(\text{Cone}((X \rightarrow Z) \rightarrow I[1])) \rightarrow J \rightarrow 0$$
should be isomorphic to the former extension. We have \( H^{-1}(\text{Cone}((X \to Z) \to I[1])) \cong \ker(X \to Z \oplus I) \), where \( X \to Z \oplus I \) is antisymmetric map of \( X \to Z \) and \( X \to I \).

\[
\begin{array}{c}
0 \rightarrow I \rightarrow (Z \oplus I)/X \rightarrow J \rightarrow 0 \\
\downarrow \text{id} \quad \sim \quad \phi \quad \downarrow \text{id} \\
0 \rightarrow I \rightarrow Y \rightarrow J \rightarrow 0
\end{array}
\]

Now let \( Z \to Y \) be the map \( \frac{(a)}{(a)} (Z \oplus I)/X \cong Y \). Then by the above diagram we have \( Z \to Y \to J \) agrees. We observe that \( \ker(Z \to (Z \oplus I)/X) \cong \ker(X \to I) \cong K \). Therefore we have \( 0 \to K \to Z \to Y \to 0 \), where \( K \to X \to Z \) are the inclusions. Note that by our construction it also follows that \( X \to Z \to Y \) coincides with \( X \to I \to Y \), which is just the map we started out with. This proves one direction.

Now assume we have \( Z \) such that \( K \leftarrow Z; Z \to J \), and \( Z/K \cong Y, X \cong \ker(Z \to J) \) in compatible ways. Furthermore we need to assume that \( X \to Z \to Y \) coincides with the original map \( X \to Y \). Now consider the following map of complexes

\[
\begin{array}{c}
0 \rightarrow X \rightarrow Z \\
\downarrow \\
K \rightarrow X \rightarrow Y
\end{array}
\]

which is commutative. Furthermore since \( 0 \to X \to Z \to J \to 0 \), we have that the above is a quasi-isomorphism. But the map \( (X \to Z) \to K[2] \) is \( 0 \), so the hut must be zero. This proves the other direction.

\[
02/18/2010
\]

**Proposition. (PSET)** Let \( X \) be a topological space with a sheaf of rings \( R_X \). Denote \( D(X) = D(Sh_{R_X - \mod}(X)) \).

Let \( \Phi: Y \to X \) be a map of topological spaces, set \( R_Y = \Phi^*(R_X) \). Thus, we have a functor \( \Phi^*: Sh_{R_X - \mod}(X) \to Sh_{R_Y - \mod}(Y) \), which is exact, and hence gives rise to a functor \( \Phi^*: D(X) \to D(Y) \).

a) \( \Phi^* \) and \( R\Phi_*: D(Y) \to D(X) \) are mutually adjoint.

b) If \( Y \) is a topological subspace of \( X \), then \( R\Phi_*: D(Y) \to D(X) \) is fully faithful.

Proof: a) We know that \( \Phi^{-1} \) and \( \Phi_* \) are adjoint to each other and \( \Phi^* \) and \( \Phi_* \) are kind of the same functors.

Now let \( \mathcal{F}^* \) an injective resolution represent an element in \( D(Y) \) and \( I^* \) be an injective resolution representing an element in \( D(X) \).

The we want to show that

\[
\text{Hom}_{D(Y)}(\Phi^*\mathcal{F}^*, I^*) \cong \text{Hom}_{D(X)}(\mathcal{F}^*, \Phi_*I^*)
\]

We know that \( D(Y) \) is equivalent to the subcategory of complexes of injectives, so any morphism in the left-hand side comes from a regular morphism \( \Phi^*\mathcal{F}^* \to I^* - \) as \( \Phi^*\mathcal{F}^* \) is injective since \( \Phi^* \) is exact. Then by adjunction, we construct the map \( \mathcal{F}^* \to \Phi_*I^* \).

Conversely, assume we have a hut \( \mathcal{F}^* \cong W^* \to \Phi_*I^* \). We produce by adjunction a map \( \Phi^*W^* \to I^* \) and a morphism \( \Phi^*W^* \to \Phi_*\mathcal{F}^* \) which is still an equivalence because \( \Phi^* \) is exact, so we get a hut \( \Phi^*\mathcal{F}^* \cong \Phi^*W^* \to I^* \).

Now let’s prove that these operations are mutually inverse to each other.

Suppose we start with a map \( \Phi^*\mathcal{F}^* \xrightarrow{f} I^* \). By adjunction we produce the map \( \mathcal{F}^* \xrightarrow{\Phi_*f} \Phi_*I^* \) which represents the hut \( \mathcal{F}^* \xrightarrow{f} \Phi_*\mathcal{F}^* \xrightarrow{\Phi_*f} \Phi_*I^* \) and which by adjunction is taken (back) to the hut \( \Phi^*\mathcal{F}^* \xrightarrow{\Phi_*f} \Phi_*\mathcal{F}^* \xrightarrow{f} I^* \) which is the same as our original morphism.

Conversely, let’s suppose we start with a hut \( \mathcal{F}^* \xrightarrow{g} W^* \xrightarrow{h} \Phi_*I^* \) and we produce the map \( \Phi^*\mathcal{F}^* \xrightarrow{\Phi^*g} \Phi^*W^* \xrightarrow{h} I^* \). Then we know there is a morphism \( f: \Phi^*\mathcal{F}^* \to I^* \) which makes the following diagram commute:
We then produce by adjunction the map $F^* \xrightarrow{f'} \Phi_* I^*$; so to prove that we’ve come back to the same we need to show that the below diagram commutes:

$$
\begin{array}{ccc}
\Phi^* W^* & \xrightarrow{\Phi^* g} & I^* \\
\Phi_* F^* & \xrightarrow{f} & \Phi_* I^* \\
\end{array}
$$

But note that the following diagram commutes, from the natural-ness of the adjunction:

$$
\begin{array}{ccc}
\Phi^* \Phi^* W^* & \xrightarrow{\Phi^* \Phi^* g} & \Phi^* \Phi^* I^* \\
\Phi_* \Phi_* F^* & \xrightarrow{\Phi_* \Phi_* f} & \Phi_* \Phi_* I^* \\
\end{array}
$$

It remains to attach to the left side of this diagram the following rectangle whose commutativity also follows from natural-ness:

$$
\begin{array}{ccc}
\Phi^* W^* & \xrightarrow{g} & W^* \\
\Phi_* F^* & \xrightarrow{\Phi_* f} & \Phi_* I^* \\
\end{array}
$$

b) As we know from the previous semester, this is equivalent to saying that the natural morphism $\Phi^* R\Phi_* \rightarrow id$ is an isomorphism.

It’s enough to show that $\Phi^* \Phi_* \rightarrow id$ is an isomorphism on sheaves, because then we can just take injective resolutions and $R\Phi_*$ is identified with $\Phi_*$ on them.

Indeed, $\Gamma(\Phi^* \Phi_* \mathcal{F}, U) = \Gamma(\Phi_* \mathcal{F}, \Phi(U))$ - this because the set $V \supset \Phi(U)$ has a minimal element $\Phi(U)$ as $V$ is a subspace of $X$, and $\Gamma(\Phi_* \mathcal{F}, \Phi(U)) = \Gamma(\mathcal{F}, \Phi^*(\Phi(U))) = \Gamma(\mathcal{F}, U)$. □

Proposition. (PSET) Let $X, Y$ be as above and assume $\Phi = i$ is a closed embedding. Let $j: U \xrightarrow{X} \xrightarrow{Y}$ be the open embedding of its complement. Consider the corresponding functors

$$
i_*: D(Y) \rightarrow D(X) \leftarrow D(U) : Rj_*
$$

a) For $\mathcal{F} \in D(X)$ the following are equivalent: i) $\mathcal{F}$ lies in the essential image of $i_*$ ii) the canonical map $\mathcal{F} \rightarrow i_* i^* \mathcal{F}$ is an isomorphism iii) $j^* \mathcal{F} = 0$

b) $D(Y), D(X), D(U)$ form an admissible triple.

c) Denote by $Ri^*: D(X) \rightarrow D(Y)$ the functor right adjoint to $i_*$. Then $Ri^* \circ i_*$ is isomorphic to the identity functor on $D(Y)$.

d) $Ri^*: D(X) \rightarrow D(Y)$ identifies with the right derived functor of the functor $i^*: Sh_{RX}$ defined as follows:

$$
\Gamma(U_1 \cap Y, i^!(\mathcal{F})) = \{ f \in \Gamma(U_1, \mathcal{F}) \mid f |_{U_1 \cap Y = 0} \}
$$

e) $D(Y) \xrightarrow{i^*} D(X) \xrightarrow{Ri^*} D(U)$

41
also form an admissible triple.

Proof: We use problems 7 and 8 from PSET 8 from the first semester.

a) i) and ii) are equivalent from the previous semester (recall that we did the same thing with sheafification). That i) and iii) are the same is the content of problem 7j) from pset 7 from last semester (we use 7j) on cohomologies, as \( i_* \) and \( j^* \) are exact)

The only part to check is that \( i_* \) is fully faithful. Indeed, as \( i_* \) is exact and fully faithful (7e, 7g), it is isomorphic to \( R_i \) which is fully faithful from problem 1b).

b) According to the previous problem, we have identified \( j^*(D(U)) \) with \( D(Y)^\perp \) - provided that \( R_j \) is fully faithful which it is, by problem 1b). This is because \( \text{Hom}(\mathcal{F}, R_j G) \cong \text{Hom}(j^* \mathcal{F}^*, G) \) so \( \text{Hom}(\mathcal{F}^*, R_j G) = 0 \) for any \( G \) if and only if \( j^* \mathcal{F}^* = 0 \) which is equivalent to \( \mathcal{F}^* \) being in the essential image of \( i_* \), by a).

So by a proposition in class, we only need to check that \( i_* \) has a right adjoint.

But \( i_* \) is exact and admits a left adjoint \( i^1 \) (8d ). Now, just like in problem 1a), these conditions are enough to conclude that \( R_i^1 \) is right adjoint to \( i_* \).

c) Like in 1b), it’s enough to show that \( i^1 \circ i_* \) is isomorphic to the identity on single sheaves. Indeed, let \( \mathcal{F} \in \text{Sh}_{R_Y \text{-mod}}(Y) \). Then \( \Gamma(i^1 i_* \mathcal{F}, U_1 \cap Y) = \{ s \in \Gamma(i_* \mathcal{F}, U_1) | \text{Res}_{U_1 \cap Y} s = 0 \} = \{ s \in \Gamma(\mathcal{F}, U_1 \cap Y) | \text{Res}_{U_1 \cap Y} s' = 0 \} \), where \( s' \) is the section on \( \Gamma(i_* \mathcal{F}, U_1) \) produced from \( s \) via \( i_* \). But of course the condition \( \text{Res}_{U_1 \cap Y} s' = 0 \) is superfluous since it’s the same as saying \( s \) vanishes when restricted to \( U_1 \cap U \cap Y = \emptyset \). So we are done.

d) Done in b).

e) From 7c), \( j_i \) is exact so it induces a map \( D(U) \rightarrow D(X) \). From here, this part is similar to parts 2a), 2b) - using \( j_i \) instead of \( i_* \), \( i_* = R_i \) instead of \( j_* \) and \( j^* \) instead of \( j^* \). Observe that \( j^* \) is right adjoint to \( j_i \) from 7a). Problem 7j) provides the analogue for part a), and so this case is absolutely similar. \( \square \)

**Proposition. (PSET)** Assume now that \( X \) is a scheme and \( R_X = \mathcal{O}_X \), and let \( Y \) and \( U \) be as in the previous proposition with \( j \) a quasi-compact isomorphism. (Note, however, that the map \( i^* (R_X) \rightarrow R_Y \) is no longer an isomorphism!). In this case we have the functors

\[
i_* : D(Y) \rightarrow D(X), j^* = j^* : D(X) \rightarrow D(U), j_* = j_* : D(U) \rightarrow D(X).
\]

Assume that \( j \) is a quasi-compact morphisms. Denote

\[
D_{Qcoh}(X)_Y = \ker(j^* D_{Qcoh}(X) \rightarrow D_{Qcoh}(U)).
\]

a)

\[
i_* : D_{Qcoh}^+(X)_Y \rightarrow D_{Qcoh}^+(X) \leftarrow D_{Qcoh}^+(U) : R_j^*
\]

form an admissible triple.

a') The above holds without the ”+” condition.

b) An object \( \mathcal{F} \in D_{Qcoh}(X) \) belongs to \( D_{Qcoh}(X)_Y \) if and only if its cohomology sheaves \( H^i(\mathcal{F}) \in Qcoh(X) \) have the following property: for every affine subscheme \( U_1 \subset X \) and a local section \( f \in \Gamma(U_1, \mathcal{F}) \) and \( a \in \alpha \) there exists some integer \( n \) such that \( a^n \cdot f = 0 \), where \( \alpha \) is the ideal in \( \Gamma(U_1, \mathcal{O}_X) \) defining \( Y \) (for some fixed scheme structure on \( Y \)).

c) Let \( \mathcal{F} \in D_{Qcoh}^+(X) \) be an object with the following property: for every \( \mathcal{F}_1 \in Qcoh(Y) \) and \( n \in \mathbb{Z} \), we have \( \text{Hom}_{D(X)}(\mathcal{F}_1[n], \mathcal{F}) = 0 \). Then \( \mathcal{F} \) lies in the essential image of the functor \( R_j^* \) (equivalently, the canonical map \( \mathcal{F} \rightarrow R_j^* j^*(\mathcal{F}) \) is an isomorphism).

Proof: a) Observe that \( R_j^* \) is still fully faithful by the same argument as before, because of 9a) from Pset 7 last semester. Now let’s show that \( (R_j^*)^\perp = D_{Qcoh}(X)_Y \). This is pretty much by definition, because \( j^* \) is left adjoint to \( R_j^* \) according to the quasi-coherent analogue of 1a) (proven in exactly the same way), and now we use the reasoning from 2b): \( \text{Hom}(\mathcal{F}^*, R_j G) = \text{Hom}(j^* \mathcal{F}^*, G) \) so \( \mathcal{F}^* \in (R_j^*)^\perp \) if and only if \( \mathcal{F}^* \in \ker(j^*) = D_{Qcoh}(X)_Y \).
Therefore it is enough to show that \( Rj_* \) has a left adjoint, which is \( j^* \).

b) By Leray, \( H^i(j^*F) = H^i(F) \) (the cohomologies are over \( Y \) and \( X \)) - note that the proof of Leray from the exercise 5 of the previous problem set adapts to quasi-coherent cohomology. So \( F \) belongs to \( D_{QCoh}(X) \) if and only if its cohomologies are in the kernel of \( j^* \).

Therefore it is enough to show that \( j^*F = 0 \) for a single sheaf \( F \) if and only if it satisfies the property mentioned in b).

Indeed, \( j^*F = 0 \) is a local condition, so we my assume \( X = U_1 = Spec(A) \) is affine. In that case, \( U \) will be a finite union of affines \( X_f \) because \( j \) is quasi-compact.

Let \( F = loc_A M \). Then \( j^*F |_{X_f} \) will be \( M \otimes_A A_f \). So \( j^*F = 0 \) if and only if every section \( m \in M \) is killed by some power of \( f_i \); i.e. \( (f_1, \ldots, f_k)^n m = 0 \) for some sufficiently big \( n \). Because the ideal defining \( Y \) has the same radical as \( (f_1, \ldots, f_k)^n \) we conclude.

c) It is enough to show that if \( F \in D_{QCoh}(X)_Y \) satisfies that condition then \( F \) is 0 (i.e. has zero cohomologies).

The reason for that is that we can insert every \( F \in D_{QCoh}(X) \) into a distinguished triangle \( F' \to F \to F'' \to F'[1] \) by a) where \( F' \in DCoh(X)_Y \) and \( F'' \) is in the essential image of \( Rj_* \). So now because \( j^* \circ i_* = 0 \) (same old problem 7 Pst 7) we conclude that everything in \( QCoh(Y) \) is perpendicular to \( F'' \) so by taking the long exact sequence of maps into we conclude that \( F'' \) satisfies the condition so if we can conclude from that that \( F' = 0 \) we will be done.

First we claim it’s enough to consider \( X \) affine. Indeed, if the cohomologies of \( F \) becomes 0 when restricted to all affines then its cohomologies are 0 globally so we are done. So assume that \( j^*_1 F \) does not become 0 when projected to some \( D(U_1) \) where \( U_1 \) is an affine subset of \( X \). If we have proven the affine case, this implies that there exists some \( F_1 \in D(U_1 \cap Y) \) such that \( Hom(F_1[r], j^*_1 F) \) is not zero. Because \( R(j_1)_* \) is fully faithful we apply it to produce a non-zero element in \( Hom(R(j_1)_*, F_1[r], R(j_1)_* j^*_1 F) = Hom(R(j_1)_*, F_1[r], F) \)-contradiction.

So let’s assume that \( X = Spec(A) \) is affine, and \( F^\bullet \) is therefore thought of as a complex of \( A \)-modules \( \ldots \to M_0 \to M_1 \to M_2 \to \ldots \).

We claim it’s enough to do if for \( F \) bounded from below. Indeed, recall that \( F \) can be written as the inverse limits of truncations bounded from below (previous pset), and now use the universal property to reduce to that case. So we may assume that \( F \) is non-zero in non-negative degrees and that its zeroth cohomology is non-zero.

Say we get \( 0 \to M_0 \to M_1 \to \ldots \) such that \( K = Ker(M_0 \to M_1) \neq 0 \). Now let \( Y = Spec(A/I) \) where \( I \sim (f_1, \ldots, f_k) \) so that the complexes \( \ldots \to (M_1)_f \to (M_2)_f \to \ldots \) are exact. Because localization is an exact functor, we conclude that \( (K)_f = 0 \) - which says that \( I^k a = 0 \) for any \( a \in K \) (for this, we can also use the previous part). Let’s choose such a minimal \( k > 0 \), and consider the module \( J = I^{k-1}a \). It is killed by \( I \) (so comes from \( Y \)) but is non-zero, by minimality of \( k \). It also obviously maps as a complex into \( 0 \to M_0 \to \ldots \) and induces a non-zero map on the cohomologies, contradiction.

Remark: if \( I \) is not finitely generated, we cannot conclude that \( I^k a = 0 \), although the previous part (in the form i couldn’t prove) does that. □

**Definition.** \( D_{QCoh}(X) \subset D(X) \) is the full subcategory consisting of objects whose cohomologies are quasi-coherent sheaves.

It is better to consider this larger subcategory then that of complexes of quasi-coherent sheaves, because the former localize better - part of the difficulty with quasi-coherent sheaves is that extension by zero \( j! \) does not send quasi-coherent sheaves to quasi-coherent sheaves (see first semester problem sets).

**Theorem.** Let \( X \) be quasi-compact and separated (in fact, quasi-separated is enough). Then

\[
D^+(QCoh(X)) \to D_{QCoh}^+(X)
\]

is an equivalence of categories.

Proof: note that the functor is always fully faithful as proven in the previous lecture. We need to show essential surjectivity. Assume \( F \in D_{\geq 0} \).
First, we can reduce to the case $\mathcal{F}^\bullet \in \mathcal{D}_{QCoh}^b(X)$ using the "holimit" construction- because $\mathcal{F}^\bullet = \text{Cone}(\oplus \tau_{\leq n}\mathcal{F} \rightarrow \oplus \tau_{\leq n-1}\mathcal{F})$ so it suffices to work with $\tau_{\leq n}\mathcal{F}$.

By induction, assume that all objects in $\mathcal{D}_{QCoh}^{\leq n, \geq 0}$ are in the image. Note that the base $n = 0$ follows from the fact that an object in $D^+$ that lives in one cohomological degree is isomorphic to a singleton.

For the induction step, consider the distinguished triangle $\tau_{\leq n}(\mathcal{F}^\bullet) \rightarrow \mathcal{F}^\bullet \rightarrow \tau_{\leq n+1}(\mathcal{F}^\bullet) \rightarrow \tau_{\leq n}\mathcal{F}^\bullet[1]$

So that $\mathcal{F}^\bullet$ is quasi-isomorphic to the cone of $\tau_{\leq n+1}(\mathcal{F}^\bullet)[-1] \rightarrow \tau_{\leq n}\mathcal{F}^\bullet$

Now $\tau_{\geq n+1}(\mathcal{F}^\bullet)[-1]$ lives in one cohomological degree so it is in the essential image, as is $\tau_{\leq n}\mathcal{F}^\bullet$ from the induction hypothesis. Because the functor $D^+(\text{QCoh}(X)) \rightarrow D^+_{QCoh}(X)$ is fully faithful, it follows that $\mathcal{F}^\bullet$ is quasi-isomorphic to the cone of this map as a map in $D^+(\text{QCoh}(X))$ so it is in the essential image. □

**Theorem.** Let $\Phi : X \rightarrow Y$ be a quasi-separated quasi-compact morphism (again, quasi-separated is enough) between schemes. Then $R\Phi_*$ sends $D^+_{QCoh}(X)$ to $D^+_{QCoh}(Y)$ and $D^\bullet_{QCoh}(X)$ to $D^\bullet_{QCoh}(Y)$

Proof: a) Take $H^i(R\Phi_*(\mathcal{F}^\bullet)) \in Sh_{\text{QCoh}(Y)}$. Consider an open embedding $\tilde{Y} \hookrightarrow Y$ and the preimage $\tilde{X} \hookrightarrow X$, so that we have the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \rightarrow & X \\
\downarrow & & \downarrow \Phi \\
\tilde{Y} & \rightarrow & Y
\end{array}
$$

We claim

$$R\Phi_*(\mathcal{F}^\bullet) |_Y \cong R(\Phi|_\tilde{X})_*(\mathcal{F}^\bullet|_{\tilde{X}})$$

Note that there is a map $\rightarrow$ by universal property. We need to show that it induces an isomorphism on cohomologies (by a principle proved before). But $H^i(R\Phi_*(\mathcal{F}^\bullet))$ is the sheaf associated to the presheaf $U \mapsto H^i(\Phi^{-1}(U), \mathcal{F}^\bullet)$ - which is a local computation, so we may assume $\tilde{Y}$ and $Y$ are affine, meaning that $X$ is a separated quasi-compact scheme.

Let $Y_f \subset Y$ be a basic open so that $R^i\Phi_*(\mathcal{F}^\bullet)$ is associated to the presheaf $Y_f \mapsto H^i(X_f, \mathcal{F}^\bullet|_{X_f})$ (for all $f$)

It suffices to show that the map

$$H^i(X, \mathcal{F}^\bullet) \rightarrow H^i(X_f, \mathcal{F}^\bullet|_{X_f})$$

is an isomorphism.

Choose a cover of $X$ by affines $X = \sqcup U_i$. We can assume, in virtue of the previous theorem, that $\mathcal{F}^b u$ is represented by quasi-coherent sheaves. Since $X$ is separated, the open embeddings $U_i \rightarrow X$ are affine maps hence the higher Cech cohomologies of $X$ with respect to $\{U_i\}$ are zero and then by a previous theorem the cohomology of $X$ can be computed via $\check{C}(X, \{U_i\}, \mathcal{F}^\bullet)$.

Similarly for $X_f$, we are left to prove that the map $\check{C}(X, \{U_i\}, \mathcal{F}^\bullet) \rightarrow \check{C}(X_f, \{U_i\}, \mathcal{F}^\bullet|_{X_f})$ induces an isomorphism on the cohomologies, which is easy to verify. (Note that localization commutes with finite direct sums).

Another way to prove this assertion is to reduce to $\mathcal{F}^\bullet$ bounded and then a single sheaf as before, and then resort to Cech cohomology.

b) Take $\mathcal{F}^\bullet \in \mathcal{D}_{QCoh}^b(X)$, as $Y$ is compact we can assume $Y$ is affine so then $\Phi$ is affine. Again this implies that $\Phi$ is affine. The cohomologies are again sheaves associated to sheaves of preimages. Because the map is affine

Now assume that $X$ can be covered by $n$ affines, and say $\mathcal{F}^\bullet \in \mathcal{P}^{\geq 0, \leq m}$.

We claim that $H^k(X, \mathcal{F}^\bullet) = 0$ for $k > n + m + 1$ - which would imply the conclusion by Leray spectral sequence.

One way is to use the alternate version of the Cech complex whose total complex dies in degree higher that $n + m$.

Another way is to perform induction on $n$. (Exercise)

It is possible to give an alternative proof of Serre’s theorem in the Noetherian case. Recall that Serre’s theorem says that $R^1\Gamma(X, \mathcal{F}) = 0$ for $X = \text{Spec}(A)$ affine, $\mathcal{F}$ a quasi-coherent sheaf. In what follows, we will assume that $A$ is a Noetherian module and more generally $X$ a Noetherian scheme.
**Proposition.** If \( I \) is an injective \( A \)-module then \( I \) is flasque as a sheaf.

Proof: first we do the case of basic opens (in fact this case will be sufficient for our purposes). Choose \( f \) a non-unit in \( A \), then we want to show \( I \to I_f \) is surjective. Let \( I^{f \cdot \infty} = \cup \text{ann}(f^k) \subset I \).

In this case the proposition is a direct corollary of the following lemma:

**Lemma:** 1) \( I^{f \cdot \infty} \) and \( I_f \) are injective
2) There exists a (canonical) short exact sequence

\[
0 \to I^{f \cdot \infty} \to I \to I_f \to 0
\]

- which in particular by injectivity, implies that

\[
I = I^{f \cdot \infty} \oplus I_f \quad \text{(non-canonically)}
\]

Proof: 1) \( I_f \) being injective over \( A_f \) is equivalent to it being injective over \( A \) by adjunction (recall that localization is adjoint to the forgetful functor that turns \( A_f \)-modules into \( A \)-modules). More generally if \( A \) is Noetherian and \( S \) a multiplicative set then \( I_S \) is injective. \( I_S \) is a direct limit over a filtered set of modules isomorphic to \( I \).

Claim: Over a Noetherian ring a filtered direct limit of injectives is injective.

We need to check the extension property of mapping into. First we claim that it’s enough to check it for finitely generated modules. This is an easy application of Zorn’s lemma: say \( M \subset N \) and we have a map \( f : M \to I \) and \( M \subset N' \subset N \) such that \( N' \) is maximal with the property that \( f \) can be extended to \( N' \). If \( N' \neq N \) choose \( a \neq N' \) then \( < a > \cap N' \) is finitely generated (as \( A \) is Noetherian) hence we can extend the map from \( < a > \cap N' \to < a > \) which means we can extend the map to \( N + a \) contradiction.

Now for finitely generated modules \( M \subset N \) the claim is also easy, since being finitely generated easily implies that the image of \( M \) lands inside one of the modules that form the (filtered!) direct limit, which is injective and then the conclusion follows.

Now we move on to \( I^{f \cdot \infty} \). More generally for any ideal \( a \) we can define \( M^{4 \cdot \infty} \subset M \) be the submodule consisting of all \( m \in M \) such that \( a^nm = 0 \) for some \( n > 0 \).

We claim that if \( I \) is injective then \( I^{4 \cdot \infty} \) is injective. Again we check the extension property for finitely generated modules \( M \subset N \).

If we map \( M \) to \( I^{4 \cdot \infty} \) then the image is killed by some power of \( a \) so we obtain a map \( M/a^nM \to I \). By Artin-Rees the \( a \)-adic topology on \( M \) equals the \( a \)-adic topology on \( N \) restricted to \( M \) - which in particular implies that \( a^nM = a^nN \cap M \) for some \( n \) - so that \( M/a^nM \to N/a^nN \) and thus by injectivity of \( I \) we can produce a larger map \( N/a^nN \to I \) that induces a map \( N \to I \) whose image is killed by \( a^n \) in particular it lands inside \( I^{a,\infty} \) as desired.

2) First assume \( I_f = 0 \) (where \( I_f \) is the kernel of \( I \xrightarrow{f} I \)). Then \( I \to I_f \) is an isomorphism as the short exact sequence \( 0 \to I \xrightarrow{f} I \to J \to 0 \) implies \( I = I \oplus J \) with \( J \) killed by \( f \) to that \( I_f = 0 \) implies \( J = 0 \) i.e. \( I \xrightarrow{f} I \) is an isomorphism and the direct limit \( I_f \) then is clearly \( I \).

Now the exactness of our sequence follows for this argument applied to \( I/I^{f \cdot \infty} \) (note that it’s localization at \( f \) is clearly \( I_f \)). \( \square \)

Let \( U \xrightarrow{i} X \xleftarrow{j} Y \) be a familiar open embedding and closed embedding of the complement.

**Claim.**

\[
R^n j_\ast j^\ast I = 0, n > 0
\]

and \( 0 \to I^{a,\infty} \to I \to j_\ast j^\ast I \to 0 \) where \( a \) is the ideal of \( Y \).

Proof: we know that \( I^{a,\infty} \) is injective therefore it splits as a direct summand, so it’s enough to prove the proposition assuming \( I^{a,\infty} = 0 \) (i.e. there are no sections supported at \( Y \)).
The statement of the proposition says \( I \cong Rj^*j^* \) because then the composite map \( I \to j_*j^*I \to Rj_*j^*I \) being an isomorphism implies the two small maps are also isomorphisms which forms the two parts of the proposition.

To prove this isomorphism we view \( X \) as a sheaf of rings, and we claim that \( D_{Sh(O_X - mod(U))} \to D_{Sh(R - mod(X))} \to D_{Sh(R - mod(U))}(U) \) is an admissible triple.

Indeed, by adjunction \( Hom(Rl_s(\mathcal{F}), Rl_s(\mathcal{F}')) = Hom(j^*RI_s(\mathcal{F}), j^*RI_s(\mathcal{F}')) = 0 \) and every object \( \mathcal{F} \) fits into a distinguished triangle \( \mathcal{F} \to j_*j^*\mathcal{F} \to Cone(\mathcal{F} \to j_*j^*\mathcal{F}) \to \mathcal{F}[1] \) because the cone of \( \mathcal{F} \to j_*j^*\mathcal{F} \) belongs to the essential image of \( Rl_s \) since the restriction of the map to \( U \) is the identity.

With respect to this pair of admissible categories, we see that \( I \to Rj^*j^*I \) if and only if \( I \) is in the essential image of \( Rj_* \) i.e. \( Hom(M^*, I) = 0 \) provided that \( M \) is set-theoretically supported on \( Y \). This is true because by chopping \( M \) as usual, it is enough to show \( Ext^k(M, I) = 0 \) for all \( k \) whenever \( M \) is \( a \)-torsion. This follows from injectivity of \( I \) - note that the case \( k = 0 \) is clear since \( I \) and \( M \) are supported on disjoint sets. \( \square \)

**Proposition. (PSET)** Let \( X = Spec(A) \) be an affine Noetherian scheme. Recall the equivalence of categories \( Loc: A - mod \to QCoh(X) \).

a) \( Loc \) sends injective \( A \)-modules to flasque sheaves.

b) Let \( Y = Spec(A/a) \) be a closed subscheme of \( X \). Let \( A - mod_{a-nilp} \) be the full abelian subcategory of \( A \)-modules consisting of modules, for which every element is annihilated by some power of \( a \). Then \( Loc \) induces an equivalence

\[
D^+(A - mod_{a-nilp}) \to D^+_{QCoh}(X)_Y
\]

c) The evident functor

\[
D^+(A - mod_{a-nilp}) \to D^+(A - mod)
\]

is fully faithful.

Proof: a) This was already done before. We have shown that if \( I \) is injective then \( I \to I_f \) is surjective, and \( I_f \) is an injective \( A_f \)-module.

Now we want to prove that \( \Gamma(U, Loc_A(I)) \to \Gamma(V, Loc_A(I)) \) is surjective if \( V \subset U \) are open in \( Spec(A) \). The aforementioned observations show that this is true when \( V, U \) are basic affines. In general, because \( A \) is Noetherian, \( U \) can be written as a finite union of \( k \) basic affines and \( V \) as a finite union of \( l \) basic affines. We use (strong) induction over \( k \) with a nested induction over \( l \) inside. The base case is \( k = l = 1 \) which was done.

Assume \( k > 1 \), and let \( U = U' \cap U_1 \) where \( U_1 \) is affine and \( U' \) is the union of \( k - 1 \) affines. Let \( t \) be a section over \( V \). According to the induction step there are sections \( s, s' \) over \( U' \) and \( U_1 \) that coincide with \( t \) on \( V \cap U' \) and \( V \cap U_1 \). If \( s \) coincides with \( s' \) on \( U' \cap U_1 \) we are done. Otherwise, their difference is a section \( \delta \) that must restrict to 0 on \( U' \cap U_1 \cap V \). Therefore if we can correct \( s \) by something which restricts to 0 on \( U' \cap V \) and \( \Delta s \) on \( U' \cap U_1 \) we will be done. But this follows from the induction step, since by compatibility 0 on \( U' \cap V \) and \( \Delta s \) on \( U' \cap U_1 \) can be glued to a section on \( U' \cap (V \cup U_1) \).

Now assume that \( k = 1 \) - so we can assume that \( U = Spec(A) \) and \( l > 1 \), let \( V = V_1 \cup \ldots \cup V_l \) and set \( V' = V_1 \cup \ldots \cup V_{l-1} \), and let \( t \) be a section of \( Loc_A(I) \) over \( V \). According to the induction hypothesis, there is a section \( s \in I \) that reduces to \( Res_{V'}t \) over \( V' \) and so by subtracting it we may assume that \( t \) reduces to 0 over \( V' \). Now say \( t \) reduces to \( t_1 \) over \( V_1 = Spec(A_1), l \in A \), then \( t_1 \) reduces to 0 over \( V_1 \cap V' \), which means that \( t_1 \in (I_1)^a \cap \cap \) where \( a_l \) is an ideal inducing a subscheme structure on \( Spec(A) - V' \). But because \( a \) is finitely generated, we deduce \( a_l = (a_l) \) and so \( (I_1)^a = (I_1)^a \). But now \( I_1 \) is injective according to what was proven in class, so in particular \( I_1 \) is injective, which shows that there exists a section \( t \in A \) that reduces to \( t_1 \in A_l \) and it also must restrict to 0 on \( V' \), which completes the induction.

b) It’s pretty clear that any module in \( A - mod_{a-nilp} \) gets annihilated when mapped to the open subscheme \( U \) of \( Spec(A) \) complementary to \( Y \) (in the language of the previous problem, this map is \( j^* \)). And in fact, it’s also quite clear that anything annihilated by \( j^* \) is in \( A - mod_{a-nilp} \).

Conversely, assume that a complex of \( A \)-modules restricts to 0 under \( j^* \). It follows that its cohomologies are in \( A - mod_{a-nilp} \). We now need to to show that it is quasi-isomorphic to a complex of modules in \( A - mod_{a-nilp} \). A
similar procedure has been carried out in a previous problem set - problem 6b) from PS 2. The key observation is that is $0 \to M \to N \to P \to 0$ is a short exact sequence in $A - \text{mod}$ and $M, P$ are $\alpha$-nilpotent, then so is $N$. From there, we proceed absolutely like in that case.

c) Follows pretty directly from part b). One should note that a priori it is not clear why $D^+(A - \text{mod}_{\alpha-\text{nilp}})$ is a full embedding because it is not clear why morphisms in $D^+(A - \text{mod}_{\alpha-\text{nilp}})$ could not be less then in the big category. However once we identify them with $D^+_{QCoh}(X)_Y$ this follows from the fact that the latter category is the kernel of the exact functor $j^*$ and in that case it is pretty clear by inspection that a hut in the big category must induce a hut in the small category. This also patches a hole in the previous part where I neglected to check that the morphisms are indeed the same in the two categories. □

**Proposition.** (PSET) Let $A$ be a Noetherian ring, and $I$ an injective $A$-module.

a) For any multiplicative subset $S \subset A$, the localization $I_S$ is injective as an $A$-module (equivalently, as an $A_S$-module), and the canonical map $I \to I_S$ is surjective.

b) $I$ can be represented as a direct sum $\oplus_{p \in \text{Spec}(A)} I^p$ where each $I^p$ is an $A_p$-module, and is $p$-torsion (i.e. belongs to $A - \text{mod}_{p-\text{nilp}}$).

c) Let $U \subset \text{Spec}(A)$ be an open subset. In terms of the above decomposition,

$$\Gamma(U, \text{Loc}(I)) = \oplus_{p \in U} I^p$$

d) Again, $\text{Loc}(I)$ is flasque.

e) Each $I^p$ is isomorphic to a direct sum of injective hulls of the residue field $k_p$, considered as an $A$-module.

Proof: a) It was already done before. $I_S$ is injective because it is the filtered direct limit of $I$ over the poset $s \in S$ indexed by divisibility, where the map from $I - "s"$ to $I - "st"$ is multiplication by $t$. As shown before, a filtered direct limit of injectives is injective (in the Noetherian case). Also shown before was that $I \to I_f$ is surjective and since $(I_f)_{f \in S}$ is another filtered set whose direct limit is $I_S$, we conclude that $I \to I_S$ is surjective.

b) For an injective $A$-module $I$ we will denote by $I_p$ the localization of $I$ at $p$, and by $I^p$ the submodule of $I^p$ consisting of $p$-torsion.

Recall that $I_p$ is injective and $I$ can be written as a direct sum of $I_f$ and $f$-torsion of $I$ if $f \in A$.

Claim 0: $I_p$ is a direct summand of $I$.

Proof: the kernel of $I \to I_p$ consists of all elements killed by some element not in $p$. This is the union of all $I_f^\infty$ for $f \not\in p$, which can be written as a direct limit once we order them by divisibility. As done in class, a filtered direct limit of injectives is injective, and also as shown in class, this implies that the short exact sequence splits.

Claim 1: $I^p = \text{Ker}(I_p \to \oplus_{p' \not\subset p} I_p)$. (In particular if $p$ is a minimal prime then $I^p = I^p$). Also, it is injective and therefore a direct summand of $I_p$.

Proof: As we know $\text{Ker}(I_p \to I'_p)$ consists of the union of $\text{Ker}(I_p \to (I_p)_f)$ over $f \in p - p'$ because $I_p$ is the filtered direct limit of $(I_p)_f$. This union is the union of all $f$-torsions, i.e. elements $a$ in $I_p$ such that $fa = 0$ for some $f \in p - p'$. Now if we take the direct sum over all $p'$, we get elements $a \in I_p$ that are annihilated by some element in $p = p'$ for all $p' \subseteq p$. That is to say that $\text{Ann}(a) \subseteq p$ is not contained in any $p' \subseteq p$. This is equivalent to saying that $\text{rad}(\text{Ann}(a))$, being the intersection of all primes containing it, is equal to $p$, which is equivalent to $a \in I^p$. The second part follows from class, as $I^p$ is just $(I_p)^\infty$ which is injective.

Claim 2: If $I$ is injective and non-zero, then $I_p \neq 0$ for some $p \in \text{Spec}(A)$.

Proof: as the kernel of $I \to I_f$ is $f$-torsion, if $I_f = 0$ then every element $a$ of $I$ is $f$-torsion for some $f \not\in p$. Like in the previous proposition, this implies that $\text{rad}(\text{Ann}(a)) = A$ which means that $a$ is killed by 1 so $I = 0$.

Claim 3: If $I$ is injective and non-zero, then $I^p \neq 0$ for some $p \in \text{Spec}(A)$.

Proof: if $I$ has no torsion at all, then in particular $A$ must be an integral domain so then $I^0 = I$ which is non-zero. Otherwise, let $a$ be torsion and let $J$ be the radical of its annihilator. Consider a maximal such $J$, which exists since $A$ is Noetherian. We claim that $J$ is prime - otherwise is $pq \in J$ and $p, q \not\in J$ then $pa$ has larger radical of annihilator. But the $a \in I^J$ so $I^J$ is non-zero.

47
Claim 4: The natural map $\bigoplus I^p \to I$ is injective.

Proof: say that $\sum a_i = 0 \in I$ where $a_i \in I^p_i$. By prime avoidance lemma, we conclude that a maximal ideal $p_1$ among $p_i$ is not a subset of $\sum p_i$. Subsequently by multiplying by something in $p_1$ but not in some $p_i$ raised to a high power, we can kill of $a_1$ but not $a_i$ and decrease the number of summands. We will eventually end up with just one non-zero summand, which will produce a contradiction since the map $I^p \to I_p \to I$ is injective.

Claim 5: $I = \bigoplus I^p$.

Proof: according to the previous claim, since $\bigoplus I^p \to I$ is injective we write $I = \bigoplus I^p \oplus J$ for $J$ injective. We then must conclude that $I^p = (\bigoplus I^p)^p = I^p \oplus J^p$ hence $J^p = 0$ for any $p$. By claim 3, $J = 0$.

c) It will be $\bigoplus p \in U \bigoplus I^p$. This is because in the above decomposition, for $f \in A$ we will have $\Gamma(I, X_f) = I_f = \bigoplus_{f \vartriangleleft p}(I^p)^f$ (because all other terms die) and $(I^p)^f = I^p$ since it is already a $A_p$-module and $f$ is invertible in $A^p$.

Subsequently, if $U = \cup_{i=1}^n X_{f_i}$ then $\Gamma(I, X_f)$ by gluing consists of all $n$-tuples of sections in $\bigoplus_{f_\vartriangleleft p}(I^p)$ that coincide on common summands - and it’s straightforward to identify this with $\bigoplus p \in U \bigoplus I^p$.

d) Immediate from c).

e) Recall that the injective hull of a module is both the smallest injective module containing it and the largest extension of it. It turns out that every module has an injective hull, which is unique up to isomorphism.

First, let’s show that this injective hull $M_p$ maps into $I^p$. We claim $I^p$ contains a copy of $k_p$. Indeed, let $p$ be generated by $f_1, \ldots, f_r$ and say $a \in I^p$ - so that $f_i^{k_i}$ kills $a$ for some $k_i$. Consider $(l_1, \ldots, l_r) \in \mathbb{N}^r$ be a minimal vector in the lexicographic order such that $\prod f_i^{l_i}$ does not kill $a$ (it exists as $l_i \leq k_i$). Then $b = \prod f_i^{l_i}a$ is non-zero and is killed by $p$. Since anything not in $p$ is invertible in $A_p$, we conclude that the submodule generated by $b$ is isomorphic to $k_p$. Therefore as $I^p$ is injective, $M_p$ maps into $I^p$. Because the kernel of this map must meet $k_p$ unless it is 0 by the property of being an injective hull, but $k_p \to I^p$ is injective, we conclude that $M_p$ injects into $I^p$ so splits as a direct summand.

Now let’s consider all configurations $S, \bigoplus_{s \in S} (M_p)_s \hookrightarrow I^p$ and consider a maximal such configuration by the natural ordering, which exists by Zorn’s lemma. If the image is not $I^p$ then $I^p = \bigoplus_{s \in S} (M_p)_s \oplus I^p$ for some injective $I'_p \neq 0$. But $M_p$ can be injected further into $I'_p$, by the previous argument, which contradicts maximality.

**Proposition. (PSET)** Let $X$ be a Noetherian scheme, and $\mathcal{J}$ be an injective object of $QCoh(X)$.

a) Let $U \subset X$ be an open subscheme, then $\mathcal{J}|_U$ is an injective object of $QCoh(U)$.

b) $\mathcal{J}$ is flasque as a sheaf on $X$.

c) All injective objects of $QCoh(X)$ are direct sums of skyscraper sheaves $I_x$ where for $x \in X$, $I_x$ is the skyscraper sheaf obtained from a torsion injective $(O_X)_x$-module.

Proof: a) Recall that restriction is $j^*$ which is exact and $j_*$ is left-exact and $j^* j_* \cong id$ (8a from PSET 7).

We will follow both the hint and the strategy from the book.

Step 0: First we claim that if $j: U \to X$ is the open embedding of an affine and $\mathcal{J}$ is injective over $U$ then $j_* \mathcal{J}$ is injective over $X$. Indeed, say $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ and $\mathcal{F}_1$ maps to $j_* \mathcal{J}$. Then as $j^*$ is exact, $j^* \mathcal{F}_1 \hookrightarrow j^* \mathcal{F}_2$ and $j^* \mathcal{F}_1$ maps to $\mathcal{J}$ by adjunction. Thus we lift it to a map $j^* \mathcal{F}_2 \to \mathcal{J}$ and get a map $\mathcal{F}_2 \to j_* \mathcal{J}$ by adjunction.

Step 1: Assume that $X$ is affine. Now let $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ be an injection of sheaves on $U$, and say $\mathcal{F}_1$ maps to $j^* \mathcal{J}$. Then $j_* \mathcal{F}_1 \hookrightarrow j_* \mathcal{F}_2$ is an injection of sheaves on $X$, and $j^* \mathcal{F}_1$ maps to $j_* j^* \mathcal{J}$. Now $j_* j^* \mathcal{J}$ is injective from the previous problem as it corresponds to the module which is the section of $\mathcal{J}$ over $U$. So we produce a map $j_* \mathcal{F}_2 \to j_* j^* \mathcal{J}$ which by restriction yields a map $j^* j_* \mathcal{F}_2 \to j^* j_* j^* \mathcal{J}$ which is the same as $\mathcal{F}_2 \to j^* \mathcal{J}$. (Of course there is a commutative diagram to do, but I feel like we’ve done this enough times).

Step 2: Now assume that $\mathcal{J} = (j_1)_* \mathcal{J}_1$ where $j_1$ is the embedding of an affine open subscheme $U_1$ and $\mathcal{J}_1$ is injective over $U_1$. In that case we need to work with $j^* (j_1)_* \mathcal{J}_1$. However it is easy to see from definition that this is the same as $(j_2)_* (j_3)^* \mathcal{J}_1$ where $j_3$ is the open embedding $U_1 \cap U \to U_1$ and $j_2$ is the open embedding $U_1 \cap U \to U$.

Now by the previous step $(j_3)^* \mathcal{J}_1$ is injective and so by step 0 $(j_2)_* (j_3)^* \mathcal{J}_1$ is also injective.

48
Step 3: We now follow the instructions in corollary III.3.6. p. 215 of Hartshorne. By choosing an open cover of $X$ by finitely many affines $U_i$, and if we let $\mathcal{I}_i = j_i^*(U_i)$ then we embed $\mathcal{I}_i$ into injectives $M_i$ so $\mathcal{I}$ embeds into $\oplus (j_i)_* M_i$. The latter is injective from step 0 so $\mathcal{I}$ splits as a direct summand of it. But then $j^* \mathcal{I}$ splits as a direct summand of $j^*(\oplus (j_i)_* M_i) = \oplus j^*(j_i)_* M_i$ which is injective by step 3.

b) Will follow immediately from what we do in part c).

c) On any affine $U$ $\mathcal{I}$ produces an injective module which therefore splits as a direct sum of injective modules $I^x$ for every point $x \in U$ according to the previous problem. Next note that $I^x$ does not actually depend on the affine chosen - it is the submodule of the stalk $\mathcal{I}_x$ which is annihilated by some power of the maximal ideal of the local ring at $x$. Therefore $I_x$ are well-defined, and we claim that $\mathcal{F}(\mathcal{I}, U) = \oplus_{x \in U} I_x$ with natural restriction maps. Indeed, this is obviously a sheaf (this was in fact checked in 5c) for affines but the method applies) and it coincides on affines with $\mathcal{I}$ - by problem 5c), so we get the conclusion.

We have yet to check that the sheaf produced is injective. This will be done in a forthcoming proposition. $\square$

Definition. (PSET) Let $X$ be a topological space with a sheaf of rings $R_X$. For $\mathcal{F}_1, \mathcal{F}_2 \in Sh_{R_X-mod}(X)$ set $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ to be a sheaf of abelian groups given by

$$\Gamma(U, (\mathcal{F}_1, \mathcal{F}_2)) = \text{Hom}_{Sh_{R_X-mod}}(\mathcal{F}_1 |_U, \mathcal{F}_2 |_U)$$

It's called the "internal Hom" between $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition. (PSET) a) $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ is indeed a sheaf.
b) For a sheaf $\mathcal{F}$ of abelian groups, we have a canonical isomorphism

$$\text{Hom}_{Sh_{A}(X)}(\mathcal{F}, \text{Hom}(\mathcal{F}_1, \mathcal{F}_2)) \cong \text{Hom}_{Sh_{R_X-mod}(X)}(\mathcal{F} \boxtimes \mathcal{F}, \mathcal{F})$$

c) If $\mathcal{F}_2$ is injective, then for any $\mathcal{F}_1$, the sheaf $\text{Hom}(\mathcal{F}_1, \mathcal{F}_2)$ is flasque.
d) For a sheaf of $R_X$-modules to be injective is a local property. That is, if $\mathcal{F}$ is such that $X$ admits a cover $U_i$ such that each $\mathcal{F} |_{U_i} \in Sh_{R_{U_i}-mod}$ is injective, then $\mathcal{F}$ is injective; and vice versa.

Proof: a) The presheaf axioms are easy, it only suffices to prove the gluing axiom. The uniqueness part in the gluing axiom, since a morphism of sheaves is determined locally. So now assume that $X$ is covered by open sets $U_i$ and we have morphisms $f_i|_{\text{colim} \mathcal{F} |_{U_i}} \rightarrow \mathcal{F}_1 |_{U_i}$ that are compatible with intersections, and we want to build a morphism $f: \mathcal{F} \rightarrow \mathcal{F}_1$ lifting them (we can consider that the open set is $X$, without loss of generality). Choose an open set $V \subset X$ and let $V_i = V \cap U_i$. For any section $s \in \mathcal{F}(X, V)$ let $s_i$ be its restriction to $V_i$. We claim that $t_i = f_i(s_i)$ are compatible on intersections. Indeed, $t_i$ on $V_i \cap V_j$ is the same as $f_i$ applied to the restriction of $s$ to $V_i \cap V_j$ - but that also equals $f_j$ applied to the restriction of $s$ to $V_i \cap V_j$ because the morphisms are compatible. So $t_i$ glue to make a section $t \in \Gamma(V, \mathcal{F})$, and we set $f(s) = t$. From the way we have constructed $f$ it is clear that it commutes with restrictions so it is indeed a sheaf morphism.

b) Let’s build the maps. Consider a section of the left-hand side: $f: \mathcal{F} |_V \rightarrow \text{Hom}(\mathcal{F}_1, \mathcal{F}_2) |_V = \text{Hom}(\mathcal{F}_1 |_V, \mathcal{F}_2 |_V)$. Then we produce a map from $(\mathcal{F} \boxtimes \mathcal{F}_1) |_V$ to $\mathcal{F}_2 |_V$ as follows: it is enough to do it locally, and on sufficiently small opens sections are just tensor products over $\mathbb{Z}$ of sections so we use the isomorphism $\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \boxtimes \mathbb{Z} B, C)$. (In fact we can decompose $X$ into connected components and work with one at a time). The converse procedure is just doing this backwards, and because locally we are just switching between $\text{Hom}(A, \text{Hom}(B, C))$ and $\text{Hom}(A \boxtimes \mathbb{Z} B, C)$ back and forth, the procedures are inverse to each other.

c) We need to show that $\mathcal{F}_1 |_V \rightarrow \mathcal{F} |_V$ lifts to $\mathcal{F}_1 |_V \rightarrow \mathcal{F} |_V$. It is enough to do it for $V = X$ as we can lift then restrict to $V$. Let $j$ be the open embedding $U \hookrightarrow X$ and recall from last semester that the restriction $j^*$ admits a left adjoint $j_!$ with $id \cong j^*j_!$. Therefore from $j^* \mathcal{F}_1 \rightarrow j^* \mathcal{F}$ we produce a map $j_!j^* \mathcal{F}_1 \rightarrow \mathcal{F}$ by adjunction which lifts to a map $\mathcal{F}_1 \rightarrow \mathcal{F}$ because $j_!j^* \mathcal{F}_1 \rightarrow \mathcal{F}_1$ is injective, as easily seen from the definition of $j_!$ (extension by 0). This is the desired map.
d) First we claim that to be acyclic for cohomologies is a local property (in fact we only use the weaker \( H^1(\mathcal{F}) = 0 \)). Indeed, using \( j_i \) as in c) we readily deduce that an injective restricts to injectives. Now if \( \mathcal{F} \) is locally acyclic then choose an injective resolution of \( \mathcal{F}, \mathcal{I}_0 \to \mathcal{I}_1 \to \ldots \) and we compute the cohomology complex map \( \Gamma(\mathcal{I}_1) \to \Gamma(\mathcal{I}_2) \to \ldots \). To show \( \mathcal{F} \) has 0 first cohomology we need to show that the kernel \( \Gamma(\mathcal{I}_1) \) into \( \Gamma(\mathcal{I}_2) \) is \( \Gamma(\mathcal{F}) \) knowing that the kernel of \( \Gamma(\mathcal{I}_1, U_i) \) into \( \Gamma(\mathcal{I}_2, U_i) \) is \( \Gamma(\mathcal{F}, U_i) \). Indeed, if we choose a section \( s \in \Gamma(\mathcal{I}_1) \) in the kernel, its restrictions to \( U_i \) must also be in the corresponding kernels so must come from \( \Gamma(\mathcal{F}, U_i) \). By compatibility they glue to a section in \( \Gamma(\mathcal{F}) \) as desired.

Take now an injection \( \mathcal{I}_1 \hookrightarrow \mathcal{I}_2 \) of sheaves - then the induce a map \( \text{Hom}(\mathcal{I}_2, \mathcal{F}) \to \text{Hom}(\mathcal{I}_1, \mathcal{F}) \) whose kernel is, as easily seen over every section, just \( \text{Hom}(\mathcal{I}_2/\mathcal{I}_1, \mathcal{F}) \). In particular it is locally flasque using the previous part since \( \mathcal{F} \) is injective locally, so it has zero first cohomology.

The map \( \text{Hom}(\mathcal{I}_2, \mathcal{F}) \to \text{Hom}(\mathcal{I}_1, \mathcal{F}) \) is locally surjective (surjective on every section over \( U_i \) so in particular surjective on \( U_i \)) as \( \mathcal{F} \) is locally injective, hence it is surjective as a map of sheaves. By taking the long exact sequence of cohomology, and since \( H^1(\text{Hom}(\mathcal{I}_2/\mathcal{I}_1, \mathcal{F})) = 0 \) we conclude that \( \Gamma(\text{Hom}(\mathcal{I}_2, \mathcal{F})) \to \Gamma(\text{Hom}(\mathcal{I}_1, \mathcal{F})) \) is a surjection, which is what we needed. \( \square \)

**Proposition (PSET)** Let \( X \) be a Noetherian scheme. The property for \( \mathcal{F} \in QCoh(X) \) to be injective is local.

Proof: One direction was given by part a) of the pre-previous proposition. Now assume \( \mathcal{F} \) is locally injective over a finite cover \( U_i \). We produce the injective modules \( \mathcal{F}^x \) over the stalks of \( x \) (which again do not depend on the \( U_i \) chosen) and then we know that locally \( \mathcal{F} \) looks like the sheaf described in c) of the pre-previous proposition. In that case \( \mathcal{F} \) is a direct sum so we may assume just that a single \( \mathcal{F}_x \) in non-zero. In that case, mapping \( \mathcal{F}_1 \) to \( \mathcal{F} \) will simply mean mapping every section over opens containing \( x \) to \( \mathcal{F}_x \) - which is the same as mapping the stalk over \( x \) to \( \mathcal{F}_x \). Since taking stalks at \( x \) is left exact, and \( \mathcal{F}_x \) is injective over the stalk of \( X \) at \( x \), we conclude that \( \mathcal{F} \) is injective. \( \square \)

**Proposition. (PSET)** Let \( \Phi: X \to Y \) be a morphism between separated quasi-compact schemes. Then the following diagram is commutative:

\[
\begin{array}{ccc}
D^+_{QCoh}(X) & \xrightarrow{R\Phi^*} & D^+_{QCoh}(Y) \\
\downarrow & & \downarrow \\
D^+(QCoh(X)) & \to & D^+(QCoh(Y))
\end{array}
\]

where the bottom horizontal arrow is the right derived functor of the functor \( \Phi_*: QCoh(X) \to QCoh(Y) \).

Proof: We follow closely the strategy for \( R\Gamma \). Like there, we need to prove that every \( \mathcal{F} \in QCoh(X) \) can be embedded into some \( \mathcal{I} \in QCoh(X) \) which is injective as an object of \( QCoh(X) \) and acyclic for \( R\Phi_* \) as an object of \( Sh_{QCoh-mod}(X) \).

Like there, we will cover \( X \) by finitely many affine \( U_i \) and inject \( \mathcal{F} \) into \( \oplus ((j_i)_*) \mathcal{I}_i \) and claim that they do the job, where \( \mathcal{I}_i \) are injective over \( U_i \).

The difference here though, is that we want \( U_i \) to be small enough so that the image of \( U_i \) lands inside some affine of \( Y \) - this can be done as \( Y \) is quasi-compact. Then because \( Y \) is separated, the map \( \Phi \circ (j_i)_* \) will be affine since it will basically be \( j^* \) of a map of affine schemes where \( j \) is an embedding of an affine scheme into \( Y \) (so \( j \) is affine).

We then conclude just like before: \((j_i)_*(\mathcal{I}_i)\) is injective in \( QCoh(X) \) because \((j_i)_*\) is right adjoint to the exact functor \((j_i)^*\).

And we use Leray (second problem 5 from PS 3) to deduce that \( R\Phi_* (R(j_i)_* \mathcal{I}_i) = R(\Phi \circ j_i)_* \mathcal{I}_i \). Now the right-hand side is just \( (\Phi \circ j_i)_* \mathcal{I}_i = \Phi_* ((j_i)_* \mathcal{I}_i) \) because the map \( \Phi \circ j_i \) is affine, and the left-hand side is \( R\Phi_* ((j_i)^* \mathcal{I}_i) \) because \( j_i \) is affine. So we deduce \( R\Phi_* ((j_i)^* \mathcal{I}_i) = \Phi_* ((j_i)_* \mathcal{I}_i) \) i.e. \((j_i)_* \mathcal{I}_i\) is acyclic, as desired. \( \square \)

02/23/2010
Let $A$ be a commutative ring, and $M$ be an $A$-module. We define the support of $M$, denoted $\text{Supp}(M)$, to consist of all prime ideals $p \in \text{Spec}(A)$ such that $M_p = M \otimes_A A_p$ is non-zero.

**Lemma.** a) $\text{Supp}(M) \subset V(\text{ann}(M))$

b) If $M$ is finitely generated, the inclusion is an equality.

Proof: If $p$ does not contain an element $x$ of $\text{Ann}(M)$, then $M_p$ is automatically zero since $x$ becomes invertible in $A_p$. So for $p$ to belong to $\text{Supp}(M)$ it must contain all elements of $\text{Ann}(M)$ which shows a). For b), assume that $M_p$ is 0 so that if $m_1, \ldots, m_r$ are generators of $M$, then $m_i$ is killed by some $b_i$ not in $p$ which immediately implies that $b_1b_2\ldots b_r \notin p$ kill $M$. □

It follows from b) that the support of a finitely generated module is a closed set. This is not true for modules which are not finitely generated: for example, if $S$ is a set of of primes of $\mathbb{Z}$ then $\oplus_{p \in S} \mathbb{Z}/p\mathbb{Z}$ has support $S$, which may not be closed. In the future, we will usually assume $A$ is Noetherian and $M$ is finitely generated.

**Definition.** An associated prime of $M$ is the generic point of an irreducible component of the support of some submodule $M'$ of $M$, i.e. one of the minimal primes in $\text{Supp}(M')$.

**Proposition.** $p$ is an associated prime of $M$ if and only if there exists $m \in M$ such that $p = \text{ann}(m)$ - equivalently there is an injection $A/p \hookrightarrow M$. (Recall that we assume $M$ is finitely generated)

Proof: the $\Leftarrow$ is clear. Converse, let’s assume $p$ is a minimal prime of $\text{Supp}(M)$ - since we can replace $M$ by a smaller $M'$ otherwise. Let $p_1 = p, \ldots, p_k$ be all the (finitely many) irreducible components of $\text{Supp}(M)$ and consider the map $M \to \oplus_i M_{p_i}$. Take the kernel of $M \to \oplus_{i>1} M_{p_i}$. We claim that the localization of the kernel at $p$ is $M_p$. It’s easy to see that this is equivalent to the following fact: for every $m \in M$ there exists $a \notin p$ such that $am$ is killed by some element $b_i$ not in $p_i$ for all $i = 2, 3, \ldots, k$. Indeed, let’s choose elements $a_i \in p_i - p, b_i \in p - p_i$. Choose $a = \prod a_i$ then for every $i, ab_i \in p \cap p_2 \cap \ldots \cap p_k = \text{rad}(\text{Ann}(M))$ hence for some $r$ (universal for all $i$) we have $(ab_i)^r m = 0$ and replacing $a$ by $a^m$ and $b_i$ by $b_i^m$ we get $b_i(am) = 0$ which implies the conclusion. Therefore we can replace $M$ by the kernel of $M \to \oplus_{i \geq 2} M_{p_i}$ and this will imply that $\text{Supp}(M) = V(p)$ is irreducible. Because $M$ is finitely generated $p^k$ kills $M$. We can assume $k = 1$. Indeed, otherwise choose an element $a \in p$ that does not kill the entire $M$ and replace $M$ by the smaller submodule $aM$, and repeat if needed. If $p$ kills $M$ then $\text{Ann}(M) = p$, and if $M$ is generated by $m_1, \ldots, m_r$ then $\text{Ann}(m_i) \supset p$ and the inclusion cannot be strict in all cases, otherwise choosing $a_i \in \text{Ann}(m_i) - p$ we get $a_1a_2\ldots a_r \notin p$ killing $M$. If $\text{Ann}(m_i) = p$ then the map $x \to xm$ is an injection $A/p \hookrightarrow M$. □

**Corollary.** If $A$ is Noetherian and $M$ is finitely generated over $A$ then there is a filtration $0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$ such that $M_i/M_{i+1} \cong A/p_i$ with $p_i$ being all the associated primes of $M$:

$$\{p_1, p_2, \ldots, p_n\} = \text{Ass}(M)$$

In particular, the set of associated primes is finite.

Proof: By the previous injection we can find an injection $A/p_1$ into $M$ and call it $M_1$, then work with $M/M_1$ and call $M_2$ the preimage in $M$ of $A/p_2$ in $M/M_1$ and so on. The process will end by Noetherian-ness. For proving that all associated primes are listed, it is enough to show two small propositions and use induction.

First, we claim that if $0 \to M \to N \to P$ then $\text{Ass}(N) \subset \text{Ass}(M) \cup \text{Ass}(P)$. Indeed, assume that $p$ is an associated prime of $N$ so that there is an injection $A/p$ into $N$. We compose this map with the the projection to $P$. If it is still injective, then $p$ is in $\text{Ass}(P)$. Otherwise, let $b \in A/p$ project to 0 i.e. it’s image lands in $M \subset N$. Then $bA/p$ gets mapped (injectively) inside $M$, but since $A/p$ is an integral domain, it is isomorphic to $A/p$ so that $p$ is an associated prime of $p$.

The second proposition is that $\text{Ass}(A/p) = \{p\}$. Indeed, if $A/q \hookrightarrow A/p$ for $q \neq p$ then the image of 1 shows that the map is $x + q \to ax + p$. In particular $aq \subset p$ is the map is well-defined. If $q \not\subset p$ then $a \in p$ which immediately implies the map is not injective. If $q \subset p$ then if $b \in p - q$ then $b + q$ will get mapped to $ab + p = p$ which again means the map is not injective. □
Corollary. If \( f \in A \) then \( M^f \neq 0 \) if and only if \( f \) is contained in one of the associated primes. (We let \( M^f \) be the kernel of \( M \rightarrow M \))

Proof: If \( f \) is contained in \( p \) and \( A/p \rightarrow M \) then the image of \( 1+p \in A/p \) via this map is non-zero and killed by \( f \). Conversely, take the submodule \( M^f \) and consider one of the irreducible components of its support - which is a prime which contains \( \text{Ann}(M^f) \) in particular \( f \).

Proposition. Let \( a \subset A \) be an ideal. Then the following are equivalent:

i) \( a \not\subset p \) for any \( p \in \text{Ass}(M) \)

ii) \( M^a = 0 \)

iii) For some \( f \in a, M \rightarrow M \) is injective

Proof: It is obvious that iii) implies ii). If \( a \) is contained in \( p \) with \( A/p \rightarrow M \) the the image of \( 1+p \) is non-zero and contained in \( M^a \) which shows that ii) implies i). It remains to show that i) implies iii). Indeed, assume \( M \rightarrow M \) is not injective for all \( f \) - which implies that every \( f \in p \) belongs to an associated prime by the previous corollary. Since there are finitely many of them, the entire \( p \) must be contained in one of the associated primes, by prime avoidance lemma. \( \square \)

Definition. Let \( A \) be a Noetherian ring and \( a \) an ideal of \( A \). If \( M \) is a finitely generated ideals then its depth, \( \text{depth}(M) \) is the maximal integer \( k \) such that for all \( i < k, \text{Ext}^i(A/a, M) = 0 \).

Example: \( \text{depth}(M) = 0 \) if and only if \( M^a = 0 \) - because \( \text{Ext}^0(A/a, M) = \text{Hom}_A(A/a, M) \cong M^a \) via the identification of a map with the image of 1 (which must be killed by \( a \)).

Corollary. \( \text{depth}(M) > 0 \) if and only if there exists \( f \in a \) such that \( M \rightarrow M \) is injective,

Definition. A sequence \( f_1,f_2,\ldots,f_n \) of elements in \( a \) is called a regular sequence if \( M \rightarrow M \) is injective, \( M/f_1M \rightarrow M/f_1M \) is injective, \( \ldots \), \( M/(f_1,f_2,\ldots,f_{n-1})M \rightarrow M/(f_1,f_2,\ldots,f_{n-1})M \) is injective.

Proposition. The following are equivalent:

1) \( \text{depth}(M) \geq k \)

2) For any regular sequence \( f_1,f_2,\ldots,f_i \) with \( i < k \) there exists an \( f_{i+1} \) such that \( f_1,\ldots,f_{i+1} \) is regular.

3) There exists a regular sequence \( f_1,f_2,\ldots,f_k \).

Proof: the implication 2) \( \iff \) 3) is clear

Lemma: Let \( f \) be regular on \( M \). Then \( \text{depth}(M) \geq k \) if and only if \( \text{depth}(M/fM) \geq k-1 \).

Proof: take the short exact sequence \( 0 \rightarrow M \rightarrow M \rightarrow M/fM \rightarrow 0 \) and use the long exact sequence of \( \text{Ext} \) that in particular gives \( \ldots \rightarrow \text{Ext}^{k-2}(A/a, M) \rightarrow \text{Ext}^{k-2}(A/a, M/fM) \rightarrow \text{Ext}^{k-1}(A/a, M) \rightarrow \ldots \)

If \( \text{depth}(M) \geq k \) then \( \text{Ext}^{k-1}(A/a, M) = \text{Ext}^{k-2}(A/a, M) = 0 \) hence we deduce \( \text{Ext}^{k-2}(A/a, M/fM) = 0 \) from the above, and similarly \( \text{Ext}^{i-2}(A/a, M/fM) = 0 \) for any \( i < k \), hence \( \text{depth}(M/fM) \geq k-1 \). Conversely, if \( \text{depth}(M/fM) \geq k-1 \) we conclude that the maps \( \text{Ext}^{i}(A/a, M) \rightarrow \text{Ext}^{i}(A/a, M) \) are injections for \( 1 \leq i \leq k \) and since \( \text{Ext} \) is functorial but \( f \) is zero on \( A/a \), these maps are 0 which is possible only for \( \text{Ext}^{i}(A/a, M) = 0 \) for \( 1 \leq i \leq k \). The same holds for \( i = 0 \) since \( f \) is regular. The lemma is proved.

This shows the implication 1) \( \iff \) 2) by induction: both the base and the induction follow from the assertion that \( \text{depth}(M) > 0 \) implies the existence of a regular element \( f \) which follows from what was done before since \( \text{depth}(M) > 0 \) implies \( M^a = 0 \) - which then implies there exists an element \( f \in a \) such that \( M \rightarrow M \) is injective i.e. \( f \) regular.

The implication 3) \( \iff \) 1) also follows from the lemma using induction, in a straightforward way. \( \square \)

Remark: we used the functoriality of transformation to show that if something kills an object, it kills its transform, too. The rigorous justification of this reasoning is as follows: if we have that \( a \in A \) kills \( X \), and \( F \) is a functor which
commutes with the action of $A$, then $a$ must kill $FX$ because we have the following commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{a} & X \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & 0
\end{array}
$$

which, after applying $F$ and using that $Fa$ is the action of $a$ on $FX$ produces the diagram

$$
\begin{array}{ccc}
FX & \xrightarrow{a} & FX \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & 0
\end{array}
$$

which exactly tells that $a$ kills $FX$. Of course, the action of $a$ on an object can more generally be replaced by any map between two objects, with commutativity conditions with $F$ if needed.

**Corollary.** Suppose $\text{depth}(M) \geq k$ and let $N$ finitely generated $A$-module be such that $\text{Supp}(N) \subset V(a)$. Then $\text{Ext}^i_A(N, M) = 0$, $\forall i < k$

Proof: We induct on $i$. For $i = 0$, $\text{Ext}^0(N, M) = \text{Hom}(N, M)$. But every element of $N$ is killed by some power of $a$ and $M$ has no $a$-torsion since $\text{depth}(M) > 0$.

For higher $i$, first assume $a$ kills $N$ - then we can choose a short exact sequence $0 \to N' \to (A/\langle a \rangle)^{\oplus n} \to N \to 0$ from which we deduce the exact sequence $\text{Ext}^{i-1}(N', M) \to \text{Ext}^i(N, M) \to \text{Ext}^i(A/a, M)^{\oplus n}$, in which the first term is zero by the induction assumption and the second term is zero according to the previous proposition.

Finally, for the general step $a^m$ killing $N$, we choose $N_1$ be the $a$-torsion, and get $0 \to N_1 \to N \to N/N_1 \to 0$ where $N/N_1$ is killed by $a^{m-1}$, and now we also induct on $m$ which immediately implies the conclusion by looking at the exact sequence $\text{Ext}^i(N/N_1, M) \to \text{Ext}^i(N, M) \to \text{Ext}^i(N_1, M)$. □

**Proposition.** Assume that $A' \xrightarrow{\phi} A$ is a surjection of Noetherian rings, and let $a' = \phi^{-1}(a)$.

Let $M$ be an $A$-module. Then

$$
\text{depth}_{A, a}(M) - \text{depth}_{A', a'}(M)
$$

Proof: Assume $\text{depth}_{A, a}(M) \geq k$. We look at $R^i\text{Hom}_{A'}(N', M) = R\text{Hom}(N', M[i])$ where $N'$ is annihilated by $a'$, and $i < k$.

We have the following pair of adjoint functors

$$
\begin{array}{ccc}
D(A' - \text{mod}) & \xrightarrow{\text{For}} & D(A - \text{mod})
\end{array}
$$

where the left adjoint to the forgetful functor is the left derived functor of $- \otimes A$ the tensor product with $A$: $N' \to N' \otimes_{A'} A$ (it can be obtained, as we know, by taking a projective resolution of $N'$ as an $A'$ modules, then tensoring it with $A$).

To prove the adjointness it is suitable to pass to the admissible subcategories of projectives - as we know tensoring with $A'$ sends projectives to projectives (because it is left adjoint to the forgetful functor which is exact). There, we use the fact that this subcategory is isomorphic to the derived category and the adjointness of the forgetful functor and the usual tensor product.

We thus have $R\text{Hom}_{A'}(N', M[i]) \cong R\text{Hom}_A(N' \otimes_{A'} A)$. 53
Now we claim $H^j(N' \overset{L}{\otimes} A)$ are supported on $V(a)$.

Indeed, by functoriality, $N' \overset{L}{\otimes} A$ is killed by any power of $a$ which implies the claim.

More generally, we have the following claim:

**Lemma.** If $N^\bullet$ is an object of $D^{\leq 0}(A)$ whose cohomologies are supported on $V(a)$, then if $M \in Mod(A)$ has
depth $\geq k$, then $Ext^i(N^\bullet, M) = 0$ for $i < k$.

Proof: we use induction on $i$. The distinguished triangle $\tau^{\leq 0}(N^\bullet) \rightarrow N^\bullet \rightarrow \tau^{>0}N^\bullet \rightarrow \tau^{\leq 0}(N^\bullet)[1]$ has $\tau^{>0}(N^\bullet) = H^0(N^\bullet)$ which is supported on $V(a)$. The long exact sequence of mapping out in particular yields the exact

$$RHom(H^0(N^\bullet), M[i]) \rightarrow RHom(N^\bullet, M[i]) \rightarrow RHom(\tau^{\leq 0}(N^\bullet), M[i-1])$$

and the extremal terms are zero by depth and induction assumption, respectively (there is no right term for $i = 0$).

The lemma is proved, and with it one direction of the proposition. \(\square\)

For the other direction, we use induction on $i$ to show that $Ext^i_A(N, M) = 0$ for any $N$ supported on $V(a)$ with $i < \text{depth}_{A', A'}(A)$.

Again, we have $RHom_{A'}(N, M[i]) \cong RHom(N \overset{L}{\otimes} A, M[i])$

and $N \overset{L}{\otimes} A$ has $N$ as its zeroth cohomology. We have the distinguished triangle

$$\tau^{\leq 0}(N \overset{L}{\otimes} A) \rightarrow N \overset{L}{\otimes} A \rightarrow N \rightarrow \tau^{>0}(N \overset{L}{\otimes} A)[1]$$

from where we get

$$RHom(N \overset{L}{\otimes} A, M[i]) \rightarrow RHom(N, M[i]) \rightarrow RHom(\tau^{\leq 0}(N \overset{L}{\otimes} A)[1], M[i])$$

Now $RHom(\tau^{\leq 0}(N \overset{L}{\otimes} A)[1], M[i]) = RHom(\tau^{\leq 0}(N \overset{L}{\otimes} A), M[i-1])$ which is zero by the induction hypothesis (for $i = 0$ it is zero because the two objects live in cohomological degrees < 0 and $\geq 0$) and $RHom(N \overset{L}{\otimes} A, M[i]) \cong RHom_{A'}(N, M[i]) = 0$ from the assumption. Therefore $RHom_A(N, M[i]) = 0$ as well. \(\square\)

Remark: the condition $A' \rightarrow A$ is needed for showing that $N \overset{L}{\otimes} A$ has $N$ as its zeroth cohomology - we need to show that $A \overset{L}{\otimes} N$ as an $A$-module is $N$. Indeed, we have the natural multiplication morphism $A \overset{L}{\otimes} N \rightarrow N$ and the inclusion morphisms $N \rightarrow A \overset{L}{\otimes} N$ which is its left inverse. The condition $A' \rightarrow A$ implies that $N \rightarrow A \overset{L}{\otimes} N$ is surjective: if $\phi(a') = a$ then $a'n$ maps to $1 \otimes \phi(a')n = 1 \otimes an = a \otimes n$, and therefore these two maps are actually inverse to each other. Thus one direction of the proposition works without the surjectivity assumption.

Note: if $A$ is a local (Noetherian) ring, the depth is tacitly assumed to be with respect to the maximal ideal, unless state otherwise. The depth of such a ring is its depth as a module over itself.

**Definition.** If $M$ is an $A$-module then the *projective*, *injective* and *Tor* dimension of $M$ is the minimal integer $k$ such that $Ext^i(M, -) = 0$, $Ext^i(-, M) = 0$ respectively $Tor_i(-, M) = 0$ for all $i > k$ (if these integers exist, otherwise the dimension is $\infty$).

**Lemma.** The following are equivalent:

i) $p.dim.(M) \leq k$

ii) There exists a projective resolution of $M$ of length $\leq k + 1$

iii) If $P_{k-1} \rightarrow \ldots \rightarrow P_1 \rightarrow P_0$ is a projective resolution of $N$, then $ker(P_{k-1} \rightarrow P_{k-2})$ is projective.
Proof: \(iii \Rightarrow ii \Rightarrow i\) is clear. For \(i \Rightarrow iii\) we use the long exact sequence of \(\text{Ext}\) (dimension shifting) repeatedly to deduce \(\text{Ext}^i(M,N) = \text{Ext}^{i-k}(\text{Ker}(P_{k-1} \to P_{k-2}), N)\) for \(i > k\) in particular \(p\text{dim}(M) \leq k\) implies \(\text{Ext}^1(\text{Ker}(P_{k-1} \to P_{k-2}), N) = 0\) for all \(N\), which is equivalent to \(\text{Ker}(P_{k-1} \to P_{k-2})\) being projective as the long exact sequence of \(\text{Ext}\) readily implies that mapping out of it is exact. In fact, this direction follows by repeated application of the following lemma plus the observation that a module of projective dimension 0 is projective. \(\square\)

**Lemma.** If the projective dimension of \(M \leq i\) and \(0 \to M' \to P \to M \to 0\) is a short exact sequence with \(P\) projective, then \(p\text{dim}(M') \leq i - 1\).

Proof: using the short exact sequence of \(\text{Ext}\) we conclude that \(\text{Ext}^i(M',N) = \text{Ext}^{i+1}(M,N)\) for all \(N\) from where the conclusion follows. \(\square\)

There are absolutely analogous statements for the injective and Tor dimensions. They are:

**Lemma.** The following are equivalent:
\[\begin{align*}
&i) \text{inj.dim.}(M) \leq k \\
&ii) \text{There exists an injective resolution of } M \text{ of length } \leq k + 1 \\
&iii) \text{If } I_0 \to I_1 \to \ldots \to I_{k-1} \text{ is an injective resolution of } N, \text{ then } I_{k-1}/\text{Im}(I_{k-2}) \text{ is injective.}
\end{align*}\]

**Lemma.** If the injective dimension of \(M \leq i\) and \(0 \to M \to I \to M' \to 0\) is a short exact sequence with \(I\) injective, then \(\text{inj.dim}(M') \leq i - 1\).

**Lemma.** The following are equivalent:
\[\begin{align*}
&i) \text{flatdim.}(M) \leq k \\
&ii) \text{There exists a flat resolution of } M \text{ of length } \leq k + 1 \\
&iii) \text{If } P_{k-1} \to \ldots \to P_1 \to P_0 \text{ is a flat resolution of } N, \text{ then ker}(P_{k-1} \to P_{k-2}) \text{ flat}
\end{align*}\]

**Lemma.** If the flat dimension of \(M \leq i\) and \(0 \to M' \to P \to M \to 0\) is a short exact sequence with \(P\) flat, then \(\text{flatdim}(M') \leq i - 1\).

Note that projective dimension is always \(\geq\) than the flat dimension because every projective module is flat, but the converse is not true, e.g. \(\mathbb{Q}/\mathbb{Z}\) is flat over \(\mathbb{Z}\) but not projective. Over Noetherian local rings though, projective=flat=free.

**Definition.** A has cohomological dimension \(\leq n\), written \(cd(A \leq n)\), if every module has projective dimension \(\leq n\). Equivalently, every module has injective dimension \(\leq n\), or \(\text{Ext}^i(M,N) = 0\) for all \(i > n\) and any \(M,N \in A - \text{mod.}\)

**Proposition.** Let \(A\) be Noetherian. The following are equivalent:
\[\begin{align*}
&i) \text{cd}(A) \leq n \\
&ii) \text{Ext}^k(M,N) = 0 \text{ for } k > n \text{ and } M \text{ finitely generated.} \\
&iii) \text{Ext}^k(M,N) = 0 \text{ for } k > n \text{ and } M,N \text{ finitely generated.}
\end{align*}\]

Proof: the forward implications are immediate.
\[\begin{align*}
&iii) \Rightarrow ii) \text{ follows because for } M \text{ finitely generated } \text{Ext}(M,-) \text{ commutes with direct limits and every module is a direct limit of finitely generated ones. To show that Ext commutes with direct limits, it is enough to choose a fixed projective resolution of } M \text{ by finitely generated free modules, and each map out of a finitely generated module commutes with direct limits.}\n\end{align*}\]

It remains to show that \(ii) \Rightarrow i)\). Let \(M\) be any module and choose a resolution \(I_0 \to I_1 \to \ldots \to I_{k-1}\) of it. We want to show that \(I_k := \text{coker}(I_{k-2} \to I_{k-1})\) is injective. The assumption \(ii)\) tells that \(\text{Ext}^1(N,I_k) = 0\) for \(N\) finitely generated, and this implies the conclusion for all \(N\) because over Noetherian rings injectivity suffices to be tested only on finitely generated modules (this was proved before - Zorn’s lemma tells us that injectivity suffices to be tested only for \(I \to A\) where \(I\) is a - finitely generated - ideal of \(A\)). \(\square\)
Let $A$ be a left Noetherian ring of cohomological dimension $\leq n$. Assume the same is true for $A^{op}$. We look at the full-subcategory $D^b_{fg}(A) \subset D(A)$ consisting of complexes with finitely generate $A$-module cohomology.

We will define a duality functor
\[
\mathbb{D}_{A \rightarrow A^{op}} \cong (D^b_{fg}(A^{op}))^{\circ}
\]
such that $\mathbb{D}_{A \rightarrow A^{op}} \circ \mathbb{D}_{A \rightarrow -A} = \text{Id}$

**Lemma.** If $A$ has finite cohomological dimension, then for every object $M^\bullet$ in $D^b_{fg}(A)$ there exists a finite complex $P^\bullet$ consisting of projective finitely generated $A$-modules with a quasi-isomorphism to $M^\bullet$.

**Proof:** First, $M^\bullet$ is quasi-isomorphic to a finite complex of finite-dimensional modules. Indeed, assuming that $M^\bullet$ lives in degrees $\geq 0$ we have the distinguished triangle $H^0(M^\bullet) \rightarrow M^\bullet \rightarrow \tau^>0M^\bullet \rightarrow H^0(M^\bullet)[1]$ thus $M^\bullet$ is the cone of $\tau^>0M^\bullet[-1] \rightarrow H^0(M^\bullet)$ so it is quasi-isomorphic to their direct sum, and now $H^0(M^\bullet)$ is a singleton of finite dimension and $\tau^>0M^\bullet[-1]$ is “shorter” than $M^\bullet$ so we are done by induction.

Next, assuming $M^\bullet$ is $M_k \rightarrow M_1 \rightarrow \ldots \rightarrow M_0$ with $M_i$ finitely generated, and let’s start building a projective resolution of $M^\bullet$. We want a sequence $\ldots \rightarrow P_k \rightarrow \ldots \rightarrow P_1 \rightarrow P_0$ with maps $P_i \rightarrow M_i$ commuting with the differentials such that the total complex $\ldots \rightarrow P_k \rightarrow P_{k-1} \oplus M_k \rightarrow \ldots \rightarrow P_1 \oplus M_1 \rightarrow M_0$ is exact. This is done by taking (finite dimensional) projective $P_0$ surjecting onto $M_1$, then $P_1$ surjecting onto $\ker(P_0 \oplus M_1)$ and so on. Now this process can continue indefinitely - but we claim that if $cd(A) \leq n$, then we can end the sequence at $P_{n+k}$ - i.e. we claim that $\ker(P_{n+k} \rightarrow P_{n+k-2})$ is projective. This follows from a previous result since $P_{n+k-1} \rightarrow P_{n+k-2} \rightarrow \ldots \rightarrow P_k$ is a projective resolution of $\ker(P_{n+k-1} \oplus M_k \rightarrow P_{k+2} \oplus M_{k+2})$. □

**Corollary.** $D^b_{fg}(A) \cong K^b_{proj,fg}(A)$

**Proof:** we need to show that the natural functor from left to right is fully faithful, but we already know it is when enlarged to the bigger categories, and essential surjectivity was proven in the previous lemma. □

We now construct $\mathbb{D}$. If we choose $\ldots \rightarrow P_1 \rightarrow P_0$ a finite projective resolution. Now if $P$ is a finite dimensional projective left $A$-module, we set $P^\vee$ be $\text{Hom}_A(P,A)$. This is a right module, and it is finitely generated and projective (using the definition of projective as direct summand of free).

We obtain a projective resolution $(P^0)^\vee \rightarrow (P^1)^\vee \rightarrow \ldots$ - and this yields the required functor. We need to show that this is well-defined, which is fairly easy, since any two quasi-isomorphic projective resolutions have quasi-isomorphisms going between them, and these produce contravariant maps between the dual projective resolutions - it is also not hard to show that they are quasi-inverse to each other, the homotopies relating the composition to the identity are dual to the homotopies relating the original compositions to the identities.

Let $A$ be a commutative Noetherian local ring with residue field $k$, $M$ a finitely generated $A$-module.

**Proposition.** The following are equivalent:

i) $\text{Tor}^i(k,M) = 0$, $i \geq n + 1$

ii) $M$ has tor-dimension $\leq n$

iii) $M$ has projective dimension $\leq n$

**Proof:** $\text{iii) } \Rightarrow \text{ii) }$ follows from $\text{proj.dim} \geq \text{tordim}$ and $\text{ii) } \Rightarrow \text{i) }$ is clear.

It remains to show $\text{i) } \Rightarrow \text{iii) }$ Indeed, let’s choose a module $M$ and choose a resolution $P_n \rightarrow P_{n-1} \rightarrow \ldots \rightarrow P_0$ of $M$ with $P_0,\ldots,P_{n-1}$ projective, and $P_n = \ker(P_{n-1} \rightarrow P_{n-2})$. We want to show that $P_n$ is projective. Using the long exact sequence of Tor, analogous to what was done before, i) implies $\text{Tor}^1(k,P_n) = 0$. Using commutative algebra, this implies that $P_n$ is flat which over local Noetherian rings is equivalent to being projective/free (for finitely generated modules). □

Let’s prove the two commutative algebra facts we referenced:

**Lemma.** Let $M$ be a finitely generated module over $A$ local Noetherian commutative ring. Then $M$ is flat if and only if it is projective if and only if it is locally free.
Proof: it suffices to imply flat implies free as the other implications are easy. Let’s choose a basis of \( M \otimes_A k \) and lift it to \( M \), producing a map \( A^m \to M \). We claim it is surjective: the cokernel dies when tensored with \( k \) because tensoring with \( k \) is right exact, and the map becomes an isomorphism when tensored with \( k \), and now by Nakayama’s lemma the cokernel is 0.

Let \( N \) be the kernel of this map, then the long exact sequence of Tor produces \( Tor^1(M, k) \to N \otimes_A k \to \mathbb{A}^m \to M \otimes_A k \to 0 \) which since \( Tor^1(M, k) = 0 \) and the last map is an isomorphism, yields \( N \otimes_A k = 0 \) which again by Nakayama’s lemma implies \( N = 0 \) thus \( A^m \cong M \). □

**Lemma.** Over a commutative local Noetherian ring \( A \) with residue field \( k \), a finitely generated module \( M \) is flat (projective, free) if and only if \( Tor^1(k, M) = 0 \).

Proof: in the proof of the above proposition we only used \( Tor^1(k, M) \) to show that \( M \) is free. □

**Corollary.** For \( A \) local commutative Noetherian with residue field \( k \), \( A \) has cohomological dimension \( \leq n \) if and only if \( Tor_i(k, M) = 0 \) for \( i > n \).

Proof: one direction is obvious. Assuming \( Tor_i(k, M) = 0 \) for \( i > n \) we deduce by the previous lemma that Tor dimension of \( k \) is \( \leq n \) thus for any \( M \) \( Tor_i(k, M) = 0 \) for \( i > n \) which, according to the previous proposition, implies that \( cd(A) \leq n \).

Let \( A \) be a local Noetherian commutative ring with maximal ideal \( m \) and residue field \( k = A/m \).

**Definition.** \( A \) is regular of dimension \( n \) if \( \dim k/m \cdot m = n \) and there exists a regular sequence \( f_1, f_2, \ldots, f_n \in m \).

(The latter is equivalent to \( \text{depth}_{A, m}(A) = n \) or \( \text{Ext}^i(k, A) = 0 \) for \( i < n \).)

**Theorem.** (Serre) If \( A \) is regular of dimension \( n \) then \( A \) has cohomological dimension \( n \).

Proof: We induct on \( n \). First, we want to construct a projective resolution of \( k \) of length \( n \). Let \( f_1, \ldots, f_n \) be a regular sequence. Set \( A_i = A/(f_1, \ldots, f_i) \).

The complex \( \mathbb{P}^i = A \xrightarrow{f_i} A \) is quasi-isomorphic to \( A_1 \) by construction. Take the map \( \mathbb{P}^i \xrightarrow{f_{i+1}} \mathbb{P}^{i+1} \). It’s cone is quasi-isomorphic to the cone of \( A_1 \xrightarrow{f_2} A_1 \) which is \( A_2 \). In a similar way, we construct \( P_{i+1} \) the cone of \( P_i \xrightarrow{f_{i+1}} P_i \), which is quasi-isomorphic to \( A_i \).

It is also clear that \( A_i \) lives in degrees \( 0, 1, \ldots, i \) and its components are free \( A \)-modules, which for \( i = n \) yields a required projective resolution of \( k \).

This resolution looks like \( \cdots \to \oplus_{i \leq n} \cdots A \to \oplus_{i+1 \leq n} \cdots A \to \cdots \to A \).

In particular, this resolution is “symmetric” which yields a non-canonical isomorphism:

\[
Tor_i(k, M) \cong \text{Ext}^{n-i}(k, M)
\]

which takes place because Tor is computed by \( P^* \otimes M \) while Ext is computed via \( \text{Hom}_A(P^*, M) = \mathbb{P}^* \otimes M \) but clearly the above resolution is non-canonically isomorphic to its dual \( P^* \cong \mathbb{P}^*[n] \) which implies the claim (note that \( \mathbb{P}^* \otimes_A M = \text{Hom}_A(P^*, M) = R\text{Hom}_A(P^*, M) \)) as for projective \( P^* \), \( R\text{Hom} = \text{Hom} \).

Particularly, \( \text{Ext}^{n}(k, k) = Tor^1_A(k, k) = k \) - which implies \( cd(A) \geq n \) but since for \( i > n \) as \( k \) has projective dimension \( \leq n \), \( Tor_i(k, k) = 0 \) which means \( cd(A) \leq n \) by a previous proposition. Thus \( cd(A) = n \). □

The identity shown in the course of the proof of the theorem has a generalization:

**Proposition.** If \( N, M \) are \( A \)-modules then

\[
\mathbb{D}(N) \xrightarrow{L} M \cong R\text{Hom}(N, M)
\]

Proof: choose a projective resolution \( P^* \) of \( N \), then the left-hand side is compute by \( \mathbb{P}^* \otimes M \) but \( M \otimes \text{Hom}(P_i, A) \cong \text{Hom}(P_i, M) \) and this complex computes the right-hand side. □
Recall that if $A$ is a local commutative ring with residue field $k$, then the projective dimension of a finitely generated module equals the Tor dimension, and is the largest integer $i$ such that $\text{Tor}_i(k,M) \neq 0$.

**Proposition.** For $M$ a finitely generated module over a regular local commutative ring $A$ of dimension $n$, 

$$pd(M) + \text{depth}(M) = n$$

Proof: Assume that $M$ has projective dimension $r$. Note that $r \leq n$ because $k$ has projective dimension $\leq n$ (in fact they are equal) as we have constructed an explicit resolution of $k$ of length $n$, and this implies $\text{Tor}_i(k,M) = 0$ for $i > n$ which therefore means $r = \text{Tor} \dim \leq n$. The isomorphism $\text{Tor}_i(k,M) \cong \text{Ext}^{n-i}(k,M)$ proven in the course of the proof of Serre’s theorem immediately implies the conclusion. □

**Proposition.** If $A$ is regular, then the projective dimension of $M$ is equal to the maximal integer $i$ such that $\text{Ext}^i(M,A) \neq 0$.

Proof: Clearly the projective dimension is greater than or equal to this maximal value of $i$. For the converse, choose a resolution $P_0 \to P_{i-1} \to P_{i-2} \to \cdots \to M$ with $P_0, \ldots, P_{i-1}$ finite projective and we have to show that $P_i$ is projective. By dimension shifting we have $\text{Ext}^i(P_i,A) = 0$ for all $j \geq 1$. Thus $\text{Ext}^i(P_i,A^n) = 0$ for all $m$. Take now the set $S$ of all modules $N$ such that $\text{Ext}^i(M,N) = 0$ for all $j > 0$. Note that is $0 \to N_1 \to N_2 \to N_3 \to 0$ is a short exact sequence with $N_1, N_2 \in S$ then we have the long exact sequence of Ext: $\cdots \to \text{Ext}^j(P_i,N_2) \to \text{Ext}^j(P_i,N_3) \to \text{Ext}^{j+1}(P_i,N_1)$ which implies that $N_3 \in S$. Since $A$ has finite cohomological dimension by Serre’s theorem, every finitely generated module has a finite free resolution, and since free modules are in $S$, all finitely generated modules are in $S$. This implies that $P_i$ is projective. Indeed, choose a short exact sequence $0 \to K \to A^m \to P_i \to 0$ then we get $0 \to \text{Hom}(P_i,K) \to \text{Hom}(P_i,A^m) \to \text{Hom}(P_i,P_i) \to \text{Ext}^1(P_i,K) = 0$ since $K \in S$, and this implies $\text{Hom}(P_i,A^m) \to \text{Hom}(P_i,P_i)$ - in particular lifting the identity we produce a splitting of the short exact sequence thus $P_i$ is a direct summand of a free module hence projective. □

**Theorem.** Let $A$ be a local Noetherian ring of finite cohomological dimension. Then $A$ is regular.

Proof: We induct on the Krull dimension of $A$. The base case is when $A$ is a field, which is clear. Assume this is true for local rings of smaller dimension.

**Lemma.** There exists $f \in m$ such that $A \xrightarrow{f} A$ is injective (i.e. a regular element)

Proof: Suppose not, then $A$ has $m$-torsion by a previous proposition, thus there is a short exact sequence $0 \to k \to A \to M \to 0$.

Assume $cd(A) = n > 0$. then $\text{Tor}_n(k,k) \neq 0$ and we write the long exact sequence

$$0 = \text{Tor}_{n+1}(M,k) \to \text{Tor}_n(k,k) \to \text{Tor}_n(A,k) = 0$$

hence $\text{Tor}_n(k,k) = 0$ contradiction.

The lemma is proved. □

Returning to the problem, we choose $f \in m$ that produces a short exact sequence $0 \to A \xrightarrow{f} A \to A_1 \to 0$.

Krull’s Hauptidealsatz tells that $A_1$ has Krull dimension at most one less than the Krull dimension of $A$, so to use the induction hypothesis it suffices to show that $A_1$ has finite cohomological dimension.

By a previous proposition, it suffices to show that $\text{Tor}_{A_1,i}(k,N)$ is 0 for sufficiently big $i$ independent of $N$, knowing that this is true for $\text{Tor}_{A,i}(k,N)$.

We want to relate them. Namely, we look at

$$k \otimes_A N \cong (k \otimes_A A_1) \otimes_{A_1} N$$
(Note that this isomorphism holds for any $M$ instead of $k$, and is easily proven manually by replacing $M$ by its projective resolution.)

The statement of the theorem will then follow immediately from the following lemma (applied to $M^\bullet = k$)

**Lemma.** Assume $M^\bullet$ is a complex of $A_1$-modules (also regarded as $A$-modules). Then as an object of $D^- (A_1 \text{mod})$, $M^\bullet \otimes_d A'$ contains $M^\bullet$ as a direct summand.

Proof: We can choose $M^\bullet$ to be represented by a free resolution $\ldots \to A^{n_1} \to A^{n_0}$ (the proof works just fine for infinite dimensional modules as well, we just replace $u_i$ by infinite sets). Since the maps between modules are represented by matrices with coefficients in $A_1 = A/fA$, these coefficients can be lifted to $A$. This will immediately produce a resolution $\ldots \to A^{n_1} \to A^{n_0}$ and it is easy to see that we have the following diagram of complexes in which the columns are short exact sequences:

\[
\begin{array}{cccc}
\cdots & A^{n_1} & \xrightarrow{f_0} & A^{n_0} \\
\downarrow & \downarrow & & \downarrow \\
\cdots & A^{n_1} & \xrightarrow{f_0} & A^{n_0} \\
\downarrow & \downarrow & & \downarrow \\
\cdots & A^{n_1} & \xrightarrow{f_0} & A^{n_0}
\end{array}
\]

In particular, $M^\bullet$ is the cone of the map between the two complexes representing the bottom rows. After we tensor this by $A_1$ modulo $A$, this cone becomes the direct sum as the vertical multiplication by $f$ maps become 0. □

**Corollary.** If $A$ is a regular local ring, and $S$ is a multiplicative set, then the localization $A_S$ is also regular.

Proof: If $cd(A) \leq n$ then $cd(A_S) \leq n$ because $\text{Ext}_i^{A_S} (M,N) \cong \text{Ext}_i^A (M,N)$. To prove the latter statement, we know that for $i = 0$ where it is just $\text{Hom}$, and it remains to remember that $\text{Ext}_i^A$ are the homologies of $\text{RHom}$ which equal $\text{Hom}$ for injective resolutions, but an injective module for $A_S$ was proven to be injective for $A$. □

**Proposition.** Let $A$ be a Noetherian commutative ring,. Then the following are equivalent:

i) $cd(A) \leq n$ ii) $cd(A_m) \leq n$, $\forall m$ maximal iii) $\forall m$ maximal, $A_m$ is regular of dimension $\leq n$ iv) $\forall p$ prime, $A_p$ is regular of dimension $\leq n$

Proof: i) $\Rightarrow$ ii), ii) $\Leftrightarrow$ 3, iv) $\Rightarrow$ iii) are clear. It remains to show that ii) implies i).

When showing that the $A$-module $\text{Ext}_m^i (M,N)$ is zero it is enough to show it’s zero when localized at any maximal ideal. We claim that the localization at $m$ is $\text{Ext}_m^i (M,N)$. Choose a projective resolution $P^\bullet$ of $M$, then $P^\bullet_m$ will be a projective resolution of $M_m$ because the localization functor is exact. The complex $\text{Hom}(P^\bullet_m, N_m)$ is the localization at $m$ of the complex $\text{Hom}(P^\bullet, N)$ - indeed, it is easy to show that $\text{Hom}_{A_m}(X_m,Y_m) = (\text{Hom}_{A}(X,Y))_m$. Because localization is exact, it follows that the cohomologies of the complexes are localizations of one another, which finishes the proof. □

**Proposition.** Let $A$ be a Noetherian regular ring of bounded cohomological dimension. Let $M$ be a finitely generated module. Then the $A$-module $\text{Ext}_m^i (M,A)$ satisfies the following assertions:

i) $\text{codim}(\text{supp}(\text{Ext}_m^i (M,A))) \geq i$ i.e. the height of every prime in the support is $\geq i$

ii) $\text{Ext}_m^i (M,A) = 0$ for $i < \text{codim}(\text{supp}(M))$

iii) If $k = \text{codim}(\text{supp}(M))$ then $\text{Ext}_m^k (M,A) \neq 0$ and is supported in codimension $k$.

Recall that the codimension of a (closed) set $S$ in $T$ is the smallest length of a maximal chain of irreducibles not contained in $S$ - equivalently the smallest height of a prime in $S$.

For example, if $A$ is a regular local ring af $M = k$, then $\text{Ext}_i^i (k,A) = 0$ for $i < n = \text{dim}(A) = \text{codim}(V(m)) = \text{codim}(\text{Supp}(M))$. 59
This is a generalization of the fact that if $M$ does not have full support then $\text{Hom}(M, A) \neq 0$ because as was proven before, if $A$ is local and regular then it has not $m$-torsion and the assertion follows by simultaneous localization.

Proof: i) Take $p$ a prime in $\text{Supp}(\text{Ext}^i_A(M, A))$ thus $(\text{Ext}^i_A(M, A))_p \neq 0$ but like in the proof of the previous proposition, this equals $\text{Ext}^i_{A_p}(M_p, A_p)$. We know $A_p$ is regular whose cohomological dimension (=Krull dimension, as seen by inspection the inductive proof of Serre’s theorem) is therefore greater than or equal to than $i$. Remark: Refer to Sherry’s notes for math 221 from Fall 2008 (on Dennis’s website).

ii) We need to show that $\text{Ext}^i(M, A) = 0$ for $i < \text{codim}(\text{supp}(M))$. By localizing simultaneously at all primes, it suffices to show $\text{Ext}^i_{A_p}(M_p, A_p) = 0$ for all $p$. Since this holds for $p \notin \text{Supp}(M)$. it suffices to assume $p$ has height greater than $i$.

We induct on height (the base case height=$i$ holds).

By replacing $A$ by $A_p$, we can assume that $A$ is a regular local ring of dimension $n > i$. $\text{Ext}^i_A(M, A)$ is then supported at the maximal ideal - if it were supported at a smaller prime $q$, localizing at $q$ would contradict the induction assumption.

It follows that it is an extension of multiple copies of $k$ - recall the theorem that says that finite generated modules are extensions of $A/p$ for $p$ associated primes.

Now recall the functor $\mathbb{D}$: $D^b_{fg}(A - \text{mod}) \to D^b_{fg}(A - \text{mod})^{\text{op}}$ with the property that

$$H^i(\mathbb{D}(M)) = \text{Ext}^i_A(M, A)$$

Now we can assume $i$ is minimal such that $\text{Ext}^i_A(M, A) = 0$. Call this module $N$. It follows that there exists a non-trivial map $N[-i] \to \mathbb{D}(M)$: we have proven before that a complex whose first non-zero cohomology is in degree $i$ is quasi-isomorphic to a complex actually living in degree $\geq i$, and in that case it’s $i$-th cohomology is a subset of its $i$-th entry so we can map into it and this produces a map into the complex. Applying the contravariant functor $\mathbb{D}$ we get a non-trivial map $\mathbb{D}(\mathbb{D}(M)) \to \mathbb{D}(N[-i])$ i.e. a map $M \to \mathbb{D}(N)[i]$. But $N$ being an extension of $k$ and since $\mathbb{D}(k) = k[-n]$ (non-canonically), $\mathbb{D}(N)[i]$ only lives in cohomological degree $n - i > 0$ while $M$ lives in cohomological degree 0. There can be non non-zero maps between such objects: a hut $M \to P^\bullet \to \mathbb{D}(N)[i]$ can be augmented with the map $M \to P^\bullet$ (which always exists) and then it can only become the zero hut.

iii) If $k$ is the codimension of the support of $M$, then we have $\tau^{<k}(\mathbb{D}(M))$ quasi-isomorphic to zero because of the previous part, and therefore $M$ can be inserted into a distinguished triangle involving $\tau^{<k}(\mathbb{D}(M))$ - isomorphic hence to $H^k(\mathbb{D}(M)) = \text{Ext}^k_A(M, A)$ and $\tau^{>k}(\mathbb{D})(M)$ - which is supported in codimension $\geq k+1$ by part i). Since $M$ itself is supported in codimension $k$, it follows that $\text{Ext}^k_A(M, A)$ is non-zero and supported precisely in codimension $k$. □

The most ”typical” regular ring is $k[x_1, \ldots, x_n]$ where $k$ is a field, as evidenced by the following proposition:

**Proposition.** If $A$ is regular, then $A[t_1, \ldots, t_n]$ is regular.

Proof: it suffices to show that $A$ regular implies $A[t]$ regular. Assume $p$ is a prime ideal of $A[t]$ and let $q = p \cap A$, then localizing at the multiplicative set $A - q$ it suffices to consider $A$ regular local ring and $p$ containing the maximal ideal $m$ of $A$ - in that case the residue field of $(A[t])_p$ will be the localization of $k[t]$ at $A[t] - p$. Consider a finite free resolution $P^\bullet$ of $k = A/m$. By tensoring it with the free $A$-module $A[t]$ we obtain a finite free resolution $Q^\bullet$ of $k[t]$ as an $A[t]$-module. Localizing at $A[t] - p$ we obtain a finite free resolution of the residue field of $A[t]_q$ which implies regularity. □

Let $A$ be a commutative ring.

**Definition.** $A$ is normal if each $A_p$ for $p \in \text{Spec}(A)$ is a domain that is integrally closed (inside its fraction field).

**Definition.** We say that a ring is $R_k$ if for every prime $p$ of height $\leq k$, $A_p$ is regular,
**Definition.** We say that a module \( M \) is \( S_k \) if for every prime \( p \) of height \( \geq k \), \( \text{depth}(M_p) \geq k \) (the depth is taken over the maximal ideal of \( A_p \)). The ring \( A \) is \( S_k \) if it’s \( S_k \) as a module over itself.

**Lemma.** \( A \) is \( S_k \) if and only if for any finitely generated \( M \) with \( \text{codim}(\text{supp}(M)) \geq k \), \( \text{Ext}^i(M, A) = 0, \forall i < k \)

Proof: Recall that \( \text{depth}_{A_p}(A) \geq k \) if and only if \( \text{Ext}^i(T, A_p) \) for all \( i < k \) and \( T \in A_p - \text{mod} \) such that \( \text{Supp}(T) = \{ p \} \). This implies the lemma, by simultaneous localization. \( \square \)

**Proposition.** \( A \) is \( S_2 \) if and only if the following geometric property takes place:

- Let \( X = \text{Spec}(A) \) and \( U \xrightarrow{j} X \) s.t. \( Y = X - U \) is of codimension \( \geq 2 \), then

\[
\mathcal{O}_X \cong j_* \mathcal{O}_U
\]

(the map is the adjunction map)

Proof: Assume \( S_2 \) holds. We want \( \mathcal{O}_X \to j_* \mathcal{O}_U \) to be an isomorphism. It’s always an isomorphism when localized at point in \( U \).

Say \( T \subset \mathcal{O}_X \) is the kernel. Then its support is contained in \( Y \), and then \( \text{Hom}(T, \mathcal{O}_X) = 0 \) by the previous lemma. This proves injectivity.

Now let \( T \) be the cokernel. Again, it is supported at \( Y \) so by lemma \( \text{Ext}^1(T, A) = 0 \). Because \( \text{Ext}^1 \) parametrizes extensions, the extension \( 0 \to A \to j_* \mathcal{O}_U \to T \) splits so we get \( T \to j_* \mathcal{O}_U \).

But \( \text{Hom}(T, j_* \mathcal{O}_U) = \text{Hom}(j^*T, \mathcal{O}_U) \) and \( j^*T = 0 \) since \( T \) is supported at \( Y \). This shows surjectivity.

Now we show the converse. Assume the geometric property holds. According to the lemma, to prove \( A \) is \( S_2 \) it suffices to show that \( \text{Hom}(T, \mathcal{O}_X) \) and \( \text{Ext}^1(T, \mathcal{O}_X) \) are both 0 when \( \text{Supp}(T) = Y \) has codimension 2.

Let \( U \) be the complement of \( Y \), thus \( \mathcal{O}_X \cong j_* \mathcal{O}_U \). Then \( \text{Hom}(T, j_* \mathcal{O}_U) \cong \text{Hom}(j^*T, \mathcal{O}_U) = 0 \).

For the second part, we have the distinguished triangle \( \mathcal{O}_X \cong j_* \mathcal{O}_U \to Rj_* \mathcal{O}_U \to \tau^{>0}Rj_* \mathcal{O}_U \to \mathcal{O}_X[1] \).

Hence using the long exact sequence of mapping into, we get \( \text{Hom}(T, \tau^{>0}Rj_* \mathcal{O}_U) \to \text{Hom}(T, \mathcal{O}_X[1]) \to \text{Hom}(T, Rj_* \mathcal{O}_U) \).

The first term is 0 because \( T \) lives in degree 0 and \( \tau^{>0}Rj_* \mathcal{O}_U \) lives in cohomological degree > 0. The second terms equals by adjunction \( \text{Hom}(j^*T, \mathcal{O}_U) = 0 \). This implies that the middle term is zero \( \square \)

**Corollary.** Assume we have a closed embedding \( Y \hookrightarrow X \) with \( Y \) of codimension \( \geq 2 \) and let \( j: U \hookrightarrow X \) be the embedding of its complement.

Let \( \mathcal{L}_U \) be a vector bundle (locally free sheaf) on \( U \). Then there exists at most one vector bundle \( \mathcal{L}_X \) on \( X \) that extends it (and if it exists, it is \( j_* \mathcal{L}_U \)).

Proof: If \( \mathcal{L}_X \) is a vector bundle on \( X \) such that \( j^* \mathcal{L}_X = \mathcal{L}_U \), we have the adjunction map \( \mathcal{L}_X \to j_* \mathcal{L}_U \). This map is an isomorphism, because it is locally so: locally, \( \mathcal{L}_X \) is free i.e. a direct sum of \( \mathcal{O}_X \) and then the map is an isomorphism from the proposition (note that being \( S_2 \) is clearly local). \( \square \)

**Proposition.** (PSET) Let \( A \) be a commutative Noetherian ring and \( M \) a finitely generated \( A \)-module.

a) \( p \) is an associated prime of \( M \) if and only if \( M_p \) has depth 0.

b) Let \( p_i, i \in I \) be the set of associated primes of \( M \). The map

\[
M \to \bigoplus_{i \in I} M_{p_i}
\]

is injective.

c) Let \( p \) be a prime and \( Y_p \) be the corresponding closed subscheme of \( X = \text{Spec}(A) \). Let \( U_p \xrightarrow{j} X \) be the complement, Then the map \( M \to j_*j^*(M) \) is injective when localized at \( p \) if and only if \( p \) isn’t an associated prime of \( M \).

d) Let \( Y \subset X \) be an arbitrary closed subscheme and \( U \xrightarrow{j} X \) its complement. \( Y \) doesn’t contain primes \( p \) such that \( \text{depth}(M_p) = 0 \) (i.e. associated primes of \( M \)) if and only if the map \( M \to j_*j^*(M) \) is an injection

e) If \( A \) is \( S_1 \), and \( X \) is reduced at the generic point of each of its irreducible components, then it’s reduced.
Proof: a) $M_p$ has depth 0 if and only if there is a non-zero (i.e. injective) map from $k_p$ to $M_p$ - that is, $M_p$ contains a non-zero element $a$ that is killed by $p.A_p$.

If $M$ contains $a$ whose annihilator is $p$ then its projection in $M_p$ will satisfy the condition above. Conversely, say $a_f$ satisfies the condition above for $f \not\in p$. Since $f$ is invertible in $A_p$ we can assume $f = 1$. Let $x_1, \ldots, x_k$ generate $p$ - then $x_i a$ are 0 in $M_p$ which means there exists $f \in A - p$ such that $f x_i a = 0$. Then $fa$ is killed by $p$ but not by anything in $A - p$, and we have proven the converse.

b) Assume that $a$ is in the kernel and let $a$ be its annihilator. Because $a$ maps to 0 in $M_p, a \not\in p$. But then by a previous proposition, $M^a = 0$ so $a = 0$.

b) Let $x_1, \ldots, x_k$ generate $p$ - then $\cup U_{x_i} = U$. The global section of $j^*M$ is the equalizer of the diagram $\oplus M_x \rightarrow M_{x_1, x_2}$ - and therefore the map $M \rightarrow j^*M$ is the module map $M \rightarrow (\oplus M_x \supset M_{x_1, x_2})$. Its injectivity is equivalent to the injectivity of the map $M \rightarrow \oplus M_{x_i}$.

Assume that $p$ is an associated prime. Then $p = ann(a)$ whence $a$ is non-zero in $A_p$ but must project to 0 in every $M_{x_i}$.

Conversely, if the map is not injective when localized at $p$ then for some $f \not\in p$ and $a$ with $Ann(a) \subset p$ we get $\frac{a}{f}$ equal 0 in every $(M_{x_i})_p$ - which says that there exists some $h \in A - p$ such that $x_i^k$ kills $ha$ hence $p^k$ kills $ha$ (and $ha$ is still not killed by anything not in $p$). So we may without loss of generality assume $h = 1, f = 1$. Now choose the minimal $c$ such that $p^c a = 0$ in $A_p$. Then $x_i^c a$ is killed by $p$ as an element of $M_p^c$ - and again by multiplying by something not in $p$ we may assume it is killed by $p$ as an element of $M_p$. On the other hand it is not killed by anything not in $p$ so we get an element whose annihilator is exactly $p$ so $p$ is an associated prime.

d) Assume that $Y$ do not contain such primes. Again let $x_1, \ldots, x_k$ be the elements generating $I$ which defines $Y$. Assume the map is not injective so that $x \in M$ gets sent to 0 in $M_{x_i}$, i.e. $x_i^c$ kills $x$. Among all elements in $Ax$ choose one with maximal annihilator $p$ - it will be prime because if $aby = 0$ then $a(by) = 0$ so either $by = 0$ or $by$ has larger annihilator. Because this annihilator will already contain $x_i$ it is in $Y$ but it’s also associated to $M$ so $depth(M_p) = 0$ contradiction.

Conversely, if $Y$ does contain such a prime $p$ then by a) it is associated so it is the annihilator of some element $x$ which must then project to 0 in $\oplus M_{x_i}$ as $x_i \in p$.

e) Condition S1 tells that $A_p$ has non-zero depth for all non-minimal primes, therefore by a) the only associated primes are the minimal ones (i.e. generic points of the irreducible components).

Assume $X$ is not reduced thus $a \not\in 0$ is nilpotent. Using b) we produce an associated thus minimal prime $p$ such that $a_p \neq 0$. But then it is nilpotent, contradicting the assumption that $A_p$ is reduced. $\square$

Proposition. (PSET) Let $p, Y, U_p$ be like in c) of the previous proposition. The map $M \rightarrow j_* j^* M$ is an isomorphism when localized at $p$ if and only if $M_p$ has depth $\geq 2$.

b) Let $Y \subset X$ be an arbitrary closed subscheme and $U \rightarrow X$ its complement. Then $Y$ doesn’t contain primes $p$ such that $depth(M_p) \leq 1$ if and only if the map $M \rightarrow j_* j^*(M)$ is an isomorphism.

c) In the setting of b), for $k \geq 2$, $Y$ doesn’t contain primes $p$ such that $depth(M_p) \leq k$ if and only if $M \rightarrow \tau^{k-1}(Rj_* j^*(M))$ is an isomorphism.

Proof: a) By c) of the previous proposition, $depth(M_p) \geq 1$ if and only if $M \rightarrow j_* j^* M$ is injective at $p$ so suffices to assume these equivalent conditions. Now take the distinguished triangle

$$M_p \rightarrow Rj_* j^* M_p \rightarrow N \rightarrow M_p[1]$$

and observe that by long exact sequence oh cohomology we get $\ldots \rightarrow M_p \rightarrow j_* j^* M_p \rightarrow H^0 N \rightarrow H^1(M_p) = 0$ so that $H^0 N = coker(M_p \rightarrow j_* j^* M_p)$. Thus we need to show that $H^0 N \neq 0$ if and only if $depth(M_p) = 0$.

Observe that $H^0(N)$ is supported on $p$: any other ideal in $A_p$ is strictly contained in $p$ so belongs to $U$ over which the map $M_p \rightarrow j_* j^* M_p$ is an isomorphism. In particular, $H^0(N)$ is an extension of $k_p$ so it it zero unless it
contains a copy of $k_p$ that is $H^0(N) = 0$ if and only if $Hom(k_p, H^0(N)) = 0$. The latter condition is equivalent to $Hom_D(A_p - mod)(k_p, N) = 0$ because $N$ lives only in non-negative cohomological degrees (since $M_p, j_*j^* M_p$ do) so we may assume $N$ is 0 in negative degrees.

Now take the long exact sequence of mapping into (in $D(A_p)$):

$$\ldots \to Hom(k_p, Rj_* j^* M_p) \to Hom(k_p, N) \to Hom(k_p, M_p[1]) \to Hom(k_p, Rj_* j^* M_p[1])$$

But $Hom(k_p, Rj_* j^* M_p[i]) = Hom(j^* k_p, j^* M_p[i]) = Hom(0, j^* M_p[i]) = 0$ so we get an isomorphism $Hom(k_p, N) \simto Hom(k_p, M_p[1]) = Ext^1(k_p, M_p)$ which finishes the problem as $Ext^1(k_p, M_p)$ is precisely the condition for $M_p$ to have depth at least 2 (once we know that $M_p$ has dimension at least 1, which we do).

b) the result of d) of the previous proposition guarantees that we can assume the map is injective and that the depth of all $M_p$ is at least 1. Now assume that the map is not an isomorphism - so for some $p$ the map $M_p \to j_* j^* M_p$ is not an isomorphism. We claim that $dim(M_p) = 1$.

Assume that the problem is false, so there exists a prime $p \in Y$ which has dimension 1 and the map is an isomorphism when localized at $p$ or it has dimension at least 2 and the map is not an isomorphism when localized at $p$. We'll show that no such prime exists by induction on height. So assume $p$ has minimal such height.

Like in the previous problem, take the distinguished triangle

$$M_p \to Rj_* j^* M_p \to N \to M_p[1]$$

and again $H^0 N = coker(M_p \to j_* j^* M_p)$. Because the map is an isomorphism on $U$, $H^0(N)$ is supported on $Y$ - so as an $A_p$ module it is supported on prime ideals contained in $Y$ and contained in $p$. But on any ideals strictly smaller than $p$ it must be 0 because of the induction assumption. So $H^0(N)$ is supported at $p$ and now we apply the method of the previous procedure to derive a contradiction.

c) We adapt the methods from the previous two parts. The already mentioned distinguished triangle

$$M_p \to Rj_* j^* M_p \to N \to M_p[1]$$

gives the long exact sequence of mapping into - and $Hom(k_p, Rj_* j^* M_p) = Hom(j^* k_p, j^* M_p) = 0$ because $j^* k_p = 0$. We therefore get the isomorphism $Hom(k_p, N[k - 1]) = Hom(k_p, M_p[k])$. As we know, homomorphism from a single object to something is the same as mapping that object into the corresponding cohomology.

Now we claim $H^k(N) = R^k j_* j^*(M_p)$ for $k > 1$ - which immediately follows from the long exact sequence of cohomology.

The condition $H^{k-1}\geq 0$ for all $i \leq k-1$ is equivalent to $M_p \to \tau^{k-1}(Rj_* j^*)$ being a quasi-isomorphism and the condition $Hom(k_p, M_p[i]) = 0$ is equivalent to $depth(M_p) \geq k$.

Thus we have established that $depth(M_p) \geq k$ if and only $Hom(k_p, H^i(N)) = 0$ for $i \leq k - 1$. This is not quite what we need - we want $H^i(N) = 0$ instead. We know that these would be equivalent if $H^i(N)$ would be supported only at $p$ - so we use induction on height.

The base case is done, now assume we have proven the equivalence for any ideals of height smaller than $p$. Then $H^i(N)$ is supported at a prime $q$ smaller than $p$ (recall $q$ must still be supported on $i$) as on $U$ the map is an isomorphism - if it is non-zero at some strictly smaller case then by the induction assumption $depth(M_q) < k$ - in which case we are done by looking at the prime $q$ (remember we are proving that the map is an isomorphism if and only if the depth is at least $k$ for all primes). Otherwise, $H^i(N)$ is indeed supported at $\{p\}$ and we are done. \[}\]

**Proposition. (PSET)** Let $A, M$ be as above.

a) Let $p_i, i \in I$ be the associated primes of $M$. Let $q_j, j \in J$ be the primes such that $depth(M_{q_j}) \leq 1$. For every $j \in J$ let $I_j$ be the subset of $I$ consisting of those $p_i$ such that $q_j$ is a specialization of $p_i$ (i.e. $q_j$ lies in the closure of $p_i$).

Then the image of $M$ in $\oplus_{i \in I} M_{p_i}$ consists of the elements $\{m_i \in M_{p_i}, i \in I\}$ that satisfy the following property:
for every \( j \), the element \( \{ m_i \in M_{p_i}, i \in I_j \} \) lies in the image of the natural map

\[ M_{q_j} \to \bigoplus_{i \in I_j} M_{p_i}. \]

b) If \( M \) satisfies \( S_2 \) and it has support on multiple irreducible components whose intersections are subschemes of codimension \( \geq 2 \), then \( M \) is a direct sum of modules, each of which is supported on its own irreducible component.

c) Assume that \( A \) is integral and \( S_1 \), and let \( K \) denote its field of fractions. Then \( M \) is the \( A \)-submodule of \( M \otimes K \) equal to the intersection of the localizations \( M_p \), where \( p \) runs over the set of height 1 primes of \( A \).

Proof: a) Obviously, every element in the image of \( M \) must project to something in the image of the map \( M_q \to \bigoplus_{i \in q} M_i \) simply because the map \( M \to \bigoplus_{i \in q} M_i \) factors through \( M_q \). Now we prove the converse, i.e. take such an element \( m = (m_i) \) and consider the set of all sets \( U \) such that \( m \mid \bigoplus_{p_i \in U} M_{p_i} \) is in the image of some section in \( \Gamma(M,U) \to \bigoplus_{p_i \in U} M_{p_i} \). Note that if there exists some section then it is unique i.e. the map \( \Gamma(M,U) \to \bigoplus M_{p_i} \) is injective: simply cover \( U \) by finitely many affines \( U_i \) and use the fact that the map \( \Gamma(M,U) \to \bigoplus \Gamma(M,U_i) \) is injective so any section must map to something non-zero in one of the \( U_i \), plus part b) of the pre-previous proposition that says \( \Gamma(M,U_i) \to \bigoplus M_{p_i} \) is injective (as irreducible components of \( U_i \) are a subset of the irreducible components of \( X \)). In particular, this uniqueness says that any such sections on \( U \) and \( U' \) must agree on the overlap so they define a section on \( U \cup U' \) - hence Noetherian-ness (or Zorn’s lemma) implies that there is a largest such set \( U \) - in particular it contains all \( p \).

We are set to prove that \( U = X \).

Assume that it’s not the case, and let \( Y \) be the closed subscheme which is not the complement of \( U \). Let \( p \) be the generic point of some irreducible component of \( Y \). First, \( M_p \) must have dimension at least 2. Indeed, otherwise either \( p \) minimal or \( M_p \) has dimension 1, in which case by the condition \( m \) comes from something in \( M_p \) (on the \( p_i \) contained in \( p \)) hence it comes from something on some open neighborhood of \( p \) (if that open neighborhood contains some \( p_i \) that are not contained in \( p \), restrict the neighborhood as an irreducible component is not contained in the union of the other irreducible components) - which contradicts the maximality of \( U \). So \( M_p \) has dimension at least 2 so it is isomorphic to \( j_i j^* M_p \) from the previous problem. Observe that the kernel and cokernel of \( M \to j_i j^* M \) are therefore not supported on \( p \) so they are not supported actually at some open neighborhood \( V \) of \( p \) (because they are finitely presented, their support is closed) - and so \( \Gamma(M,V) = \Gamma(j_i j^* M,V) = \Gamma(j^* M ,U \cap V) = \Gamma(M ,U \cap V) \). In particular, restricting the section \( s \in \Gamma(M,U) \) that projects onto \( m \in \Gamma(M,U \cap V) \) we see that there is a section in \( \Gamma(M,V) \) that is the same as \( m \) at every \( p_i \in V \) - so \( V \subseteq U \) by the maximality of \( U \). But \( p \in U \) contradiction.

b) Let \( Y \) be the support of \( M \) with the scheme structure chosen from \( M \), and \( i: Y \hookrightarrow X \) be the inclusion of \( X \) into \( M \). First we claim \( M = i_* i^* M \). This is clearly an isomorphism when localized at points outside \( Y \) as both are 0 there. Now let’s take \( p \in Y \) then \( i^* M \) will be associated to the module \( M \otimes_A A/I = M/IM = M \) as \( I = \text{ann}(M) \) so now it’s clear both localizations are \( M_p \) at \( p \).

This fact allows us to assume that \( \text{Supp}(M) = X \) - because otherwise we just work with \( Y \) instead of \( X \) and then just write \( i^* M \) as a direct sum of what we need and then just apply \( i_* \) to get the decomposition for \( M \).

Take now \( Y_i \) be the irreducible components, \( U \) the complement of \( \cup(Y_i \cap Y_j) \) - which has codimension at least 2, and \( j: U \hookrightarrow X \) be the corresponding open embedding. Then the map \( M \to j_* j^* M \) is an isomorphism at every point of \( U \) but also at every point outside of \( U \) because of \( 2a \) - so it is an isomorphism. However \( U \) decomposes as the disjoint union of \( U \cap Y_i \) thus \( j^* M \) decomposes as a direct sum of modules \( M_i \) supported over \( U \cap Y_i \). This means that \( j_* j^* M \) decomposes as a direct sum of modules \( j_* M_i \) and to finish we only need to show that \( j_* M_i \) are supported over (at most) \( Y_i \). Indeed, if \( p \notin Y_i \) then it has a neighborhood \( V \) disjoint from \( Y_i \) (and \( U \cap Y_i \)) hence the sections of \( j_* M_i \) are 0 because they equal the sections of \( M_i \) over \( V \cap U \) which only contains points not in \( Y_i \) so the localizations at these points are zero.

c) That is a direct consequence of b) of the previous proposition: in this case, the irreducible component is \((0)\) and \( M_{(0)} = M \otimes_A K \). Remark: we need \( S_1 \) in that case to conclude that the only associated primes are the minimal primes of \( A \) i.e. 0. Otherwise it’s plain wrong for example \( \mathbb{Z}/2\mathbb{Z} \) over \( \mathbb{Z} \). □
**Proposition. (PSET)** Let $A$ be a Noetherian ring.

a) If $A$ is $S_1 + S_2$ and is reduced at each generic point, then it is a disjoint union of integral schemes.

b) If $A$ is $R_1$ and $R_2$, it’s also $S_1$.

c) If $A$ satisfies $S_2$ and $R_1$ and $X$ is connected, then $A$ is integrally closed.

d) The $S_2$ condition on a module $M$ can be rephrased as follows: $M$ doesn’t have submodules supported on subschemes which are not unions of irreducible components, and any extension

$$0 \rightarrow M \rightarrow M' \rightarrow T \rightarrow 0$$

with $\text{codim}(\text{supp}(T)) \geq 2$, splits.

e) Let $A$ be a domain. Normality of $A$ is equivalent to the following property: for any $A \twoheadrightarrow A'$ such that $A'$ is a domain and finite as an $A$-module and $K \rightarrow A' \otimes K$ is an isomorphism, we have $A = A'$. Also, $S_2 + R_1$ implies normality, from this point of view.

Proof: a) We apply part b) of the previous proposition. For this, let’s show that the irreducible components of $\text{Spec}(A)$ meet in subschemes of codimension at least 2. Indeed, assume that $p$ belongs to two irreducible components - so $p$ contains two minimal primes $p_1, p_2$. It does not have codimension 0 because it is not a minimal prime, so assume it has codimension 1, therefore it has height 1 and $\text{depth}(A_p) = 1$. In that case $A_p$ is integral by assumption therefore it has a minimal 0 ideal whose preimage in $A$ (i.e. all elements killed by something in $A - p$) will therefore contain both $p_1$ and $p_2$ which contradicts the minimality of at least one of these two ideal - contradiction.

So we can apply previous proposition part b) and write $A$ as a direct sum of submodules (i.e. ideals) $A_i$ each supported at its own irreducible component. This means that each $A_i$ is a ring. This ring is also reduced according to pre-pre-previous proposition part e) (note that $S_2$ and the integrality at primes of height 1 imply $S_1$ because an integral domain has non-zero dimension). It remains to show it is integral. More precisely, every ring with a unique minimal ideal which is reduced is integral: this is because the intersection of prime ideals will be the minimal ideal, and since the ring is reduced, it must be 0.

b) Choose an ideal of height at least 1. It its height is 2, the conclusion of S1 is satisfied in virtue of S2. If its height is 1, then R1 tells us that the ring $A_p$ is regular - however the Krull dimension of the maximal ideal is then 1 therefore $A_p$ is a DVR. In particular it is an integral domain, and integral domains have non-zero depth with respect to the maximal ideal (because the annihilator of anything is only zero).

c) According to b), $X$ is a disjoint union of integral schemes and since it is connected, it is a single integral scheme. It remains to check that $A$ is closed in its field of fractions. Indeed, inside $K$ it is $\cap A_p$ for $p$ of height 1 according to the previous part, so it suffices to show $A_p$ are integrally closed. Indeed, if $p$ has height 1 then by $R_1$ $A_p$ is regular - and hence it has Krull dimension 1 so its maximal ideal is principal so it is a DVR. But DVRs are integrally closed.

d) Here is a proof for finitely generated $M,M'$ (which is enough for applying it for e) )

First, we claim the following: a short exact sequence of (finitely generated) $A$-modules $0 \rightarrow M \rightarrow N \rightarrow P$ splits if and only if it splits when localized at every prime.

One direction is obvious, now assume $0 \rightarrow M_p \rightarrow N_p \rightarrow P_p \rightarrow 0$ splits for every prime $p$. Then there is a splitting map $N_p \rightarrow M_p$ for any $p$ and it’s easy to see this implies a splitting map $N_f \rightarrow M_f$ for some $f \notin p$ - by Noetherian-ness we now get splitting maps $j_i: N_{f_i} \rightarrow M_{f_i}$ for $f_1, \ldots, f_k$ generating the unit ideal in $A$. By replacing the $f_i$ by suitable high powers of themselves, and as $N$ is finitely generated, we produce maps $j_i: N \rightarrow M$ such that $j_i \circ \phi = f_i \circ \text{id}$ where $\phi$ is the map $M \twoheadrightarrow N$ ( $j_i$ is $f_i$ times the splitting map). Now if $\sum c_i f_i = 0$ we take $\sum c_i j_i$ our splitting map.

Now we return to the problem. The $S_2$ condition is equivalent to 2 stipulations: if $p$ has height at least 2, then:

i) $\text{Hom}(k_p, M_p) = 0$ i.e. $(M_p)^{\mathbb{P}A_p} = 0$ which is equivalent to $M$ having no element killed by $p$ (any element in $M_p^{\mathbb{P}A_p}$ corresponds to an element $x$ with $\text{Ann}(x) = p$ via $\frac{x}{f} \leftrightarrow x$)

ii) $\text{Ext}^1(k_p, M_p) = 0$
Condition i) is clearly equivalent to $M$ having no submodules supported on subschemes of codimension 2.

Condition ii) is equivalent to $\text{Ext}^1(N, M_p) = 0$ for anything supported on $V(p)$ as shown in class - i.e. that any short exact sequence involving $M_p$ as the first terms splits. Via the discussion above, this is the same as any short exact sequence $0 \to M \to N \to P$ where $P$ is supported on subschemes of codimension 2 splits (because if we localize at a prime of height 1 it is not in the support of $P$ so $P$ becomes 0 and then the sequence splits).

Another proof for the ii) part, from class:

Assume that $0 \to M \to M' \to N \to 0$ is a short exact sequence with $N$ supported in codimension $\geq 2$. By applying the left exact functor $j_!$ to that (which kills $N$) we obtain $0 \to j_! j^* M \to j_! j^* M' \to 0$ hence $M \cong j_! j^* M \cong j_! j^* M$ so the map $M' \to j_! j^* M' = M$ is the splitting map.

For the converse, assume that every such sequence splits. If we form the short exact sequence $0 \to M \to j_! j^* M \to N \to 0$ where $N$ is non-zero and supported at $p$ and $j$ is the embedding of $U(p)$ into $X$, then the assumption tell that the sequence splits - hence that $j_! j^* M$ has $p$-torsion which implies that $j^* M$ has $p$-torsion but that is of course impossible since it’s supported at $p$.

Yet another way: assuming that the sequence splits is the same as saying $\text{Ext}^1(N, M) = 0$ for all $N$ supported on codimension at most 2. Assume it does. By pre-previous proposition d), we need to show that $M \to j_! j^* M$ is an isomorphism for appropriate $j$. It’s injective by i), so take $N$ be the cokernel. Then $\text{Ext}^1(N, M) = 0$ and $\text{Hom}(N, j_! j^* M) = \text{Hom}(j^* N, j^* M) = 0$ so by long exact sequence of Ext we get $\text{Hom}(N, N) = 0$ so $N = 0$.

conversely, assume $M \to j_! j^* M$ is an isomorphism for all appropriate $j$ - in particular for the embedding of the complement of the support of $N$. We want to show $\text{Ext}^1(N, M) = 0$ i.e. $\text{Ext}^1(N, j_! j^* M) = 0$.

Now take the distinguished triangle $j_! j^* M \to Rj_! j^* M \to \tau_{\geq 1} Rj_! j^* M \to j_! j^* M[1]$ and take the LES:

$$
\text{Hom}(N, \tau_{\geq 1} j_! j^* M) \to \text{Hom}(N, j_! j^* M[1]) \to \text{Hom}(N, Rj_! j^* M[1]) \to \ldots
$$

The last map is 0 by adjunction as $j^* N = 0$, the first map is 0 because $N$ lives in cohomological degree 0 while $\tau_{\geq 1} j_! j^* M$ in cohomological degree at least 1, so the middle map is 0 which means $\text{Ext}^1(N, j_! j^* M) = 0$.

e) Assume that $A$ is normal. The condition $A' \otimes_A K = K$ allows us to regard $A'$ as a submodule of $K$ (containing $A$). In that case, because $A = \cap A_p$ inside $K$ (previous proposition c), $A' \notin A_p$ for some $p$. But because $A'$ is finite as a $A$-module, Cayley-Hamilton tells us that any element in $A'$ satisfies a monic polynomial equation over $A$ - hence over $A_p$, as well, and as $A_p$ is closed in its field of fractions - which is also $K$ by integrality we deduce that $A'$ is contained in $A_p$ contradiction.

Conversely, assume that $A$ is not normal. Because $A$ is a domain, so are $A_p$ so some $A_p$ must not be closed inside $K$ - which implies that for some $a \in K - A_p$, $a$ satisfies a monic polynomial equation over $A_p$. By multiplying if by something appropriate in $A - A_p$ we produce an element $x$ in $K - A_p$ that satisfies a monic polynomial equation over $A$. We claim that this polynomial can be chosen to be irreducible over $A$. Indeed, any factor of it must also be monic hence as $K$ is a field if we decompose the polynomial as the product of two factors $x$ must be a root of one of them so we can reduce the degree. If the polynomial is irreducible then $A[x] \subset K$ is an integral domain, finite over $A$, and contained in $K$ so this would contradict the assumption.

Now that we have proven this equivalence, let’s prove that S2 and R1 implies normal.

First, let’s show that $A$ is a direct sum of integral domains, using a). For this, we need to check that it is reduced at minimal primes - indeed, at minimal primes (of height 0) $A_p$ must be regular of dimension 0 so it must be a field. Ar primes of height 1 $A_p$ must be regular of dimension 1 so it must be a DVR - integral. So we can apply 4a) and thus by passing to one of the components we may assume $A$ is an integral domain.

In that case we apply the above definition of normality. So assume $A'$ is a domain containing $A$ and finite over $A$, embedded into the field of fractions $K$ of $A$. We need to show $A' = A$.

Consider the short exact sequence $0 \to A \to A' \to A'/A \to 0$ (of $A$-modules). We claim that $A/A'$ is only supported at primes of height at least 2. Indeed, say $A/A'$ was supported at the minimal prime (0). By localizing we get $0 \to K \to (A')_0 \to (A/A')_0 \to 0$ which implies $(A/A')_0 = 0$ since $K \simto (A/A')_0 = A' \otimes_A K$. Assume it was supported at a prime $p$ of height 1. We get $0 \to A_p \to A'_p \to (A/A')_p \to 0$. But $A_p$ is a DVR, so it is integrally closed in its field of fractions, which implies $(A')_p = A_p$ whence $(A'/A)_p = 0$. 

66
Therefore we conclude by d) that the sequence splits so \( A' = A \oplus J \) as \( A \)-modules. Of course this is impossible: as \( J \) embeds into \( K \) it has no \( A \)-torsion, and every two torsion-free \( A \)-submodules of \( K \) intersect (just take any two elements \( u, v \in J \) then \( au = vb \) if \( u/v = b/a \)). □

**Proposition. (PSET)** If \( A \) is an integrally closed Noetherian domain, then it satisfies S2.

**Proof:** Assume \( A \) is local or else replace \( A \) by \( A_p \) where \( p \) is the ideal of height 2 in question. Note that this localization will still make it integrally closed, because anything in \( K \) that satisfies a monic equation over \( A_p \) can be multiplied by something in \( A - p \) to satisfy a monic equation over \( A \).

Now assume that \( A \) does not satisfy S2, more precisely we have \( \text{Ext}^1(k, A) \neq 0 \) - as \( \text{Hom}(k, A) = 0 \) because \( A \) is integral.

It follows that we have a non-split short exact sequence \( 0 \to A \to M \to k \to 0 \) and so \( M \) is finitely generated. We can assume \( M \) is torsion-free. Indeed, if \( b \in A - m \) then \( bx \notin M \) but if \( c \in m \) then \( cx \in A \) as \( c \) kills \( k \) so as \( A \) is torsion-free \( cx = 0 \) which shows that \( x \) is a copy of \( k \) inside \( M \) and so the sequence splits. So assume that \( x \) is torsion-free.

Because \( K \) is an injective \( A \)-module, we have a map \( M \to K \) lifting \( A \to K \), let’s show that it is injective. Indeed, assume \( x \in M - K \) is in the kernel. For the very same reason as above \( x \) must be killed by \( m \) but by nothing not in \( m \) so the sequence would split again.

Now consider \( x \in M \) lifting a non-zero element of \( k \). Observe that \( xM \subset A \) so it is an ideal of \( A \) which is either contained in \( m \) or is everything. The second case is impossible, because then \( qx = 1 \) for some \( q \in m \) so if \( a \in m \) then \( ax = b \in m \) so \( ax = bq \) hence \( a = bq \) as \( M \) is torsion-free but this implies that \( m \) has one generator so has height at most 1. So \( xM \subset m \) and hence \( x \) defines a linear map on the \( A \)-module \( m \) hence by Cayley-Hamilton it is integral over \( m \) - in particular over \( A \). But \( A \) is integrally closed in \( K \), and \( x \notin A \). Contradiction. □

**Proposition. (PSET)** Let \( A \) be a Noetherian ring of finite cohomological dimension. For \( M \) a finitely generated \( A \)-module, its projective dimension is the minimal integer \( i \) such that for \( i' > i \) we have \( \text{Ext}^{i'}(M, A) = 0 \).

**Proof:** (it was in fact proven in the proof of a previous proposition). We know that every module \( N \) has finite projective dimension. We will show that for \( i' > i \), \( \text{Ext}^{i'}(M, N) = 0 \) by induction on the projective dimension of \( N \) (for all \( i' \) simultaneously). It is enough to consider \( N \) finitely generated because Ext factors though something finitely generated (if \( M \) is itself finitely generated, but we can assume that as every module is a limit of finitely generated modules and Ext commutes with limits).

The base: if \( N \) is projective then \( N \) is a direct summand of \( A' \) hence \( \text{Ext}^{i'}(M, N) = \text{Ext}^{i'}(M, A') = 0 \)

The induction step: consider \( P \) projective such that we have a short exact sequence \( 0 \to Q \to P \to N \to 0 \) with \( Q \) of smaller projective dimension. A part of the Ext long exact sequence reads \( \text{Ext}^{i'}(M, P) \to \text{Ext}^{i}(M, N) \to \text{Ext}^{i'+1}(M, Q) \) and the boundary modules are \( 0 \) from the induction hypothesis, hence the middle one is, too. □

**Proposition. (PSET)** Let \( A \) be a Noetherian ring of finite cohomological dimension.

a) Any locally free coherent sheaf over \( A \) is \( S_k \) for any \( k \).

b) \( \text{Ext}^{i}(N, A) = 0 \) for a f.g. \( A \)-module \( N \) with \( \text{codim}(\text{supp}(M)) > i \)

c) For a f.g. \( A \)-module, its projective dimension is \( \geq \) the codimension of its support.

**Proof:** a) Consider a prime \( p \) of height \( k \). Then \( M_p \) is free, \( M_p = A_p^m \). It follows that the depth of \( M_p \) equals the depth of \( A_p \) since it is expressed by vanishing of certain Ext functors which decomposed as direct sums of copies of the corresponding Ext functors for \( M_p \), so it is enough to prove the claim for \( M_p = A_p \). We know from lecture that \( A_p \) is regular.

In general, let’s prove that for a local regular Noetherian ring \( A \) where the maximal ideal \( m \) has Krull dimension \( n \), the depth of \( A \) is also \( n \). If fact it is \( n \), and this fact will follow from the fact that \( A \) is free hence projective, so its projective dimension is zero, once we have established the following fact:
If $A$ is a regular local Noetherian ring of dimension $n$, then the projective dimension of a finite dimensional module $N$ plus the depth of a module equals $n$.

Proof of the fact (this was proven before, in fact): We’ve exhibited a non-canonical isomorphism $\text{Ext}^i(k, N) = \text{Tor}^{n-i}(k, N)$ which immediately tells that the cohomological dimension plus the Tor dimension equals $n$. It remains to prove that the Tor dimension equals the projective dimension.

On one hand, Tor dimension is always less than or equal to the projective dimension - because if an object has a projective resolution of length at least $m$, then this resolution can be computed to give Tor. For the inequality in the other direction, we argue by induction on Tor dimension.

If Tor dimension is 0 then $\text{Tor}^i(N, k) = 0$ - this implies by the long exact sequence of Tor for the short exact sequence $0 \to m \to A \to k \to 0$ that $0 \to mN \to N \to N \otimes_A k \to 0$ is also a short exact sequence, meaning that $N/mN$ as a vector space over $k$ is $N \otimes_A k$ and choosing a basis for that vector space we lift it to form a basis for $N$ by Nakayama’s lemma (details in problem 8)- so $N$ is free hence projective.

The induction step is performed as follows: if we choose a short exact sequence $0 \to Q \to P \to N \to 0$ with $P$ projective hence flat, by the Tor long exact sequence we deduce that the Tor dimension of $N$ is 1 plus the Tor dimension of $Q$, and from the Ext long exact sequence that the projective dimension of $N$ is 1 plus the projective dimension of $Q$.

b) This fact was proven before, here is another proof. As argued before (for example, in pre-pre-previous proposition part d ) it is enough to localize at every prime $p$ - if $N_p$ is not zero then $p$ has height at least $i + 1$ hence $\text{Ext}^i(N_p, A_p) = 0$ by the previous part.

c) This will follow from what was done in a), once we show that the projective dimension is greater than or equal to the largest of the projective dimensions of the localizations. Indeed, the projective dimension is the length of the smallest projective resolution and clearly any such projective resolution induces a projective resolution of the same length in the localization. □

**Proposition. (PSET)** Let $A$ be a Noetherian ring, such that for every maximal ideal $m$ the localization $A_m$ is regular of dimension $n$. Let $M$ be a f.g. $A$-module, which is $S_n$. Then $M$ is locally free.

Proof: We will show that $M$ is projective at every maximal ideal $m$. Because localizing at every prime $p$ is obtained by first localizing at a maximal ideal $m \supset p$ and then localizing at the prime $pA_m$ in $A_m$, this will imply that $M$ is projective at every prime ideal $p$, but projective over a local ring means free by Kaplansky’s theorem.

Indeed, we know from the proof of a) of the previous proposition that projective dimension plus depth equals dimension, in the case of regular local rings, and $S_n$ tells us that $\text{depth}(M_m)$ is at least $n = \text{dim}(A_m)$ - which implies that projective dimension of $M_m$ is 0 so $M_m$ is projective over $A_m$, as desired.

Remark: it is easy to prove that projective (in fact, just flat) over a local ring implies free in the finitely generated case. Indeed, say $P$ is projective in particular flat, so that we can tensor by $N$ to get the short exact sequence $0 \to mN \to N \to N \otimes_A k \to 0$ (like in c) of the previous proposition). Now take a basis of $N \otimes_A k$ and lift it to $N$ - it will therefore yield a basis for $N/mN$ which by Nakayama’s lemma must be a basis for $N$. More precisely, we have a surjection $A^m \to N$ that becomes an isomorphism when tensored with $k$. This means that its kernel $K$ must also become 0 when tensored with $k$, by the long exact sequence of Tor and the fact that $N$ is flat. But then $K$ is also flat by the long exact sequence of Tor so we have like before $K \otimes_A k \cong K/mK$ which implies $K = mK$ so by Nakayama’s lemma $K = 0$. □

03/04/2010

**Divisors**

Let $X$ be a quasi-compact scheme that is regular in codimension 1 (i.e. the local rings at points of height 0 or 1 are regular).
Weil divisors

**Definition.** The semigroup of effective Weil divisors is \( \text{Div}^W \) the free semi-group on primes of height 1 as generators. The group of Weil divisors \( \text{Div}^W \) is its group completion, the free group on priems of height 1.

Now let \( K \) be the field of fractions of \( X \) (recall that an integral scheme has a generic point - every affine does as it corresponds to an integral domain, and these points must coincide for two different affines).

There exists a natural map \( K^\times \to \text{Div}^W \)

\[
f \to \sum_{\text{ht}(p)=1} n_p \cdot p
\]

where \( n_p \) is the valuation of \( f \) in the discrete valuation ring \( \mathcal{O}_{X, p} \).

We need to show that the sum is finite. By quasi-compactness, it suffices to consider \( X = \text{Spec}(A) \) affine in which case \( f \) is a ratio of two non-zero elements from \( A \). So it suffices to show that if \( f \in A \), there are finitely many primes of height 1 containing \( f \).

Indeed, consider the closed subscheme \( Y = \text{Spec}(A/fA) \subset \text{Spec}(A) = X \). Then the primes \( p \) of \( A \) such that \( \nu_p > 0 \) correspond to points of \( Y \), and Krull’s Hauptidealsatz tells that the minimal primes of \( A/fA \) are precisely thes of height 1. But there are finitely many of those, since they are the irreducible components of \( Y \).

Now take \( \Gamma(X, \mathcal{O}_X)^\times \hookrightarrow K^\times \). These invertible elements are in the kernel of the map, so that we get a map \( K^\times /\Gamma(X, \mathcal{O}_X)^\times \to \text{Div}^W \).

**Lemma.** If \( X \) is normal (i.e., every affine is normal) then the map

\[
K^\times /\Gamma(X, \mathcal{O}_X)^\times \to \text{Div}^W
\]

is injective.

Proof: Invertibility is local so we can assume \( X = \text{Spec}(A) \) and let \( f \in K^\times \) such that \( (f) \) - this is the standard notation for the map - is 0. We show that \( f \in A \), and analogously \( f \in A \) which will imply the claim. Indeed, \( f \in A_p \) for every \( p \) of height 1 so that \( f \in \cap_{\text{ht}(p)=1} A_p \subset K \) but as shown above the intersection \( \cap_{\text{ht}(p)=1} A_p \) inside \( K \) is just \( A \) itself, when normality holds. \( \square \)

The class group of Weil divisors is the quotient \( Cl^W = \text{Div}^W / K^\times \)

Cartier divisors

**Definition.** The semigroup \( \text{Div}^C \) of Cartier divisors has as elements the isomorphism classes of line bundles \( \mathcal{L} \) together with injective sections \( \mathcal{O} \rightarrowtail \mathcal{L} \) - or equivalently, by duality, line bundles \( \mathcal{L} \) equipped with an injection \( \mathcal{L} \rightarrow \mathcal{O} \). The multiplication is given by the tensor product.

To show that it makes sense, we have to show that the tensor product of the sections stays injective (note that associativity and also commutativity follows directly from the properties of the tensor product). Indeed, say we have two effective Cartier divisors \( \mathcal{O} \rightarrowtail \mathcal{L}_1, \mathcal{O} \rightarrowtail \mathcal{L}_2 \). The section \( s_1 \otimes s_2 \) decomposes as \( \mathcal{O} \rightarrowtail \mathcal{L}_1 \otimes_{\mathcal{O}} \mathcal{L}_2 \rightarrowtail \mathcal{L}_1 \otimes \mathcal{L}_2 \). The first map is injective, and so is the second, as it is obtained from the injection \( s_2 \) by tensoring up with the invertible sheaf \( \mathcal{L}_1 \).

**Definition.** The group of Cartier divisors \( \text{Div}^C \) is the enveloping group of \( \text{Div}^C \): i.e. the free group on elements of \( \text{Div}^C \) modulo the relations \( a_1 + \ldots + a_n = b_1 + \ldots + b_m \) that hold in \( \text{Div}^C \).

The map \( \text{Div}^C \to \text{Div}^C \) is injective. Note that this does not hold for all semigroups - for example if \( a + b = a + c \) then \( b \) will equal \( c \) in the enveloping group, but not necessarily in the semigroup. In fact, this is the only situation which can contradict injectivity - if \( a = b \) in the enveloping group then the expression \( a - b \) (in the free group) equals a finite sum of differences \( (a_1 + \ldots + a_n - b_1 - \ldots - b_m) \) and by grouping terms by signs we will get \( a + \sum x_i = b + \sum y_i \) with \( \sum x_i = \sum y_i = 1 \). So we must show that this situation does not occur.
That is, we need to show that if $s_1 \otimes s = s_2 \otimes s$ then $s_1 = s_2$.

Indeed, assume that $\mathcal{L}_1 \otimes \mathcal{L} \hookrightarrow \mathcal{L} \rightarrow \mathcal{O}$ and $\mathcal{L}_2 \otimes \mathcal{L} \hookrightarrow \mathcal{L} \rightarrow \mathcal{O}$ represent the same Cartier divisor. This means that $\mathcal{L}_1 \otimes \mathcal{L}$ and $\mathcal{L}_2 \otimes \mathcal{L}$ represent the same subsheaf of $\mathcal{L}$, and by tensoring with $\mathcal{L}^v$ we deduce that $\mathcal{L}_1$ and $\mathcal{L}_2$ represent the same subsheaf of $\mathcal{O}$ i.e. $s_1 = s_2$. \hfill $\Box$

03/09/2010

Briefly recall what was done last time.

Weil divisors

Consider $X$ a Noetherian scheme regular in codimension 1.

We have defined $\text{Div}^{W,\text{eff}} = \text{Div}^W + \text{Div}^W$. If $X$ is connected and integral we take $K$ its fraction field and we get a map

$$K^*/\Gamma(X, \mathcal{O}^X) \rightarrow \text{Div}^W,$$

which is injective for $X$ normal.

The group $\text{Cl}_W$ is $\text{Div}^W / K^*$.

Cartier divisors

We define a sheaf of semigroups $\text{Div}^{C,\text{eff}}$ defined by $\text{Div}^{C,\text{eff}}(U)$ consisting of isomorphism classes of $(\mathcal{L}, \mathcal{L} \hookrightarrow \mathcal{O}_U)$ where $\mathcal{L}$ is a vector bundle over $U$. Multiplication is given by taking tensor products. It is easy to see that this is indeed a sheaf.

The sheaf $\text{Div}^C$ is the sheaf of groups completion.

More generally, if in a category $\mathcal{C}$ we have $V^+$ a semigroup object, its completion $V$ is an object with a map $V^+ \rightarrow V$ such that any map $V^+ \rightarrow V'$ where $V'$ is a group object, factors through a unique map $V \rightarrow V'$. (Note that this universal property determines $V$ up to isomorphism).

It is fairly easy to see that group completions exist in the category of presheaves, constructed by group completion of every section. But the presheaf group completion of a sheaf is not always a sheaf - the sheaf group completion is obtained by sheafification.

Indeed, if $\mathcal{F}^+$ is a presheaf of semigroups, $\mathcal{F}$ is its presheaf group completion, and $s(\mathcal{F})$ is the sheafification of that, endowed with the composite map $\mathcal{F}^+ \rightarrow \mathcal{F} \rightarrow s(\mathcal{F})$ from $\mathcal{F}^+$, we have for any sheaf of groups $\mathcal{G}$,

$$\text{Hom}_{\text{Sh}(X)}(\mathcal{F}^+, \mathcal{G}) = \text{Hom}_{\text{PSh}(X)}(\mathcal{F}^+, \mathcal{G}) \cong \text{Hom}_{\text{PSh}(X)}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\text{Sh}(X)}(s(\mathcal{F}), \mathcal{G})$$

It follows that an element of $\Gamma(\text{Div}^C, U)$ is given by a choice $\bigcup U_i = U$, and on each $U_i$ the formal ratio

$$\frac{\mathcal{L}_1 \mid_{U_i \cap U_j} \otimes \mathcal{L}_2 \mid_{U_i \cap U_j}}{\mathcal{L}_2 \mid_{U_i \cap U_j}} \rightarrow \mathcal{O}_{U_i \cap U_j},$$

with the compatibility condition

$$\mathcal{L}_1 \mid_{U_i \cap U_j} \otimes \mathcal{L}_2 \mid_{U_i \cap U_j} \hookrightarrow \mathcal{O}_{U_i \cap U_j} \cong \mathcal{L}_2 \mid_{U_i \cap U_j} \otimes \mathcal{L}_1 \mid_{U_i \cap U_j} \rightarrow \mathcal{O}_{U_i \cap U_j} .$$

Define also the sheaf of non-zero divisors $\mathcal{O}^{\text{ndz}} \subset \mathcal{O}$ whose sections on $U$ are the non-zero divisors in the ring $\Gamma(\mathcal{O}_X, U)$. The sheaf $\mathcal{K}^X$ is defined to be the sheaf group completion of $\mathcal{O}^{\text{ndz}}$ - note that if $X$ is integral (connected) it is the constant sheaf $K$ where $K$ is the fraction field of $X$.

**Proposition.** There are natural isomorphisms

$$\mathcal{O}^{\text{ndz}} / \mathcal{O}^X \cong \text{Div}^{C,\text{eff}}$$

$$\mathcal{K}^X / \mathcal{O}^X \cong \text{Div}^C$$

70
Proof: the map from \( \mathcal{O}^{ndz} \) to \( \text{Div}^{C,\text{eff}} \) is given locally by \( f \rightarrow (\mathcal{O} \xrightarrow{\sim} \mathcal{O}) \). It is easy to see the "kernel" is \( \mathcal{O}^* \) - more rigorously (since the notion of kernel in semigroups is not defined), if \( \mathcal{O} \xrightarrow{f_1} \mathcal{O} \) and \( \mathcal{O} \xrightarrow{f_2} \mathcal{O} \) are isomorphic, then there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{f_1} & \mathcal{O} \\
\downarrow{g} & & \downarrow{g} \\
\mathcal{O} & \xrightarrow{f_2} & \mathcal{O}
\end{array}
\]

with \( g \in \mathcal{O}^* \) and then \( f_1 = gf_2 \).

The sheaf quotient, defined as the sheafification of the presheaf \( \mathcal{O}^{ndz} / \mathcal{O}^* \) then maps injectively into \( \text{Div}^{C,\text{eff}} \). As before, the quotient of \( X \) by \( Y \) := \( X \) in a category is given by the universal property \( \{ \beta \in Hom(X, Z) \mid \beta \circ \alpha = 0 \} = Hom(X/Y, Z) \) - and like before, we take quotients of sections and then sheafify and deduce the universal property by adjunction.

It remains to see that this map is a surjection of sheaves - which is equivalent to saying that any injection of line bundles \( \mathcal{L} \hookrightarrow \mathcal{O} \) is locally given by compatible maps \( \mathcal{O} \xrightarrow{\sim} \mathcal{O} \). That is easy since locally \( \mathcal{L} \) is trivial, and we choose local trivializations \( \mathcal{L} \cong \mathcal{O} \) in which case we obtain maps \( \mathcal{O} \xrightarrow{\sim} \mathcal{O} \) (with \( f_1 \) a non-zero divisor because of injectivity). Note that since trivializations differ (locally) by elements in \( \mathcal{O}^* \), these \( f_i \) are compatible modulo \( \mathcal{O}^* \) on intersections, so they define an element in \( \mathcal{O}^{ndz} / \mathcal{O}^* \). Caution: in particular, the map \( \mathcal{O}^{ndz} \to \text{Div}^{C,\text{eff}} \) may not be surjective as presheaves, even though a map from its quotient is - the map from a sheaf to a quotient of it is a surjection of sheaves but not necessarily of presheaves.

The second case is done in absolutely the same way - or alternatively, we can prove manually that since the group completion of \( \mathcal{O}^{ndz} \) is \( \mathcal{K}^* \) it follows that the group completion of \( \mathcal{O}^{ndz} / \mathcal{O}^* \) is \( \mathcal{K}^* / \mathcal{O}^* \) which implies the second part.

\[ \square \]

Remark: the first condition can be reformulated as follows: the line bundles \( \mathcal{L} \subset \mathcal{O} \) in \( \text{Div}^{C,\text{eff}} \) are given locally by \( f_i \in \Gamma(X, U_i) \) such that \( \frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \Gamma(X, (U_i \cap U_j)^\times) \). Two such line bundles are the same if (after some refinement) they are given by sections \( f_i, f_j \) such that \( \frac{f_i}{f_j} \in \Gamma(X, U_i)^\times \). The same definition works for \( \text{Div}^{C,\text{eff}} \) with ratios \( \frac{f_i}{g_i} \) instead.

We have the short of exact sequence of sheaves \( 0 \to \mathcal{O}^* \to \mathcal{K}^* \to \text{Div}^C \to 0 \) whence we get the long exact sequence of (Cech) sheaf cohomology

\[ 0 \to \Gamma(X, \mathcal{O}^*) \to \Gamma(X, \mathcal{K}^*) \to \text{Div}^C \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{K}^*) \]

**Proposition.** \( \text{Pic}(X) := \frac{H^1(X, \mathcal{O}^*)}{H^1(X, \mathcal{K}^*)} \) is the group of isomorphism classes of vector bundles of \( X \).

Proof: We use the Hartshorne version of Cech cohomology. An element of \( H^1(X, \mathcal{O}^*) \) is given by a cover \( X = \bigcup U_i \) and elements \( f_{ij} \in \Gamma(X, U_i \cap U_j)^\times \) subject to the cocycle condition

\[ f_{ij} |_{U_i \cap U_j \cap U_k} \cdot f_{jk} |_{U_i \cap U_j \cap U_k} \cdot f_{ki} |_{U_i \cap U_j \cap U_k} = 1 \]

This can be interpreted as giving a line bundle on \( X \), as follows:

A line bundle \( \mathcal{L} \) with local trivializations \( \phi_i : \mathcal{L} \mid_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i} \) produces \( f_{ij} \) as equal to the isomorphism \( \phi_i \circ \phi_j^{-1} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j} \) - which is equivalent to an invertible section of \( \mathcal{O}_{U_i \cap U_j} \). The cocycle condition is obtained automatically from cancellation.

Conversely, given such a choice of \( f_{ij} \) we can construct \( \mathcal{L} \) by taking \( \Pi_i \mathcal{O}_{U_i} \) modulo the subsheaf which is the direct sum of \( \text{Im}(1, -f_{ij}) : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j} \oplus \mathcal{O}_{U_i \cap U_j} \) (the first is regarded as an open subsheaf of \( \mathcal{O}_{U_i} \) and the second of \( \mathcal{O}_{U_j} \) - note that in this particular case the presheaf quotient is already a sheaf. There is a map \( \phi_i : \mathcal{O}_{U_i} \to \mathcal{L} \mid_{U_i} \) and the cocycle condition tells that this map is an isomorphism. For example, on stalks, we have a direct sum \( \oplus_{i \in I} \mathcal{O}_{X,x} \) modulo \( \phi_i - f_{ij} \phi_j \) and it’s easy to see this ensures the map \( \phi_i \) from \( \mathcal{O}_{X,x} \) is an isomorphism.
These two operations are mutually inverse. It remains to see when two line bundles are isomorphic. By passing to a refinement, we may assume they are given by maps \( \phi_i, \phi'_i \) on a common cover \( \{ U_i \} \). The maps \( \phi_i \circ \phi'_i : \mathcal{O}_{U_i} \to \mathcal{O}_{U_i} \) correspond to invertible sections \( g_i \in \Gamma(\mathcal{O}_{X, U_i}) \) and then we deduce that \( \frac{f'_i}{f_i} = \frac{g_i}{g_i' \circ \phi'_i} \) which tells that they differ by a cocycle. Conversely, if they differ by a cocycle then the line bundles are isomorphic, because that’s equivalent to replacing \( \phi_i \) by \( g_i \phi_i \).

Defining the group \( Cl^C = Div^C / \Gamma(X, \mathcal{K}^\times) \) we deduce the exact sequence \( 0 \to Cl^C \to Pic(X) \to H^1(X, \mathcal{K}^\times) \).

Assume now that \( X \) is integral. Then \( \mathcal{K}^\times = \mathcal{K}^\times \) is a constant sheaf, so it is flasque thus it has zero higher cohomologies thus \( H^1(X, \mathcal{K}^\times) = 0 \). Therefore, we get

\[
0 \to \Gamma(X, \mathcal{O}^\times) \to \mathcal{K}^\times \to Div^C \to Pic(X) \to 0 \quad \text{and} \quad Cl^C \cong Pic(X)
\]

On an integral scheme, \( Div^C \) can be though as consisting of line bundles \( \mathcal{L} \) together with an isomorphism \( \mathcal{L} \otimes K \isom K \). Indeed, as \( \mathcal{L} \hookrightarrow \mathcal{L} \otimes K \) (because \( \mathcal{L} \) is flat the map \( \mathcal{L} \otimes \mathcal{O}_X \to K \) is injective) by quasi-compactness we may modify the isomorphism by a global section such that it lands inside \( \mathcal{O} \subset \mathcal{K} \) in which case we recover the usual definition of \( Div^C \).

**Proposition.** If \( X \) is regular in codimension 1 (so that Weil divisors make sense), there exists a natural map from Cartier divisors to Weil divisors, i.e. a diagram

\[
\begin{array}{ccc}
Div^{C, eff} & \longrightarrow & Div^{W, eff} \\
\downarrow & & \downarrow \\
Div^C & \longrightarrow & Div^W
\end{array}
\]

Also, this induces a map \( Cl^C \to Cl^W \).

**Proof:** A vector bundle \( \mathcal{L} \), regarded as a subsheaf of \( \mathcal{O}_X \), is locally given by section \( f_i \in \Gamma(\mathcal{O}_{X, U_i}) \) which then gets mapped to a Weil divisor on \( U_i \). We can “glue” these divisors to form a Weil divisor on \( X \) (note that they are compatible). Also, anything in \( \Gamma(X, \mathcal{K}^\times) \) will get sent to 0 in \( Cl^W \) by definition.

**Lemma:** If \( X \) is normal, then the above map \( Div^C \to Div^W \) is injective.

**Proof:** Regarding a Cartier divisor given by non-zero divisor sections \( f_i \) on \( U_i \), and another one given by \( f'_i \) (again we can refine the covers so that they are given on the same cover), they produce the same divisor if and only if \( f_i \) and \( f'_i \) have the same valuations at primes of height 1 of \( U_i \). We can assume \( U_i \) are affine - then \( f_i \) and \( f'_i \) having the same divisor mean \( \frac{f'_i}{f_i} \) is an invertible section of \( U_i \) - we have encountered that before as \( \frac{f'_i}{f_i} \) lies in \( \bigcap_{ht(p)=1} A_p \subset K \). For this, we need to assume that \( U_i \) is integral - i.e. we need to know that at least some refinement of \( U_i \) consists of integral schemes, knowing from normality that all local rings are integral. Indeed, choose a prime ideal \( x \) of \( U_i = \text{Spec}(B) \) then since \( B_x \) is integral, each element of the ideal \( I \) of \( B \) generated by all zero-divisors is killed by something not in \( x \) - but since \( B \) is Noetherian, \( I \) is finitely generated so that some \( a \in B - x \) kills \( I \) meaning \( \text{Spec}(A_a) \) is a neighborhood of \( x \) which is integral and this is what we need.

Alternatively, if \( \mathcal{L} \hookrightarrow \mathcal{O} \) is in the kernel, then the line bundle is isomorphic to \( \mathcal{O} \) at each prime of height 1 hence in some neighborhood of it. The union of all these neighborhoods is an open subscheme \( U \) whose complement has codimension \( \geq 2 \), and where the line bundle is isomorphic to \( \mathcal{O} \). But normality implies S2 which implies the geometric property that there exists a unique line bundle of \( X \) extending \( \mathcal{L}_U = \mathcal{O}_U \) - and this means \( \mathcal{L} = \mathcal{O} \).

**Corollary.** The map \( Cl^C \to Cl^W \) is injective if \( X \) is normal.

**Proof:** Follows immediately from the injectivity of the previous map as the kernels of \( Div^C \to Cl^C \) and \( Div^W \to Cl^W \) are the same.

Let \( A \) be a domain.
**Definition.** A is a UFD (unique factorization domain) if any element $f$ can be written as $g \cdot f_1 \cdot \ldots \cdot f_k$ where $g \in A^\times$ and $f_i$ are prime (i.e. they generate prime ideals) Note that this decomposition is unique up to permutation and multiplication by elements in $A^\times$ - the proof is just like the proof of Euclidean factorization. Such an $A$ is also called factorial.

**Proposition.** If $A$ is a Noetherian domain, then it is a UFD if and only if every prime ideal of height 1 is principal.

Proof: suppose $A$ is a UFD and let $p$ be a prime of height 1, $f \in p$ non-zero. Then writing $f = f_1 \cdot \ldots \cdot f_n$ (note that $g$ can be eliminated by multiplying by it one of the $f_i$) with $f_i$ generating prime ideals, we have $f_i \in p$ for some $i$ thus $(f_i) \subset p$ but since $(f_i)$ is a prime of height at least 1 we deduce $(f_i) = p$.

Conversely, assume every prime of height 1 is principal. Take $f$ and look at its divisor $(f) = \sum p_n p \cdot p$. If $p$ is generated by $f_p$ so that $(f_p) = p$ we get $g = \prod f_p$ be an element of $K$ with zero divisor i.e. it is an invertible element of $A$, as shown before. □

**Corollary.** Let $A$ be a Noetherian domain. Then $A$ is a UFD if and only if $A$ is normal and $CI^W = 0$.

Proof: Assume $A$ is a UFD. It must then be integrally closed inside its field of fractions $K$. Indeed, if $x \in K$ and $v_p(x) < 0$ and $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ with $a_i \in A$ then $x^n = -a_{n-1}x^{n-1} - \ldots - a_0$ which cannot happen because the left hand side has strictly larger valuation than the right hand side - note that being a UFD gives a valuation of $A$ by letting $v_p(a)$ be the exponent of $f_p$ in the representation of $a$ as a product of primes and this extends to $K$, with $a \in A$ if and only if $v_p(a) \geq 0$ for all $p$ since $v_p(a) \geq 0$ is equivalent to $a \in A_p$. Then each localization $A_p$ is integral and integrally closed inside $K$ too. Indeed, if $a \in K$ satisfies an equation $a^n + c_{n-1}a^{n-1} + \ldots + c_0$ with $c_i \in A_p$, we can find some $b \in A - p$ such that $c_i b \in A$ and then $ab$ satisfies $(ab)^n + c_{n-1}b(ab)^{n-1} + \ldots + b^n c_0 = 0$ thus $ab \in A$ so $a \in A_p$.

Conversely, assume $A$ is normal and $CI^W = 0$. It suffices to show every prime $p$ of height 1 is principal. Indeed, since $CI^W = 0$, the divisor $p$ can be represented as $(f)$ or some $f \in K$, and normality implies that $f \in A$ since its divisor is effective. □

**Theorem.** Let $A$ be a normal Noetherian domain, regular in codimension 1. Then the following are equivalent:

i) $DIV^C \rightarrow DIV^W$ is an isomorphism

ii) $A$ is locally factorial i.e. every localization $A_p$ is a UFD.

iii) For any prime $p$ of height 1 with $Y_p$ the corresponding subscheme, the sheaf of ideals $I_{Y_p}$ that defines $Y_p$ is a line bundle.

Proof: ii) $\iff$ iii): both are local so we can assume that $A$ is local. i) says $A$ is factorial - it follows that every prime $p$ of height 1 is principal generated by some $f_p$, whence $I_{Y_p}$ is a line bundle, being generated by $f_p$ (which is a non-zero divisor). Conversely, if $I_{Y_p}$ (which is just $p$) is a line bundle then is is generated by one section $f_p$, hence $p$ is generated by one element.

iii) $\implies$ i): iii) implies ii) which means $A$ is locally factorial so the map $DIV^C \rightarrow DIV^W$ is injective. We want surjectivity. It’s enough to write $p \in DIV^W$ as the locus of vanishing of a line bundle - namely this line bundle is $I_{Y_p}$.

i) $\implies$ iii): Let $p$ be a prime of height 1.

By assumption there exists a line bundle $\mathcal{L} \rightarrow O$ that gets sent to $p \in DIV^W$.

We have the short exact sequence $0 \rightarrow \mathcal{L} \rightarrow O \rightarrow O/\mathcal{L} \rightarrow 0$ and $O/\mathcal{L}$ vanishes everywhere except at $p$ (because at ideals different from $p$ the vector bundle is generated by an invertible section)

It follows that $O/\mathcal{L}$ maps to $O_{Y_p}$. Indeed, in terms of ideals we want $A/I$ to map to $A/p$ where $I$ defines $\mathcal{L}$ which is equivalent to $I \subset p$ - which is true since otherwise $I$ would contain something not in $p$ hence the map $I \rightarrow A$ would be an isomorphism when localized at $p$, but it’s supposed to be zero.

It follows that we can complete the following diagram with a map $\mathcal{L} \rightarrow I_{Y_p}$: 73
indeed, the map \( \mathcal{L} \to \mathcal{O} \to \mathcal{O}_{Y_p} \) equals by diagram commutativity \( \mathcal{L} \to \mathcal{O} / \mathcal{L} \to \mathcal{O}_{Y_p} \) which is 0 hence \( \mathcal{L} \to \mathcal{O} \) factors through \( I_{Y_p} \).

It remains to show that this map is an isomorphism. A priori it is injective. We deduce a short exact sequence

\[
0 \to \mathcal{L} \to I_{Y_p} \to \mathcal{O} / \mathcal{L} \to 0.
\]

Because both line bundles are isomorphic to \( \mathcal{O} \) outside \( Y_p \), the map \( \mathcal{L} \to I_{Y_p} \) is an isomorphism outside \( Y_p \) which means \( I_{Y_p} / \mathcal{L} \) is supported (at most) at \( Y_p \). But note that \( \mathcal{L} \to I_{Y_p} \) is an isomorphism at \( p \) as well, because both \( \mathcal{L} \) and \( Y_{p_0} \) are given locally at \( p \) by an element of valuation 1. Thus \( I_{Y_p} / \mathcal{L} \) is supported on a subset of \( Y_p - \{p\} \) which is necessarily of codimension \( \geq 2 \).

It follows, by a previous proposition that \( Ext^1(I_{Y_p}, \mathcal{L}) = 0 \) thus \( \mathcal{L} \) splits as a direct summand of \( I_{Y_p} \) but they both being line bundles, it is only possible if there are actually isomorphic. \( \Box \)

**Theorem.** Let \( X \) be a normal scheme, regular in codimension 1. Then it's locally factorial if and only if for any \( Y \hookrightarrow X \not\supset U \) with \( codim(Y) \geq 2 \) and a line bundle \( \mathcal{L}_U \) on \( U \) there exists a (unique) line bundle \( \mathcal{L}_X \) on \( X \) extending it.

Proof: We will prove a stronger version with vector bundles instead of line bundles.

Recall the geometric property of \( S2 \) (which holds by normality): if \( X \) is a normal scheme and we have \( Y \hookrightarrow X \not\supset U \) with \( codim(Y) \geq 2 \) then if \( \mathcal{L}_U \) is a line bundle on \( U \) it admits at most one extension to a vector bundle on \( X \) - namely \( j_* \mathcal{L}_U \) (if this is a line bundle). This shows uniqueness.

So it suffices to prove the equivalence of existence and being locally factorial.

First we show the equivalence of the existence of extensions of vector bundles and the weaker condition of existence of extensions of line bundles - note that if a line bundle extends to a vector bundle, then that vector bundle is a line bundle too because \( X \) is connected (we can replace \( X \) by its connected components to reduce to that case).

Indeed, let’s show lifting of line bundles implies lifting of vector bundles. Because being a vector bundle is local, we can assume that \( X = Spec(A) \) is a affine and and \( \mathcal{L} \) is free on \( U \). Then \( \mathcal{L} \) is a direct sum of copies of \( \mathcal{O}_U \), but \( j_!(\mathcal{O}_U) \) is a line bundle on \( X \) by the line bundle condition hence \( j_!(\mathcal{L}) \) is a direct sum of line bundles thus it is a vector bundle.

Therefore, we can work with line bundles instead.

Existence of lifting of line bundles is equivalent to \( Div^C(X) \to Div^C(U) \) being surjective.

If \( X \) is locally factorial then \( Div^C(X) \cong Div^W(X) \) and \( Div^C(U) \cong Div^W(U) \) but the map \( Div^W(X) \to Div^W(U) \) is obviously surjective - in fact an isomorphism as \( Y \) contains no points of height 1.

Conversely, if line bundles can be lifted, we will use the previous proposition, so it suffices to show \( Div^W \to Div^C \) is an isomorphism. Because \( X \) is normal, it is already injective, so it suffices to show injectivity. We want to construct a line bundle \( \mathcal{L} \to \mathcal{O} \) that gets sent to the divisor \( p \) for some given \( p \) of codimension 1. Such a line bundle will be \( j_*j^*I_{Y_p} \) provided that it is indeed a line bundle, where \( U \) is some open whose complement has codimension 2. Indeed, then this line bundle will coincide with \( I_{Y_p} \) on \( U \) which will readily imply that it does indeed get sent to \( p \). It remains to find such a \( U \). For any \( q \neq p \) of codimension 1, we can find an open \( U_q \) around \( q \) not containing \( p \) - locally it says that if \( p, q \) are two ideals of \( A \) of height 1, there exists \( f \in p - q \) which is clear. When restricted to \( U_q \) \( I_{Y_p} \) will be a line bundle namely equal to \( \mathcal{O}_{U_q} \). Also, we can pick an open \( U_p \) around \( p \) where \( I_{Y_p} \) is principal, because \( (I_{Y_p})_p = m_p \) is principal since \( X \) is regular at \( p \). Thus we can take \( U = \cup U_q \cup U_p \). \( \Box \)

03/11/2010

Let \( X \) be a Noetherian scheme and recall the sheaf \( K^X \).
To each line bundle $L$ we associate the sheaf $K_L$, which is associated to the (separated) presheaf $U \rightarrow \{ \frac{f}{g} \mid s: L| U \rightarrow O_U, f: O_U \rightarrow O_U \}$. 

If $X$ is connected and integral it is $K \otimes_O (L - 0)$

A Cartier divisor is a pair $(L'; s \in \Gamma(X, K_{L'})$. 

Indeed, recall that a Cartier divisor is locally given by compatible ”fractions” $\frac{f}{g}$ over $\mathcal{O}_{K_{L'}}$. Such a fraction can be well described by the line bundle $L_1 \otimes L_2'$ with the map to $\mathcal{K}$ which is the ratio of the two injections from $L_1$ and $L_2$ into $\mathcal{O}$: more rigorously, locally $L_1$ and $L_2$ are subsheaves of $\mathcal{O}$ generated by single elements $a, b$ and then $L_1 \otimes L_2'$ is the subsheaf $\frac{a}{b} \mathcal{O}$ of $\mathcal{K}$. Obviously the map lands in $\mathcal{O}$ when multiplied by a non-zero divisor section of $\mathcal{O}$ (locally) - for example $b$ is such a section, and hence it can be represented as a ratio $\frac{b}{a}$ with $s: L_1 \otimes L_2' \twoheadrightarrow \mathcal{O}, f: \mathcal{O} \rightarrow \mathcal{O}$. Now the compatibility condition ensures that the line bundles $L_1 \otimes L_2'$ glue to a single line bundle, and the ratios $\frac{b}{a}$ are also compatible.

Now let $X$ be integral and regular in codimension 1.

We have proven that if $X$ is normal, then $\text{Div}^C \hookrightarrow \text{Div}^W$.

**Proposition.** The following are equivalent: 

i) $X$ is normal 

ii) For any open subscheme $\hat{X} \subset X$ the map $\text{Div}^C(\hat{X}) \rightarrow \text{Div}^W(\hat{X})$ is injective 

iii) If we have an CO pair (closed embedding and open embedding of the complement) $Y \subset \hat{X} \supset U$ and $L_U$ is a line bundle over $Y$, then if $Y$ has codimension $\geq 2$ then there exists at most one line bundle $L$ on $\hat{X}$ extending $L_U$. 

Proof: i) $\Rightarrow$ ii), iii) was done before. 

iii) $\Rightarrow$ ii) was essentially done before too. If $L$ is a line bundle on $\hat{X}$ with an injection $s$ into $\mathcal{O}_{\hat{X}}$ that gets sent to zero in $\text{Div}^W$, then $s$ is an isomorphism at all primes of codimension 1, so the map of line bundles is an isomorphic away from codimension $\geq 2$, which by iii) implies that it is actually an isomorphism, so $L = \mathcal{O}_{\hat{X}}$. 

ii) $\Rightarrow$ iii) Say $L_U$ has extensions $L_1, L_2$. Choose a meromorphic section $s$ of $L_U$ (i.e. an element of the stalk at the generic point). They define meromorphic sections of $L_1, L_2$ which then define two Cartier divisors. These two Cartier divisors get mapped to the same Weil divisor because $U$ contains all primes of height 1 in $\mathcal{O}$ and they are the same non $U$. Because of condition ii), $L_1$ and $L_2$ (together with the corresponding sections) must be isomorphic to each other - which says $L_1$ and $L_2$ are the same line bundle inside of $\hat{K}$. 

Finally, it remains to show iii), ii) $\Rightarrow$ i). 

Since $X$ is regular in codimension 1, it is $R1$. It remains to show $S2$. Indeed, let $p$ be a prime of minimal height that fails $S2$ - the $p$ has height $\geq 2$ since in codimension 1 $A$ is regular, in particular all localizations at primes of codimension 1 are DVR’s so normal hence $S2$. We can assume $X = Spec(A)$ local, $p = m$. Let $U = SpecA - \{m\}$. 

Recalling a previous proposition, non-normality implies the existence of a ring $A' \subset K$ containing $A$ and a finite module over it. However at any prime $q \neq m$ in $Spec(A)$, the map $A_q \rightarrow A'_q$ must be an isomorphism, because $q$ is $S2$ by minimality of $p$. Thus, the map $A \rightarrow A'$ is an isomorphism over $U$. 

Now we claim that $A'^X \subset K$ the set of is strictly larger than $A^X$. Assume first that the residue field $k$ of $A$ is infinite. We may assume, by shrinking $A'$ otherwise, that $A' = A[t]$ for some $t \in K - A$. Let $p$ be the minimal monic polynomial of $t$ over $A$, and $\overline{p}$ be its reduction modulo $m$. In fact, by shrinking $A'$ again, we may assume that $deg(p)$ is minimal among all such choices of $A'$ and $t$. Then $p \neq 0$ hence there is $\overline{t} \in k$ (because $k$ is infinite) such that $\overline{p}(\overline{t}) \neq 0$ and lifting $\overline{t}$ to $a \in A$ we get $p(a) \in A - m$ hence $p(a)$ is invertible but $p(a)$ is the free coefficient of the characteristic polynomial $p(x + a)$ of $t - a$ hence $t - a$ is invertible in $A'$ as desired. 

It remains to investigate the case when $k$ is a finite field, say with $q$ elements. In that case $\overline{p}$ may be identically zero on $k$, if it’s divisible by $x^q - x$. The idea is to replace $t$ by $t^{(q-1)m}$ where $m = deg(p)$. A polynomial annihilating $t^{(q-1)m}$ is $p_m$ given by the condition $p_m(t^{(q-1)m}) = p(t)p(wt) \cdots p(w^{(q-1)m-1}t)$ where $w$ is a $(q - 1)m - 1$-th root of 1 adjoined to $A$ (and $k$ in the residue field) - which will land in $A[t]$. This time, $\overline{p}_m$ cannot be divisible by $t^q - t$ - otherwise $\overline{p}_m(t^{(q-1)m})$ would be divisible by $t^{q(q-1)m} - t^{(q-1)m}$ in particular by $t^{(q-1)m+1}$ - meaning that some $\overline{p}(w^*t)$
would have \( w' \) as a root, where \( w' \) is some primitive \((q - 1)^{m+1}\)-th root of unity (in \( k \)) - but this would mean \( \overline{p}(t) \) would have \( w\overline{w'} \) as a root which is also a primitive \((q - 1)^{m+1}\)-th root hence adjoining a root of \( \overline{p}(t) \) would produce an extension of \( k \) of degree at least \( m + 1 \). This is impossible because \( p \) is divisible by \( t \) and by removing this factor we get a polynomial of degree \(< m \) which cannot produce an extension of degree \( m + 1 \).

It follows that by successively raising \( t \) to power \( q - 1 \) we will eventually run into either of the following two cases: i) \( t \in A \) and ii) \( t \notin A \) and \( \overline{p}(\overline{t}) \neq 0 \) for some \( a \in A \) hence \( t - a \) is invertible in \( A' \) but does not belong to \( A \). If we reach case ii) we are done. If we reach case i), let’s stop right before it, so that \( t^{q - 1} \in A \) but \( t \notin A \). Now recall that \( t \) was chosen to have smallest possible minimal polynomial over \( A \) (raising \( t \) to any power does not increase the degree of its minimal polynomial). Since \( t^{q - 1} \in A \), it follows that \( \text{deg}(p) \leq q - 1 \) but then \( \overline{p} \) cannot be divisible by \( t^q - t \) and so we reach case ii) again.

This finishes the construction of an element \( t \in A' \) which is invertible over \( A' \). Let’s write \( t \in K \) as \( \frac{u}{v} \) where \( u, v \in A \) and consider the Cartier divisor \( A \xrightarrow{u} A - \) in terms of effective divisors, it is the ratio of \( A \xrightarrow{u} A \) and \( A \xrightarrow{v} A \). The map \( A \xrightarrow{t} A \) is well-defined on \( U \) (as \( t \) is invertible there since \( A' \cong A \) on \( U \)) and induces an isomorphism there. Thus this divisor gets sent to the zero Weil divisor - but it is non-trivial, since otherwise we would have to produce an isomorphism as follows:

\[
\begin{array}{c}
A \xrightarrow{u} A \\
\sim \\
A \xrightarrow{id} A
\end{array}
\]

hence the left vertical isomorphism \( A \xrightarrow{\sim} A \) would have to be given by an invertible element of \( A \), hence \( t, \frac{1}{t} \) inside \( K \) would have to belong to \( A \), contradicting condition ii).

Alternatively, this contradicts condition iii) as well as the line bundle \( u\mathcal{O}_U \subset \mathcal{O}_U \) on \( U \) admits two extensions to \( X \): namely \( u\mathcal{O}_X \) and \( v\mathcal{O}_X \). □

Remark: \( K^x/\Gamma(X, \mathcal{O}^x) \hookrightarrow \text{Div}^W \) does not imply normality. For example, the pinched affined plane \( X = \mathbb{A}^2 = \text{Spec}[X^2, XY, Y^2, X^3, XY^2, X^2Y, Y^3, X^4, \ldots] \) is not normal since localizing at \((X, Y)\) does not make it integrally closed - \(((X^2)/X)^2 = X^2 \) but \( X^3 \) does not belong to the localization. However if \( a \in K^x \) gets mapped to 0 in \( \text{Div}^W_{\mathbb{A}^2} \), it is easy to see that the \( a \) gets mapped to 0 in \( \text{Div}^W_{\mathbb{A}^2} \) as well, but this implies \( a \) is a unit since \( \mathbb{A}^2 \) is normal.

It is obvious that a locally factorial ring is normal. However, the converse is not true. To construct a counterexample, we introduce two important notions:

**Definition.** Let \( X \) be a scheme of dimension \( n \) which is of finite type over a field \( k \) (i.e. over \( \text{Spec}(k) \)). We say that \( X \) is LCI (locally complete intersection) if locally (i.e. on an affine cover) it looks like \( \text{Spec}(k[x_1, \ldots, x_m]/(f_1, f_2, \ldots, f_{m-n})) \) where \( f_1, \ldots, f_{m-n} \) is a regular sequence.

**Definition.** If \( A \) is a local Noetherian ring and \( M \) is a finitely generated module over \( A \), then \( M \) is Cohen-Macaulay or CM if \( \text{depth}(M) = \text{dim}(\text{Supp}(M)) \). \( A \) is CM of dimension \( n \) is it is such as a module over itself, and a scheme is CM if and only if every of its local rings is.

Remarks: As shall be proven later, the dimension of every irreducible component of \( \text{Supp}(M) \) is \( \geq \text{depth}(M) \), so \( M \) is CM if all irreducible components of \( \text{Supp}(M) \) have the same dimension \( \text{depth}(M) \). Also \( A \) is CM if and only if its depth equals its Krull dimension. \( A \) field is always CM, as is a regular ring hence any ring of finite cohomological dimension.

**Proposition.** If \( X \) is LCI then it is CM.

Proof: if \( A = k[x_1, \ldots, x_m]/(f_1, \ldots, f_{m-n}) \) we need to show \( A \) is CM over itself. We claim it’s enough to show it’s CM over \( k[x_1, \ldots, x_m] \). Indeed, as shown before, the depth over every localization stays the same, and the dimension of the support is also the same - as the supports are actually equal.
In general, we claim that if \( f_1, \ldots, f_{m-n} \) form a regular sequence with respect to a module \( M \), then if \( M \) is CM the so is \( M/(f_1, \ldots, f_{m-n})M \) and the dimension drops by \( m-n \). Indeed, the depth drops by \( m-n \)-this has been shown a while ago. It remains to show that when replacing \( M \) by \( M/fM \) the dimension of the support drops by exactly 1. A priori it drops by at least 1 because every prime ideal in \( A/ann(M/fM) \subset A/ann(M) \) contains \( f \) which is a non-zero divisor in \( A/ann(M) \) so it has height 1 with respect to \( A/ann(M) \) - this shows that all minimal primes in the support of \( M \) are removed so the dimension drops. We are left to show that it does not drop by more than 1 - which is ensured by the inequality depth \( \leq \dim(supp) \) which we will prove later. □

Now we construct a ring which is normal but not locally factorial. We will construct a ring which is regular in codimension 1 and \( CM \), and it will later be showed that \( CM \) implies \( S2 \) hence it will be normal.

Take \( A = \mathbb{C}[X,Y,Z]/(X^2 + Y^2 + Z^2), X = SpecA \).

We claim that \( X \) is integral, regular in codimension 1, and normal.

Integrality is easy. For regularity, we use the criterion that a local ring is regular if its Krull dimension equals \( \dim_A m/m^2 \) - apparently it was done in 221. Let’s choose a prime \( p \) of height 1 - identified with a prime (denoted by \( q \)) of \( \mathbb{C}[x,y,z] \) of height 2 that contains \( x^2 + y^2 + z^2 \). Since \( \mathbb{C}[x,y,z] \) is regular, the ring \( A = \mathbb{C}[x,y,z]_q \) is a regular local ring of dimension 2. Its residue field \( k \) and maximal ideal \( m \) satisfy \( \dim_k m/m^2 = 2 \). Now in general if we replace \( A \) by \( A/FA \) where \( A \) is regular, the Krull dimension will become 1 while the maximal ideal will become \( m/fA \). It’s easy to see that \( m/m^2 \) will be replaced by \( m/(m + fA) \). It dimension over \( k \) will either stay the same or drop by 1 depending whether \( f \) belongs to \( m \) or not. Thus we only need to show that \( f = x^2 + y^2 + z^2 \) does not belong to \( m^2 \) if \( q \) has height 2 - i.e. that there is no \( g \notin q \) such that \( fg \in q^2 \). Moreover by regularity \( p \) is generated by two elements \( u,v \) which we can assume belong to \( \mathbb{C}[x,y,z] \) therefore we can actually get \( fg \in (u^2,v^2,we) \). By Hilbert’s Nullstellensatz there exists a point \((a,b,c) \in C^3 \) such that \( g(a,b,c) \neq 0 \), \( u(a,b,c) = 0 \), \( v(a,b,c) = 0 \). It then follows that \( f \) vanishes at 0 with degree \( \geq 2 \) hence \( (a,b,c) = 0 \) so all its first derivatives at \( a,b,c \) are 0 too. Since its derivatives are \( 2x,2y,2z \) we deduce that \((a,b,c) = (0,0,0) \). It follows that the free term of \( g \) is non-zero so we can assume it is 1. Then by reducing modulo terms of degree higher than 3 we deduce that \( x^2 + y^2 + z^2 = au_1^2 + bu_1v_1 + cv_1^2 \) where \( u_1,v_1 \) are the linear parts of \( u,v \). However the last part is impossible: there is a common zero \((a,b,c) \neq (0,0,0) \) of \( u_1,v_1 \) and as we’ve shown before there cannot be such a double zero for \( x^2 + y^2 + z^2 \).

So it is regular in codimension 1, and obviously an \( LCI \), hence it is also CM and thus \( S2 \), which tells us that the ring is normal. It is not locally factorial, though. If we localize at the maximal ideal \((x,y,z) \), then the prime ideal \((x,y + iz) \) (that contains \( x^2 + y^2 + z^2 \)) has height 2 but is not principal. Indeed, if some single \( u \) generates it then \( p_1 x - p_2 u \) must be divisible by \( x^2 + y^2 + z^2 \) for some \( p1 \) with \( p1(0,0,0) \neq 0 \) and similarly \( q_1(y + iz) - q_2 u \) must be divisible by \( x^2 + y^2 + z^2 \) for \( q_1(0,0,0) \neq 0 \). We can now remove multiples of \( x^2 + y^2 + z^2 \) from \( p_1 \) and \( u \) so we can assume that \( u = u'(y,z) + x u'' (y,z) \). Because \( u \in (x,y + iz) \) we can deduce \( u'(y,z) \) is divisible by \( y + iz \). We then get \( p_1^2 x^2 + p_1 x - p_2 u' - x(p_2 u'' + p_2 v') = x^2 p_2 u' \) is divisible by \( x^2 + y^2 + z^2 \) hence \( p_2 u' + p_2 u' - p_1 = 0, p_2 u' = (y^2 + z^2)(p_2 u'' - p_2 u'') \). It follows that \( p_2, u'' \) are not in the maximal ideal \((x,y,z) \).

Similarly, we get \( q_1'(y + iz) + x q_1''(y + iz) - (q_2' + xq_2')(u' + x u'') = q_1'(y + iz) - q_2 u' - x^2 u'' q_2'' + x(q_1'(y + iz) - q_2 u' - q_2 u') \) is divisible by \( x^2 + y^2 + z^2 \) from where we get \( q_1'(y + iz) - q_2 u'' = 0 \) and \( q_1'(y + iz) - q_2 u' + (y^2 + z^2) u' q_2'' = 0 \). It follows from the first identity that \( q_2' \) is divisible by \( y + iz \) - but then we can simplify the second identity by \( y + iz \) to get \( q_1' - q_2' u' + (y - iz) u'' q_2'' = 0 \) but this is impossible as then since \( u'' \) is divisible by \( y + iz \), it would imply that \( q_1' \) is in the maximal ideal, which is false.

Alternatively, the ring is not locally factorial because the map \( Div^C \to Div^W \) is not surjective. Namely the Weil divisor associated to the prime \( p \) above is not given by any line bundle - if this line bundle were given above the maximal ideal \((x,y,z) \) by some element \( f \), and above \( p \) by some element \( f' \), then because \((x,y,z) \) is in the closure of \( p \), we can readily deduce \( f = f' \) and similarly for any other ideal contained in \((x,y,z) \), which will tell that the only prime contained in \((x,y,z) \) that contains \( f \) is \((f, p \notin p^2) \), but this would mean that \( p \) is principal in the local ring at \((x,y,z) \) as it is generated by \( f \).

**Theorem.** Let \( X \) be regular in codimension 1 and integral. Then the following are equivalent:

i) \( X \) is locally factorial
ii) $Div^C \to Div^W$ is an isomorphism for all $\overset{\circ}{X}$ open subschemes of $X$

iii) For any $\overset{\circ}{X}$ open subscheme of $X$ and a CO pair $Y \hookrightarrow \overset{\circ}{X} \supset U$ with $codim(Y) \geq 2$, any line bundle $\mathcal{L}_U$ on $U$ admits an extension $\mathcal{L}_X$ to $\overset{\circ}{X}$ and it is unique.

Proof: because of a previous proposition, normality of $X$ is equivalent to injectivity in ii) and uniqueness in iii), so we can assume these conditions already hold. In that case, the equivalence of i) and iii) has already been established before, and so was the equivalence of i) and ii). □

**Proposition. (PSET)** Let $A$ be a local Noetherian ring. Let $M$ be a finitely generated $A$-module.

a) $\text{depth}(M) \leq \dim(A/p)$ for any associated prime $p$, by induction on depth.

Base is obvious.

Now let’s do the induction step. If the depth of $M$ is zero we have nothing to prove. Otherwise, we find $f \in A$ such that multiplication by $f$ is injective on $M$. We also find a submodule $M'$ isomorphic to $A/p$ and let $M'' = \{m \in M \mid \exists n, f^nm \in M'\}$

We have $0 \to M'' \to M \to M/M'' \to 0$ and now $M/M''$ is obviously $f$-torsion free, which implies that $M''/fM''$ injects into $M/fM$ because if $m \in M''$ is such that $m = fm'$ then $m' \in M''$ by definition of $M''$.

Also we have that the support of $M''$ is contained in $\text{Spec}(A/p)$ also pretty much by definition, so that the support of $M''/fM''$ is contained in $\text{Spec}(A/(p,f))$. Note that the ring $A/(p,f) = (A/p)/f$ has Krull dimension strictly less than the dimension of $A/p$ - because, for example, any chain of ideals containing $p,f$ can be completed with $p$ as $f$ clearly is not in $p$. Therefore any of the irreducible components of $\text{Supp}(M''/fM'')$ has dimension at most one less than the dimension of $A/p$. This component is an associated prime of $M/fM$ which as we know has depth one less, so now the induction base finishes.

b) If $A$ is Artinian then $m^k = 0$ for some $k$, because the sequence $m \supset m^2 \supset \ldots$ must stabilize and if $m^k = m^{k+1}$ then $m^k = 0$ by Nakayama’s Lemma. Then the depth of $M$ is zero, just like in the previous part. Also, every irreducible component of $\text{Supp}(M)$ has the form $V(A/I)$ and it must have dimension 0 because the ring $A/I$ has one maximal ideal which is also nilpotent so belongs to all prime ideals therefore it has only one prime ideal and the Krull dimension is zero.

c) We have proven that the projective dimension plus the depth equals $n$. Thus a module $M$ is CM if and only if its depth equals $n$ and its support has dimension $n$. The first condition is equivalent to $M$ having projective dimension 0 i.e. $M$ being projective i.e. $M$ being free (since $A$ is local Noetherian). Moreover if $M$ is free the first condition holds by equivalence and the support of $M$ is $\text{Spec}(A)$ which has dimension $n$.

e) Nothing to prove, really. The depth of the module over $A$ and over $A'$ is the same as proven in class because the map $A \to A'$ is surjective. The dimensions are also the same, because every chain in $\text{Supp}(M)$ as a module over $A$ yields a chain in $\text{Supp}(M)$ as a module over $A'$ and vice versa, as every prime must contain the kernel of $A \to A'$ since it annihilates $M$. □

**Proposition. (PSET)** Let $A$ be a Noetherian ring and $M$ a f.g. $A$-module. Let $A'$ be another Noetherian ring, and $A' \to A$ a homomorphism. Let $p'$ be a prime of $A'$. 

78
a) Assume $A$ is f.g. as an $A'$-module. Let $p_i$ be the prime ideals of $A$ above $p$ (note that the latter are in a bijection with the maximal ideals of the Artinian ring $A \otimes k'$). Then the depth of $M_p$ as an $A_p'$-module equals the minimum over $i$ of the depths of $M_{p_i}$ as an $A_{p_i}$-module.

b) Assume $M$ is finitely generated over $A'$ (but we no longer assume $A$ is f.g. over $A'$). Let $p_i$ be the primes of $A$ over $p'$ such that $M_{p_i} \neq 0$. Then the same conclusion as in a) holds.

Proof: a) First it is enough to replace $A'$ by $A'$ and $A$ by $A_{A' - p}$ (localization). Also it is enough to consider the case where $A'$ is a subring of $A$ because we can replace $A'$ by its image and this will not change the depth. We claim that $\text{depth}_{m'}(M) > 0$ if and only if all $\text{depth}_{p_i}(M_{p_i}) > 0$. This is enough to prove the claim by induction, as follows.

The base is proven, and is $\text{depth}_{m'} M > 0$ then there is $f \in m$ injective on $M$. We claim it will also be injective on $M_{p_i}$. Indeed, if $fm = 0$ in $M_{p_i}$, then $fcm = 0$ for $c \in A - p_i$ then as $f$ is injective we get $cm = 0$ so that $m = 0$ in $M_{p_i}$. Moreover $M_{p_i}/fM_{p_i}$ is naturally isomorphic to $(M/fM)_{p_i}$. Here is how we show this: the first ring receives a surjective map from $M_{p_i}$. The second ring also does, because $M_{p_i} = M \otimes_A A_{p_i}$ and the map $M \otimes_A A_{p_i} \to (M/fM)_{p_i}$ is bilinear. So we have to show that the kernels of these maps coincide. The kernel of the first map is $fM_{p_i}$. The kernel of the second map consists of all elements $\frac{m}{x}$ such that $\frac{mx}{x^2} = 0$ i.e. $mc = fm'$ meaning $\frac{m}{x} = \frac{m'}{\frac{x}{c}}$ i.e. the kernel of the second map is $fM_{p_i}$. Now we can conclude by the induction on $M/fM$ as all relevant depths decrease by 1.

So it remains to show the claim. One direction is easy: assume $\text{depth}_{p_i}(M_{p_i}) = 0$ so that there is an element $m \in M_{p_i}$ whose annihilator is $p_i$. We may assume $m \in M$ as it can be multiplied by something invertible in $M_{p_i}$ to become an element of $M$ (or rather, represented by an element in $M$). In this case we claim that $m$ is annihilated by $p \in A$. Indeed, first of all any element in $a \in p$, $am$ must be zero in $M_{p_i}$, so that $cam = 0$. In this case we can replace $m$ by $cm$ and since $p$ is finitely generated we can "correct" $M$ so that $p$ kills it. Conversely if $am = 0$ then it must also be 0 in $M_{p_i}$, meaning that $a$ must be in $p_i$ but as $a \in A'$ we must have $a \in p$ which concludes. An alternative proof is done by using the element $f$ from above.

Finally let’s prove that if $\text{depth}(M) = 0$ then one of the depths of $M_{p_i}$ is also 0. There is an element $m \in M$ whose annihilator in $A'$ is $p$. We claim that one of its annihilators in $A_{p_i}$ is precisely $p_i$.

Recall from a problem set in the previous semester that $p_i$ correspond to all prime (and maximal) ideals of the algebra $B = A \otimes_{A'} k$ which is a finite vector space over $k$, and $A_{p_i}/p_i A_{p_i}$ is a finite vector space over $k$ as well.

Consider the $A$-submodule $M_1$ generated by $m$ in $M$. Because $p$ kills $m$, it factors as a module over $B = A \otimes_{A'} k$ and being monogenerated it is isomorphic as $B/J$ where $J$ is an ideal of $B$ which is not all of $B$ because $k$ injects into $M_1$. Then $J$ is contained in a maximal ideal corresponding to some $p_i$. We claim that his $p_i$ works. For this, we need to show that $(B/J)_{p_i} \cong (B/J)_{q_i}$, is isomorphic to $A_{p_i}/p_i A_{p_i}$, which is naturally isomorphic to $B/q_i$ where $q_i$ is the maximal ideal of $B$ corresponding to $p_i$. Consider an element $b \in B/J$. Assume first that $b \in q_i/J$ then it will go to 0 in the localization. Indeed, by the Chinese remainder theorem there exits $c \in \bigcup_{i \neq j} q_i - q_i$ meaning that $c$ is invertible in the localization but $cb$ is nilpotent - therefore $(cb)^k = 0$ for $k > 1$ so that $b(c - e^k b^{k-1}) = 0$ and $c - e^k b^{k-1} \notin q_i$ so that it goes to 0 in the localization. Conversely, if $b \notin q_i/J$ it may not go to 0 because if we multiply it by anything not in $q_i$ we will get an element not in $q_i$. This finishes the proof.

Remark we implicitly used the fact that $B/q_i$ is naturally isomorphic to the residue field of $A/p_i$. This comes from construction: $p_i$ gives is a prime of $A/pA$ which does not meet $A' - p$ so produces a prime in $A/pA \otimes A'_p = B$ which is $q_i$. The first step (passing from a prime in $A$ to a prime in $A/pA$ does nothing to the residue ring) while the second localizes it with respect to $A' - p$. As we know this already produces a field so this is the residue field of $p_i$.

b) The primes $p_i$ such that $M_{p_i}$ is non-zero are those that contain $\text{ann}(M)$. So it suffices to consider $A/\text{ann}(M)$ and $A'/\text{ann}_{A'}(M)$ instead. We will be done by part a) once we show that $A/\text{ann}(M)$ is finite over $A'/\text{ann}_{A'}(M)$ - or over $A'$ which is the same thing. Indeed, $A/\text{ann}(M)$ injects into $\text{End}_{A'}(M)$ which is a quotient of $M_k(A')$ where $(A')^d$ surjects onto $\text{ann}(M)$ and this is a finite $A'$-module. □

**Proposition. (PSET)** Let $k$ a field, $A$ a f.g. $k$-algebra, and $M$ a f.g. $A$-module.

We say that $M$ is $CM$ of dimension $n$ if $M_m$ is CM over $A_m$ of dimension $n$ for every maximal ideal $m$ of $A$. 79
Let \( A' \to A \) be a homomorphism such that \( M \) is f.g. as an \( A' \)-module. Then \( M \) is CM of dimension \( n \) over \( A \) if and only if it is such over \( A' \).

Proof: Like before, it suffices to consider the case when \( A \) is finite over \( A' \) - this is because we can replace \( A \) by \( A/\text{ann}(M) \). If that were to mess up some of the maximal ideals \( m \) i.e. \( \text{ann}(M) \not\subseteq m \) then \( M \) would be zero, a case which is impossible (but we’ll return to it later). For the same reason, is suffices to regard \( A' \) as a subring of \( A \).

In that case \( A \) is finite over \( A' \) so it is integral over \( A' \) and we quote theorem 5.8. from Atiyah-McDonald which says for \( q \in \text{Spec}(A) \), \( p \in \text{Spec}(A') \), \( q \cap A' = p \) we have that \( p \) is maximal if and only if \( q \) is maximal.

Assume now that \( M \) is CM of dimension \( n \) for \( A \). Choose a maximal ideal \( m \in A' \). According to the theorem just quoted, every prime ideal above \( m \) in \( A \) is maximal, so by the previous problem we conclude \( \text{depth}(M_m) = n \). Then the minimal dimension of an irreducible component of \( \text{Supp}_{A'_m}(M_m) \) is at least \( n \) - we want to show that the dimension of an irreducible component of \( \text{Supp}_{A'}(M) \) is on the other hand at most \( n \), which would imply that \( M \) is CM of dimension \( n \) for \( A' \). Indeed, if we had a chain \( q_1 \subset \ldots \subset q_{n+1} \) inside \( m \) then by the going up theorem we would have produced a chain \( p_1 \subset \ldots \subset p_{n+1} \) in \( A \). Also if \( M_{q_1} \neq 0 \) then \( M_{p_1} \neq 0 \) - because we took \( A/\text{ann}(M) \) in the beginning. This would have implies that dimension of \( \text{Supp}_{A_p}(M_p) \) is at least \( n + 1 \) for some \( p \supset p_{n+1} \) maximal, which is false.

Assume that \( M \) is CM of dimension \( n \) for \( A' \). Choose a maximal ideal \( m \in A \) and let \( m' = m \cap A' \) then \( m' \) is maximal in \( A' \). From the previous problem, the depth of \( M_m \) is at least \( n \). We claim that it in fact equal to \( n \), by induction. More precisely, we claim that the dimensions of \( M_m \) for \( m, n \) above \( m' \) are all equal to the dimension of \( M_m \).

Indeed, recall how we did problem 2. We showed that \( \text{depth}(M_{m'}) = 0 \) if and only if one of \( \text{depth}(M_{m_i}) \) is 0 and then we used induction to show that the depth equals the MINIMUM. To show that the depth equals the COMMON value of the depths of \( M_m \), we need to modify the statement to \( \text{depth}(M_{m'}) = 0 \) if and only if all of \( \text{depth}(M_{m_i}) \) is 0.

Again, one direction is clear so let’s prove that \( \text{depth}(M_{m'}) = 0 \) implies \( \text{depth}(M_{m_i}) = 0 \), for example. We chose an \( x \in M \) such that \( \text{Ann}(x) = m' \) and then deduced that \( Ax \) is isomorphic to \( BJ \) for some proper ideal \( J \). If \( J \) is not contained in \( m \) then there is an element \( u \in A - m \) that kills \( x \), and since \( A/m \) is a finite extension of \( A'/m' \), \( u \) as an element of \( \text{Hom}_{A'}(M) \) must satisfy a monic irreducible equation with coefficients in \( A' \) whose free term \( c_0 \) is not in \( m' \). But then \( c_0 \) kills \( M \) but is not in \( m' \) - impossible. Thus \( J \subset m \) and this allowed us to conclude that \( \text{Depth}_m(M_m) = 0 \), as desired.

Now we need to show the second part - that the dimension of any irreducible component of \( \text{Supp}(M_m) \) is exactly \( n \). We already know it’s at least \( n \). Assume it were larger so that we had a chain \( q_0 \subset \ldots \subset q_n \) of prime ideals with \( q_0 \) in \( \text{Supp}(M_m) \). Intersecting them with \( A' \) yields a chain of length \( n + 1 \) of prime ideals contained in \( m' \) and they are clearly in the support of \( M_{m'} \). The ideals cannot become equal to each other because if say \( q_0 \cap A' = q_1 \cap A' = p \) then there are two ideals above \( p \) contained in each other which is impossible as inclusion would be preserved for their residue fields and they are the residue fields of the maximal ideals of \( A/pA \) which are not contained in each other (not in the natural way). Contradiction which shows the claim.

Now we come back to the idea in the beginning: why we are allowed to quotient \( A \) by \( \text{ann}(M) \). This is allowed as long as all maximal ideals of \( A \) contain \( \text{ann}(M) \) and similarly for \( A' \), otherwise we would lose some maximal ideals. But if \( \text{ann}(M) \) is not contained in \( m \), then \( M_m \) is immediately 0 so its depth is 0. And because \( A/m \) is a finite extension over \( A'/m' \) at least when projected to how it acts on \( M \), this easily implies that something not in \( m' \) must kill \( M \) (just like above with \( c_0 \) hence \( M_{m'} \) is also 0. Thus \( n \) can only be 0 and these case mutually imply (or exclude) each other.

Finally, we can also replace \( A' \) by its image in \( A \) because of 1e). Again, we need to make sure we do not lose any maximal ideals - but in fact we don’t need this as each maximal ideal projects to a maximal ideal in the quotient. □

**Proposition. (PSET)** Let \( A \) be a finitely generated algebra over a field \( k \), and \( M \) a finitely generated \( A \)-module.

a) \( M \) is CM of dimension \( n \) if and only if for some \( A' \) which is regular of dimension \( n \) with a map \( A' \to A \) such that \( M \) is f.g. as an \( A' \)-module (such an \( A' \) exists by the Noether normalization lemma), \( M \) is locally free over it. In
this case, $M$ is locally free for any such choice of $A'$.

b) If $\text{Supp}(M) = \text{Spec}(A)$ and $M$ is CM, then $M$ is $S_k$ for all $k$.

Proof: a) If $M$ is locally free over $A'$, then it is CM of dimension $n$ over it, and by the previous proposition it is CM of dimension $n$ over $A$. The converse follows by the same route (which is invertible).

The any part is also clear because we simply took "any" $A'$ to begin with.

b) Consider a prime $p$ of height $k$. We want to show that $M_p$ has depth at least $k$. Note first that it is non-zero. Now consider any $A'$ like in part a) - moreover which is integrally closed as $k[y_1, \ldots, y_d]$ is, which exists by Noether normalization lemma, and let $q$ be the preimage of $p$. According to part a), $M$ is locally free over $A'$ in particular $M_q$ is isomorphic to a non-zero direct sum of copies of $A_q$. Observe also that $A_q$ is regular and has at least the same height as $A_p$ - because from every chain of prime ideals in $A$ (ending in $p$ for this particular case) we build a chain of primes from their contractions, and two primes contained in each other cannot contract to the same thing, because then as in a) of the pre-previous proposition they would correspond to maximal ideals in some Artinian algebra but such ideals do not contain each other. If follows that the depth of $M_q$ is equal to the depth of $A_q'$ which is at least $k$, and $M_q$ is therefore CM over $A_q'$.

Now we apply the previous problem to the map of rings $A_q' \to A_p$ which implies that the depth of $M_p$ is the same as the depth of $M_q$, so is at least $k$. □

**Proposition. (PSET)** Let $A$ be a f.g. algebra over a field $k$

a) $A = k[x, y]/xy$ is CM of dimension 1.

Assume that $A$ is CM of dimension $n$.

b) $X = \text{Spec}(A)$ is equidimensional (all irreducible components of $X$ have the same dimension).

c) $A$ satisfies $S1$, and if it is reduced at its generic points, then it’s reduced.

d) $A$ satisfies $S2$ (in fact $S_k$ for any $k$)

e) If $A$ is regular in codimension 1, then it’s normal.

Proof: a) We claim that the ring has depth 1. Indeed, it does not have an ideal isomorphic to $k$, because there is no element which is killed simultaneously by $x$ and $y$. On the other hand, it does not contain a regular sequence of length 2, for the following reason: assume $u, v$ are elements that form a regular sequence, so that $u$ is not divisible by $x$ nor $y$. Let’s regard $u$ and $v$ as polynomials in $k[x, y]$ via some lifting . If $u$ and $v$ share a common factor $p$, then the equation $uf = vg$ has solutions with $g = \frac{u}{v}$ thus $v$ sends $g$ to 0 in $k[x, y]/(xy, v)$ - which mean $g$ is itself 0 there so that $g = xuv + uq_1$. In particular this means $g \mid q_1$ as $u$ and $g$ are not divisible by $x$ or $y$ so that then $1 = xy\frac{uq_1}{g} + up$. This is impossible because then $u$ is invertible. It remains to see that $u$ and $v$ can be made to be not coprime, because for example they can be made divisible by $xy - 1$ by adding a certain multiple of $xy$.

It remains to prove that $k[x, y]/(xy)$ has Krull dimension 1. Indeed, the sequence $(x) \subset (x, y)$ yields Krull dimension at least 1. Assume now that there exists a chain $p_1 \subset p_2 \subset p_3$ of primes containing $xy$ - in particular $p_1$ contains either $x$ or $y$ so we may assume $(x) \in p_1$. By adding 0 to it we produce a sequence of four primes in $k[x, y]$ but $k[x, y]$ has Krull dimension 2. Contradiction.

b) Follows from a) of a previous proposition (the first one from this PSET). Note that the irreducible components of $X$ are among the associated primes of $A$ as an $A$-module so that $n \leq$ the minimum of these dimensions. On the other hand, by definition $n$ equals the maximum of these dimensions - so they must be all equal.

c) $S1$ follows from b) of the previous proposition and now the problem follows from 1e) of the previous problem set.

d) Follows from b) of the previous proposition.

e) If $A$ is regular in codimension 1 then every $A_p$ for $p$ of height 1 is regular of height 1 so is a DVR, in particular $R_1$ follows, once we show the $R_0$ part as well, i.e. that $A_p$ is a field for all minimal primes $p$. Indeed it is, for each
minimally prime is contained in a prime of height 1 so we are basically left to show that the localization of a DVR is a field - which is obvious. Now we have $R_1$ and $S_2$ which imply normality. □

**Proposition. (PSET)** Let $A$ be a local Noetherian ring, and $M$ a f.g. $A$-module of depth $n$. Let $p \subseteq A$ be a prime ideal such that $\text{dim}(A/p) = k$. Then $\text{depth}(M_p) \geq n - k$. Also if $M$ is CM of dimension $n$, then $M_p$ is CM of dimension $n - k$.

Proof: it is enough to consider the case $k = 1$ and then proceed by induction. We also perform induction on $n$. The case $n = 1$ is clear. Assume $n > 1$. Because of a previous proposition, every associated prime of $M$ has codimension at least $n$, in particular $p$ cannot be contained in any such prime because it has codimension at least 1, so by a previous proposition there is a regular element $f \in p$ of $M$. Note that it will then be a regular element of $M_p$ as well: if $f^n = 0$ for $x \in A - p$ then $fmy = 0$ for some $y \not\in p$ thus either $my = 0$ in which case $f^n = 0$ in $M_p$ or $my \neq 0$ which contradicts the regularity of $f$ on $M$.

Thus the dimension of $M/fM$ and $M_p/fM_p$ drops exactly by 1, and we now apply the induction step to $M/fM$ and the ring $A/fA$. Note that the codimension of $p$ in $A/fA$ stays the same as the quotient $A/p$ stays unchanged.

The second part follows since $M_p$ will have dimension $n - k$ but each irreducible component of the support of $M_p$ has dimension at most $n - k$ - any associated prime $q$ of $M_p$ corresponds to a prime $q' \subseteq p$ in $\text{Supp}(M)$ and its codimension is the length of the longest chain from $q'$ to $p$ - which is at most $n - k$ since we can augment a chain of length $k$ to create a chain from $q'$ to $m$ the maximal ideal of $A$ but the length of this chain cannot be greater than the codimension of $\text{Supp}(M)$ (which is $n$) since clearly $q' \in \text{Supp}(M)$. Thus the depth equals at least the dimension of any irreducible component of $M_p$, and since the reverse inequality holds, this is actually an equality so $M_p$ is CM. □

Recall the notion of $\text{Pic}(X) = H^1(X, \mathcal{O}^\times)$, which for a Noetherian scheme describes the isomorphism classes of line bundles on $X$. If $X$ is integral, then the sheaf $\mathcal{K}^\times = \mathcal{K}_X^\times$ is flasque, so the exact sequence

$$0 \rightarrow \mathcal{O}^C \rightarrow \text{Pic}(X) \rightarrow H^1(X, \mathcal{K}^\times)$$

implies $\text{Pic}(X) \cong \mathcal{Cl}^C(X)$. If $X$ is in addition locally factorial, $\text{Pic}(X) \cong \mathcal{Cl}^C(X) \cong \mathcal{Cl}^W(X)$.

We propose to compute $\text{Pic}(\mathbb{P}^n) = \mathcal{Cl}^C(\mathbb{P}^n) = \mathcal{Cl}^W(\mathbb{P}^n)$.

**Proposition.** $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$

Proof: first we prove a lemma:

**Lemma.** Let $U$ be an open subscheme of $X$ whose complement $Y$ is an irreducible subscheme of codimension (height) 1. Then we have the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \text{Div}^W(X) \rightarrow \text{Div}^U(X) \rightarrow 0$$

where the first map is given by $1 \rightarrow (p)$ where $p$ is the prime corresponding to the complement of $U$, in particular passing to $\text{Cl}$ we get

$$\mathbb{Z} \rightarrow \text{Cl}^W(X) \rightarrow \text{Cl}^W(U) \rightarrow 0$$

The proof of the lemma is obvious, as $U$ contains all primes of height 1 but $p$.

In particular, we apply the lemma to $U = \mathbb{A}^n \subseteq \mathbb{P}^n$. Note that $\text{Cl}^W(U) = 0$ because $\mathbb{A}^n$ is globally factorial - $k[x_1, \ldots, x_n]$ is a UFD.

Thus $\mathbb{Z} \rightarrow \text{Cl}^W(\mathbb{P}^n) \cong \mathcal{Cl}^C(\mathbb{P}^n)$. We claim the map is 1 $\rightarrow \mathcal{O}(-1)$. Indeed, by definition 1 gets sent to $\mathcal{O}_{\mathbb{P}^{n-1}}$. If $x_0$ is the linear function whose locus of vanishing is $\mathcal{O}_{\mathbb{P}^{n-1}}$ and we identify $\mathbb{P}^n$ with $\text{Proj}(k[x_0, \ldots, x_n])$ then by definition 1 gets mapped to the line bundle which locally on $\text{Spec}(k[x_0^2, \ldots, \frac{x_0}{x_1}])$ is generated by $\frac{x_0}{x_1}$. This line bundle is isomorphic to $\mathcal{O}(-1)$ via the map $\mathcal{O}(-1) \xrightarrow{x_0} \mathcal{O}$. Finally, we need to prove that the map $\mathbb{Z} \rightarrow \mathcal{Cl}^C(\mathbb{P}^n)$ is injective - that amounts to saying that no $\mathcal{O}(n)$ is isomorphic to $\mathcal{O}$. This is indeed the case by cohomology: if $n > 0$, then $H^0(\mathcal{O}(n))$ has $k$-dimension $> 1$ and if $n < 0$ then $H^0(\mathcal{O}(n))$ has $k$-dimension 0, while $H^0(\mathcal{O})$ has dimension 1. □

82
When \( f: X \to Y \) is a morphism between two schemes, if \( \mathcal{L}_Y \) is a line bundle over \( Y \) then \( f^* \mathcal{L}_Y \) is a line bundle over \( X \), moreover if this line bundle is isomorphic to \( \mathcal{O}_Y \) then \( f^* \mathcal{L}_Y \) is isomorphic to \( \mathcal{O}_X \). Thus we can speak of the functor \( f^* \) that sends \( \text{Pic}(Y) \) to \( \text{Pic}(X) \)

**Proposition.** If \( S \) is integral and regular in codimension 1, then

\[
\text{Pic}(S) \times \mathbb{Z} \cong \text{Pic}(S \times \mathbb{P}^n)
\]  

(\( \mathbb{P}^n \) is taken over a field).

**Proof:** more precisely, considering \( p_1, p_2 \) the two projections \( \mathbb{P}^n \times S \to \mathbb{P}^n, S \). We introduce the operation "\( \boxtimes \)" from \( \text{Pic}(\mathbb{P}^n) \times \text{Pic}(S) \to \text{Pic}(\mathbb{P}^n \times S) \) by \( \mathbb{P}^n \boxtimes S = p_1^* \mathcal{L}_1 \boxtimes p_2^* \mathcal{L}_2 \).

We claim that every line bundle on \( \mathbb{P}^n \times S \) is isomorphic to \( \mathcal{O}(m) \boxtimes \mathcal{L} \) for some line bundle on \( S \). This will show that the map \( \text{Pic}(\mathbb{P}^n) \times \text{Pic}(S) \to \text{Pic}(\mathbb{P}^n \times S) \) given by \( \boxtimes \) is surjective. This map is also injective, as \( \mathcal{O}(m) \boxtimes \mathcal{L} \) can be recovered from \( \mathcal{O}(m) \boxtimes \mathcal{L} \). Indeed, there is a map \( Spec(k) \to \mathbb{P}^n \) which produces a map \( q: S \to \mathbb{P}^n \times S \) such that \( p_2q = id_S \) and \( p_1q \) is the "zero map" from \( S \to Spec(k) \to \mathbb{P}^n \) and then \( q^*(\mathcal{O}(m) \boxtimes \mathcal{L}) = q^*p_1^*\mathcal{O}(m) \boxtimes q^*p_2^*\mathcal{L} = (p_2q)^*\mathcal{O}(m) \boxtimes (p_1q)^*\mathcal{L} = \mathcal{O} \boxtimes \mathcal{L} = \mathcal{L} \). To recover \( \mathcal{O}(m) \), we tensor with \( p_2^*(\mathcal{L}') \) so we need to know how to recover \( \mathcal{O}(m) \) from \( p_2^*\mathcal{O}(m) \).

Let \( s \in S \) be a point, and \( k_s \) be its residue field, Then \( k \to k_s \) and we have \( S \to Spec(k_s) \) hence we produce a map \( S \times_k \mathbb{P}^n k_s \to \mathbb{P}^n k_s \). But since \( S \times_k \mathbb{P}^n k_s = (S \times_k \mathbb{P}^n k) \times_k Spec(k_s) \) we produce a map \( S \times_k \mathbb{P}^n k_s \to S \times_k \mathbb{P}^n k \) such we can produce \( r^*p_1^*\mathcal{O}(m) \) and then \( \alpha^*r^*p_1^*\mathcal{O}(m) \) which will equal \( \mathcal{O}(m) \) for \( k = k_s \) because we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{P}^n k_s & \to & \mathbb{P}^n k_s \\
\downarrow & & \downarrow \\
\mathbb{P}^n k \times S & \to & \mathbb{P}^n k
\end{array}
\]

From there, we recover \( m \), which is what we want.

Now show surjectivity. Let \( \mathcal{L} \) be a line bundle on \( \mathbb{P}^n \times S \). Let \( s \in S \) be a point and consider \( \mathcal{L}_s \in \text{Sh}(\mathbb{P}^n k_s) \) be as above, then \( \mathcal{L}_s \cong \mathcal{O}(m) \) on \( \mathbb{P}^n k_s \). Therefore by tensoring with \( \mathcal{O}(m) \) we can assume that \( \mathcal{L}_s \) is trivial.

We will show then that \( \mathcal{L} \) is locally - i.e. on some \( U \supset p_1^{-1}(s) \) - equal to \( p_1^*((p_1)_*\mathcal{L}) \) and that \( (p_1)_*\mathcal{L} \) is a line bundle when restricted to \( U \). This will be enough for the following reason: this assertion proves that every line bundle is locally of the form \( \mathcal{O}(m) \boxtimes \mathcal{L}' \) and injectivity implies that the points \( s \in S \) where \( m \) equals a given \( r \) form an open disjoint cover of \( S \) and \( S \) can be assumed connected, so therefore this \( m \) is universal and since for the original \( s \) it is 0, we get that \( \mathcal{L} \) equals \( p_1^*((p_1)_*\mathcal{L}) \) everywhere and \( (p_1)_*\mathcal{L} \) is a line bundle.

Therefore is suffices to prove this.

We prove that there is a neighborhood \( U \) of \( s \) in \( S \), such that if we restrict \( \mathcal{L} \) to \( \mathbb{P}^n \times U \) and denote \( p \) be the projection \( \mathbb{P}^n \times U \to U \) (which is just the restriction on \( U \)), then:

\[
\begin{align*}
&i) \ R^i p_*(\mathcal{L}) = 0 \text{ for } i > 0 \\
&ii) \ p_*(\mathcal{L}) = \mathcal{L} \text{ is a line bundle.} \\
&iii) \ The \ adduction \ map \ p^* p_*(\mathcal{L}) \to \mathcal{L} \text{ is an isomorphism.}
\end{align*}
\]

**Proof:** note that \( p^*, p_* , R p_* \) are the restrictions of \( p_1^* , (p_1)_* , R(p_1)_* \). Let \( i \) be the largest number such that \( R^i p_*(\mathcal{L}) \neq 0 \). In the proof of Serre's theorem from last semester it was shown that the individual cohomologies are finite dimensional and terminate - apparently this can be proven for \( R^i p_* \) as well.

Then by Nakayama \( k_s \otimes R^i p_*(\mathcal{L}) \neq 0 \).

We claim that \( H^j (k_s \otimes \mathcal{L}) \cong H^j (\mathbb{P}^n \times k_s, \mathcal{L}) \) - this is true as long as \( \mathbb{P}^n \times S \) is flat over \( S \), which it is.

Module this claim: \( R p_*(\mathcal{L}) \) lives in cohomological degree \( \leq i \). As \( \otimes \) is right exact, the tail \( t < i \) does not matter for \( H^j (\otimes) \) so it is \( k_s \otimes R^i p_*(\mathcal{L}) \). But in the right hand side, \( \mathcal{L} \otimes k_s \cong k_s \) therefore \( H^i (\ldots) = 0 \) contradiction.
It follows thus that \( p_*(L) \cong Rp_*(L) \).

We want to show it is a line bundle. We claim it is the same as proving that \( k_s \otimes p_*(L) \) zero in dimension 0 i.e. \( \text{Tor}_i(k_s, p_*(L)) = 0 \) for \( i < 0 \), \( \text{Tor}_0(k_s, p_*(L)) \cong k_s \). Indeed, as done before, checking Tor against the residue field implies that \( p_*(L) \) over \( \mathcal{O}_s \) is flat i.e. free, and the dimension must be 1.

Take negative cohomologies in the big identity from before, then the cohomologies in the right-hand side are clearly 0, and \( H^0(\mathbb{P}^n, \mathcal{O}) \) is 1-dimensional, which shows the claim.

Finally, for iii), let \( L = p_*L \). We need \( p^*p_*L \rightarrow L \) to be an isomorphism (shrinking \( U \) if necessary).

It is enough to show it is an isomorphism over the fiber at \( s \), because then it will be an isomorphism over \( \mathcal{O}_s \) by Nakayama, and then we can produce a small open over which the isomorphism takes place. But that is true. \( \square \)

Remark: we have used the affirmation that if \( F \) is a right exact functor, then the \( < i \) tail of an object \( M^* \) does not matter for computing \( H^i(RFM^*) \).

Indeed, let’s prove it for \( i = 0 \) for example. Assume \( M^* \) is \( M^{-1} \rightarrow M^0 \rightarrow M^1 \rightarrow \ldots \). Then \( RF \) is computed by constructing a projective resolution \( \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow \ldots \)

If we take the truncation \( M^0 \rightarrow \ldots \) then a projective resolution of it will be \( \rightarrow P^0 \rightarrow P_1 \rightarrow \ldots \) where \( P^i \) surjects onto \( \text{Ker}(P^0 \rightarrow M_0) \). The zeroth cohomology in the first way will be \( \text{ker}(FP^0 \rightarrow FP^1)/\text{Im}(FP^1 \rightarrow FP^0) \) and in the second way it will be \( \text{ker}(FP_0 \rightarrow FP')/\text{Im}(FP' \rightarrow FP_0) \). Therefore it suffices to show that \( \text{Im}(FP^{-1}) \) equals \( \text{Im}(FP') \) inside \( FP_0 \). Note first that \( \text{Im}(P_1 \rightarrow P_0) = \text{Im}(P' \rightarrow P_0) = K = \text{Ker}(P_0 \rightarrow M_0) \), and here right-exactness comes into play: if \( K = \text{Im}(X \rightarrow Y) \) then \( FK = \text{Im}(FX \rightarrow FY) \). Indeed, since \( X \rightarrow Y \) factors as \( X \rightarrow K \rightarrow Y \) then \( FX \rightarrow FY \) factors as \( FX \rightarrow FK \rightarrow FY \) so \( \text{Im}(FX \rightarrow FY) \subset FK \). But since \( X \rightarrow K \) by right-exactness \( FX \rightarrow FK \), and this finishes the proof. There is an analogue for left-exact functors.

**Proposition. (PSET)** Let \( S \) be a normal scheme over a field \( k \). Then \( S \times \mathbb{A}^1 \) is also normal, and

\[
\text{Pic}(S \times \mathbb{A}^1) \cong \text{Pic}(S)
\]

the isomorphism being provided by \( \pi^*: \text{Pic}(S) \rightarrow \text{Pic}(S \times \mathbb{A}^1) \) where \( \pi: S \times \mathbb{A}^1 \rightarrow S \) is the natural projection.

**Proof:** (by Jonathan Wang): For normality, note that it is local, so it suffices to show that if \( A \) is a normal domain, so is \( A[x] \). Integrality is clear, it remains to show it is integrally closed. Let \( K \) be the ring of fractions of \( A \), and let \( f \in F(x) \) which is integral over \( A[x] \). Because \( F[x] \) is a UFD, it is integrally closed in its fraction field \( F(x) \) so \( f \in F[x] \). The leading coefficient \( a_n \) of \( f \) must then be algebraic over \( A \), so it must land in \( A \). Then since the algebraic closure of \( A[x] \) is a ring, \( f - a_nx^n \) is also algebraic over \( A[x] \). Therefore we deduce \( a_{n-1} \in A \) as well, and continuing like that, we deduce \( f \in A[x] \).

For the second part, we can assume \( S \) is connected otherwise work with each connected component separately. Considering the map \( \text{Spec}(k) \rightarrow \mathbb{A}^1 \) corresponding to \( k[t] \rightarrow k \), \( t \rightarrow 0 \) (this map corresponds to the point \( 0 \in \mathbb{A}^1 \)), and by base change we produce a map \( S \rightarrow S \times \mathbb{A}^1 \) such that \( \pi \circ i = id \), thus \( i^*\pi^* = id \) to \( \pi^ \) is injective.

If \( S \) is connected and normal, it is integral, and it locally equals \( \text{Spec}(A) \) with all \( \text{Spec}(A) \) sharing the same generic point, it follows that \( S \times \mathbb{A}^1 \) it the union of \( \text{Spec}(A[t]) \) and all share the same generic point corresponding to the zero ideal in \( A[t] \), so \( S \times \mathbb{A}^1 \) is integral with fraction field \( K(t) \) where \( K \) is the fraction field of \( S \).

We are left to show surjectivity of \( \pi^* \). We claim that the affine case will imply the general case. Indeed, assume we have proven this for \( S \) affine, and let \( S \) be covered by affines \( U_i \) so that \( U_i \times \mathbb{A}^1 \) form an affine cover of \( S \times \mathbb{A}^1 \). Let \( L \) be a line bundle on \( S \), so that \( L|_{U_i} = \pi_{i*}(L_i) \) for \( L_i \) line bundles on \( U_i \). Also, for any \( U_i \cap U_j \) if we denote by \( \pi_{ij} \) the projection of \( (U_i \cap U_j) \times \mathbb{A}^1 \) to \( U_i \cap U_j \) (note that it coincides with \( \pi_{U_i \cup U_j} \) with restricted target), then \( \pi_{ij} \) is injective as above, and thus \( \pi_{ij*}(L_i|_{U_i \cap U_j}) = \pi_{ij*}(L_j|_{U_i \cap U_j}) \) and this implies \( L_i|_{U_i \cap U_j} = L_j|_{U_i \cap U_j} \) thus the line bundles glue to a line bundle on \( S \). Alternatively, note that \( L \otimes \mathcal{O}_S(K) \cong K(t) \) so that we can produce an embedding of \( L \) into \( K(t) \) thus \( L \) can be regarded as a subsheaf of \( K(t) \). Then passing to actual modules, \( L \) is a submodule of \( K(t) \) and then \( \pi_{ij*}L_i = \pi_{ij}L_i = A_1[t] \) (if \( U_i = \text{Spec}(A_i) \)) so that \( L_i \subset \pi_{ij*}L_i \subset K(t) \) thus since \( \pi_{ij*}L_i \) and \( \pi_*L_j \) coincide on the intersection - in particular at the generic point, then are actually equal as submodules of \( K(t) \).
so that $\mathcal{L}_i = \pi^*_i \mathcal{L}_i = \pi^*_j \mathcal{L}_j = \mathcal{L}_j$ as a submodule of $K \subset K[t]$ thus they agree and once again glue to form a global line bundle.

Thus is suffices to prove the affine case. This tells that every module $I$ which is locally free of rank 1 over $A[t]$ - where $A$ is a normal domain thus $S_2, S_1, R_1$ can be represented as $M \otimes_A A[t] \cong A[t]$ for $M$ locally free of rank 1 over $A$. We know $I \otimes_{A[t]} K[t] \cong K[t]$ - because $K[t]$ is a UFD so $Pic(K[t]) = 0$. So we can assume $I \subset K[t]$ and by multiplying by the common denominator we can assume $I \subset A[t]$ i.e. $I$ is an ideal of $A[t]$ We can divide by a suitable power of $t$ so we can assume $I \cap A \neq \{0\}$. Denoting $M = I \cap A$ we claim $I = M[t]$.

By a previous proposition,

$$I = \bigcap_{ht(p) = 1, p \subset A[t]} I_p$$

Consider an ideal $p$ of height 1. Let $q = p \cap A$. If $q \neq 0$ then $p = q[t]$ otherwise $q[t]$ is not minimal but has smaller height. If $q = 0$ then $p$ corresponds to an ideal $p'$ of $A[t]_{A-\{0\}} = K[t]$ which is principal since $K[t]$ is a UFD, and since $I \cap A \neq 0$ it must be that $I_p = K[t]_{p'} = A[t]_{p'}. The intersection of all $A[t]_{p}$

Hence we deduce that

$$I = A[t] \cap \bigcap_{ht(q) = 1, q \subset A} I \cdot A_q[t]$$

If $p = q[t]$ we claim that $I_p = (I_p \cap A_q)[t]$. Indeed, by $R_1$, $A_q$ is a DVR hence $I_p$ is principal in $A_q[t]$. Its generator must be in $A_q$ since $I \cap A \neq 0$. Thus $I_p = (I_p \cap A_q)[t]$ as desired. Therefore

$$I = A[t] \cap \bigcap_q (I \cdot A_q[t] \cap A_q)[t] = \left( A \cap \bigcap_q I \cdot A_q[t] \cap A_q \right) = (I \cap A)[t]$$

where we use $\cup A_q = A$. Since $I$ is a line bundle, $I/tI$ is a line bundle over $A$, but $I/tI$ is isomorphic to $M = I \cap A$ via since $I = M[t]$. It follows that $I = \pi^*(M)$ which shows the assertion in the affine case. $\square$

**Proposition. (PSET)** Let $A$ be a Noetherian ring. Recall the definition of a locally complete intersection, which in this particular case means that there is a regular local ring $A'$ and a surjection $A' \rightarrow A$ whose kernel is generated by a regular sequence $f_1, \ldots, f_n$. In this case we have:

a) $n = \dim(A') - \dim(A)$

b) If $A$ is an lci, it is CM

Proof: a) It is enough to show it for $n = 1$ by induction, as $f_1, \ldots, f_n$ a regular sequence in $m_{A'}$ implies $f_2, \ldots, f_n$ project to a regular sequence in $m_{A'}/(m_{A'} \cap f_1 A')$.

In that case we need to show that $\dim(A'/f_1 A') = \dim(A') - 1$. Indeed, by Krull’s Hauptidealsatz, any minimal prime containing $f_1$ has height $1$ hence any maximal chain in $A'/f_1 A'$ can be extended with at least one more element implying that $\dim(A'/f_1 A') \leq \dim(A') - 1$. It remains to see, that $\dim(A'/f_1 A') \geq \depth(A'/f_1 A')$ but the latter depth equals $\depth(A') - 1$ as $f_1$ is regular and $\depth(A') = \dim(A')$ because $A'$ is regular.

b) Since $\dim(A) = \dim(A') - n$ from the previous part, we need to show that $\depth(A) \geq \dim(A') - n$ (as a module over itself). But $\depth(A)$ as a module over itself equals $\depth(A')$ as a module over $A'$ because the map $A' \rightarrow A$ is surjective. The result now follows from repeatedly applying the result that if $f \in A_{A'}$ acts faithfully on $B$ then $\depth_{m_{A'}}(B/fB) = \depth_{m_{A'}}(B) - 1$. $\square$

**Proposition.** Let $A \rightarrow B$ be a local map between regular local rings. Then it’s flat if and only if $\dim(B \otimes_A k_A) = \dim(B) - \dim(A)$ and in the latter case $B' = B \otimes_A k_A$ is an lci.

Proof (probably by Jonathan Wang): take a regular sequence $f_1, \ldots, f_n$ of $A$ where $n = \dim(A)$ that generates $m_A$. Let $g_1, \ldots, g_n$ be their images in $B$ which land in $m_B$. Then $k_A = A/m_A = A/(f_1, \ldots, f_n)$ and $B \otimes_A k_A = B/(g_1, \ldots, g_n)$. 85
Assume $B$ is flat over $A$. Then tensoring $B$ with $0 \to A/(f_1, \ldots, f_i) \xrightarrow{f_{i+1} A/(f_1, \ldots, f_i)} B/(g_1, \ldots, g_i)$ gives $0 \to B/(g_1, \ldots, g_i) \xrightarrow{g_{i+1}} B/(g_1, \ldots, g_i)$ thus $g_1, \ldots, g_n$ form a regular sequence in $B$ and $B'$ is an lci. Then $B'$ is CM, so $\dim(B') = \dim(B) - n$ since $B'$ is a quotient of $B$ by a regular sequence of length $n$.

Assume $\dim(B') = \dim(B) - \dim(A)$. We claim this implies $g_1, \ldots, g_n$ a regular sequence in $B$. Let $B_i = B/(g_1, \ldots, g_i)$. We show by induction on $i$ that $g_1, \ldots, g_i$ is regular, (base is $B_0 = B$). Assuming the induction hypothesis, for $i$, $B_i$ is an lci hence CM. If $g_{i+1}$ is a zero divisor on $B_i$, then it belongs to some associated prime $p$ of $B_i$, but every such associated prime $p$ satisfies $\dim(B_i/p) = \dim(B) - i$ since $B_i$ is CM - as $f_{i+1} \in p, B_i/f_{i+1} = B_i/p \Rightarrow \dim(B_{i+1} + \dim(B) - i$. This is impossible because then by Krull’s Hauptidealsatz the dimension of $B_i = B'$ will drop by at most $n - i - 1$ so it will be at least $\dim(B) - n + 1$ contradiction. Thus $f_{i+1}$ is not a zero-divisor, which shows the induction step.

We now apply the Local Criterion for Flatness (Eisenbud Theorem 8.8, pg,168 - available on Googlebooks), we know that $B$ is flat over $A$ if and only if $\text{Tor}_1(k_A, B) = 0$. To compute Tor, take the Koszul complex/resolution $K(f_1, \ldots, f_n, A) \to k_A$. By tensoring up with $B$, we obtain the Koszul complex $K(g_1, \ldots, g_n, B)$ and since $g_1, \ldots, g_n$ is a regular sequence, this complex is quasi-isomorphic to $B/(g_1, \ldots, g_n) = B'$ in particular its positive cohomologies are zero, which implies $\text{Tor}_i(k_A, B) = 0$ for $i > 0$, so $B$ is flat.

Here is the statement of the Local Criterion for Flatness:

**Proposition. (Local Criterion for Flatness)** Let $R \to S$ be a map of local Noetherian rings, and $M$ a finitely generated $S$-module. Then $M$ is flat as an $S$-module if and only if $\text{Tor}_1(k_R, M) = 0$.

**Proposition. (problem Hartshorne, II.6.10, the Grothendieck Group)** Let $X$ be a Noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on $X$, by the subgroup generated by all the expressions $\mathcal{F}' - \mathcal{F}'' + \mathcal{F}'''$ for all short exact sequences $0 \to \mathcal{F}' \to \mathcal{F}'' \to \mathcal{F}''' \to 0$ on $\text{Coh}(X)$. If $\mathcal{F}$ is a coherent sheaf, we denote by $\gamma(\mathcal{F})$ its image in $K(X)$.

a) If $X = A_k^1$ then $K(X) \cong \mathbb{Z}$

b) If $X$ is any integral scheme, and $\mathcal{F}$ a coherent sheaf, we define the rank of $\mathcal{F}$ to be $\text{dim}_K(\mathcal{F}_x)$ where $\xi$ is the geometric point of $X$, and $K = \mathcal{O}_x$ is the function field of $X$. The rank function defines a surjective homomorphism $\text{rank}: K(X) \to \mathbb{Z}$.

c) If $Y$ is a closed subscheme of $X$, there is an exact sequence $K(Y) \to K(X) \to K(X - Y) \to 0$ where the first map is the extension by zero, and the second map is restriction.

Proof: a) $A_k^1 = \text{Spec}(k[t])$, and every coherent sheaf over $A_k^1$ is a finitely generated $k[t]$-module $M$. Consider the $k(t)$-vector space $M \otimes_{k[t]} k(t)$ and lift a basis of it to $M$ (we can multiply so that it can be lifted to $M$). This produces a map $(k[t])^p \to M$ and this map is injective because it becomes injective when localized at the generic point. Thus $[M] = [k[t]]^p + [N] = p[k[t]] + [N]$ where $N$ is the cokernel of the map. Note that $N$ is still finitely generated, but it is also torsion - and since every torsion element satisfies a polynomial of finite degree, we deduce that $N$ is a finite dimensional vector space over $k$. We claim that this implies $[N] = 0$.

Take a set $v_1, \ldots, v_n$ of generators of $N$ as a $k$-vector space. They induce a surjection $k[t]^n \to N$. Also, $tv_i$ is a linear combination of $v_1, \ldots, v_n$ over $k$, hence $tv_i + \sum c_{i,j} v_j$ are in the kernel. They are obviously linearly independent (the determinant is $t^n + \text{something of smaller degree}$). In fact, they also pretty clearly generate the entire kernel as we can remove all powers of $t$ from any linear relation using them. Whence the kernel is isomorphic to $k[t]^n$ so $[N] = n[k[t]] - n[k[t]] = 0$.

We have implicitly used that $[k[t]^n] = n[k[t]]$ via induction and short exact sequences $0 \to k[t]^{n-1} \to k[t]^n \to k[t] \to 0$. We have just proved that $[k[t]]$ generated $K(A_k^1)$ i.e. $K(A_k^1)$ is a quotient of $\mathbb{Z}$. That it is actually $\mathbb{Z}$ will follow from the next part.

b) Taking stalks at every point is an exact functor on coherent schemes. This is because restricting to every open is exact, and on affine opens stalks is just localizing. On the other hand over a field, taking dimension is clearly "additive" on short exact sequences of finite dimensional vector spaces. Thus this implies $\mathcal{F} \to \text{dim}_K(\mathcal{F}_x)$ induces a map $K(X) \to \mathbb{Z}$. It is surjective because $O_X$ gets mapped to 1.
c) The surjectivity of the last map is the content of problem 3 from PS 11 last semester (= Hartshorne 5.15.)

The first map ("extension by zero") is actually $i_*$ where $i: Y \hookrightarrow X$ is the embedding. Because $i_*$ is exact, it induces a map $K(Y) \rightarrow K(X)$.

Exactness at the middle will follow from the following lemma: if $\mathcal{F}$ is a coherent sheaf supported at $Y$ then there is a filtration $\mathcal{F} = \mathcal{F}_n \supset \mathcal{F}_{n-1} \supset \cdots \supset \mathcal{F}_1 = 0$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is an $\mathcal{O}_Y$-module i.e. is in the essential image $i_*$. Indeed, then $[\mathcal{F}] = [\mathcal{F}_n/\mathcal{F}_{n-1}] - [\mathcal{F}_{n-1}/\mathcal{F}_{n-2}] + \cdots$ which in this form is in the image of $K(Y)$. Conversely, the composite of the two maps is clearly zero.

It remains to prove the lemma. Let $\mathcal{I}$ be the ideal sheaf of $Y$. For any sheaf $\mathcal{F}$ we can consider the sheaf $\mathcal{I} \cdot \mathcal{F} = \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{F}$. We claim that if $\mathcal{I} \cdot \mathcal{F} = 0$ then $\mathcal{F}$ is in the essential image of $i_*$. Indeed, we need to show $\mathcal{F} \rightarrow i_i^* \mathcal{F}$ is an isomorphism which is a local property, and on affines this is standard algebra.

Now if $\mathcal{F}$ is supported on $Y$ then $\mathcal{I} \cdot \mathcal{F} = 0$. Indeed, on affines this is standard algebra again (using Noetherian-ness) and $X$ is quasi-compact. Hence the filtration can be obtained by the sheaves $\mathcal{I}^k \cdot \mathcal{F}$ as $\mathcal{I}^k \cdot \mathcal{F}/\mathcal{I}^{k-1} \cdot \mathcal{F}$ is killed by $\mathcal{I}$: again it is obvious on affines. $\square$

**Proposition. (problem Hartshorne II.6.11., the Grothendieck Group of a non-singular curve)** Let $X$ be a smooth curve over an algebraically closed field $k$: this means all the local rings of $X$ are regular local rings. Then

$$K(X) \cong \text{Pic}(X) \oplus \mathbb{Z}$$

as follows:

a) For any divisor $D = \sum n_i P_i$ on $X$, let $\psi(D) = \sum n_i \gamma(k(P_i))$ where $k(P_i)$ is the skyscraper sheaf $k$ at $P_i$ and 0 elsewhere. If $D$ is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated sub-scheme of codimension 1. Then $\psi(D) = \gamma(\mathcal{O}_D)$. For any $D$, $\psi(D)$ depends only on the linear equivalence class of $D$, so $\psi$ defines a homomorphism $\psi: Cl(X) \rightarrow K(X)$

b) (the determinant map) For any coherent sheaf $\mathcal{F}$ on $X$, there exist locally free sheaves $\mathcal{E}_0$ and $\mathcal{E}_1$ and an exact sequence $0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$. Let $r_0 = \text{rank}(\mathcal{E}_0), r_1 = \text{rank}(\mathcal{E}_1)$ and define the line bundle $\text{det}(\mathcal{F}) = (\Lambda^{r_0} \mathcal{E}_0) \otimes (\Lambda^{r_1} \mathcal{E}_1)^{-1} \in \text{Pic}(X)$ where $\Lambda$ denotes the exterior power. The determinant is independent of the resolution chosen (i.e. there exist canonical isomorphisms between any two such constructed determinants), and it gives a homomorphism $\text{det}: K(X) \rightarrow \text{Pic}(X)$. Moreover, if $D$ is a divisor, the $\text{det}(\psi(D)) = \mathcal{L}(D)$.

c) If $\mathcal{F}$ is any coherent sheaf of rank $r$, there is a divisor $D$ on $X$ and an exact sequence $0 \rightarrow \mathcal{L}(D)^{\oplus r} \rightarrow \mathcal{F} \rightarrow \mathcal{L}$ where $\mathcal{L}$ is a torsion sheaf (i.e. of rank 0). As a corollary, if $\mathcal{F}$ is a sheaf of rank $r$, then $\gamma(\mathcal{F}) - r\gamma(\mathcal{O}_X) \in \text{Im}(\psi)$.

d) The maps $\psi, \text{det}, \text{rank}$ and $1 \rightarrow \gamma(\mathcal{O}_X)$ from $\mathbb{Z}$ to $K(X)$, define the isomorphism $K(X) \cong \mathbb{Z} \oplus \text{Pic}(X)$

Proof: a) We will prove the first statement by induction on the degree of the effective divisor $D$.

Base: assume $D = p$ where $p$ is a closed point i.e. irreducible subset of codimension 1. We claim $k(p)$ (the skyscraper sheaf) is equal to $\mathcal{O}_p$ hence the statement will follow. Indeed, the stalk of $\mathcal{O}_p$ at $p$ is $k$ - this is because it equals the residue field of some maximal ideal of a finite $k$-algebra, which must be $k$ by the Nullstellensatz. Hence we can map $\mathcal{O}_p$ to $k(p)$ and this map is an isomorphism on stalks (both sheaves have zero stalks outside $p$) so is an isomorphism.

Induction step: assume $D = D_0 + p$. We will show that $[\mathcal{O}_D] = [\mathcal{O}_p] + [\mathcal{O}_{D_0}]$ and we will be done by the induction step.

Indeed, for different points $p_i$ the skyscraper sheaves and $\mathcal{O}_D$ have nothing to do with each other at all -i.e. if $D_1, D_2$ do not have common points then $\mathcal{O}_{D_1+D_2} = \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}$. Therefore it is enough to prove the assertion in the case $D = np$. In that case the stalk of $\mathcal{O}_D$ at $p$ will me the local ring at $p$ modulo the maximal ideal to the $n$-th power. In particular it maps to $k$ which is the residue field at $p$ so we have a map $\mathcal{O}_D \rightarrow \mathcal{O}_p$. We now claim that the kernel is $\mathcal{O}_{(n-1)p}$ which will finish the claim. It suffices to consider an open around $p$ because the stalks are 0 outside $p$. So we can assume the scheme is affine in which case we have the map $A/p^n \rightarrow A/p$. Its kernel is $p/p^n$. Now remember that the curve is regular hence $p$ is generated by one element in $A_p$. Since $A$ is Noetherian, $p$ will be also generated
by one element in some localization \( A_f \) and so restricting \( A \) is necessary we may assume \( p \) is principal. In that case \( p/p^n \) is obviously isomorphic to \( A/p^{n-1} \) and we are done.

Now we show that \( \psi \) only depends on the equivalence class of \( D \). Since \( \psi \) is a homomorphism, it suffices to show it sends principal divisors to 0 - and this amounts to showing that if \( D_1, D_2 \) are effective and equivalent, then \( \psi(D_1) = \psi(D_2) \). But by what was just proved, \( \psi(D_1) = [\mathcal{O}_{D_1}] \) and \( \psi(D_2) = [\mathcal{O}_{D_2}] \).

By proposition 6.18. in Hartshorne (something similar was done before) \( \mathcal{O}_{D_1} = \mathcal{L}(-D_1) \) and \( \mathcal{O}_{D_2} = \mathcal{L}(-D_2) \). Because they are equivalent, \( \mathcal{L}_{-D_1} \) and \( \mathcal{L}_{-D_2} \) are actually equal so we are done.

b) Let’s prove that every coherent sheaf is the quotient of two locally free sheaves. First we do it over every point. If this point is a generic point then we have nothing to prove, otherwise the point is of codimension 1 and the local ring is a DVR.

Sub-lemma: in a DVR (more generally a Noetherian principal ideal domain) any submodule of a free (more generally torsion-free) finitely generated module \( M \) is also free.

Proof: for a DVR, we just lift a basis of the module tensored up with the residue field and use Nakayama’s lemma. In general, we can generalize the approach for \( \mathbb{Z} \): choose a generating set, and if there is a relation between them we may assume one coefficient divides all the others and then make a substitution to decreases the size of the set. The sub-lemma is proved.

Corollary: every torsion-free coherent sheaf is locally free - where torsion-free means there are no sections over any open that restrict to 0 over the generic point.

Proof: over any point, the sheaf is torsion-free.

Sub-lemma: if \( 0 \rightarrow \mathfrak{E}_1 \rightarrow \mathfrak{E}_0 \rightarrow \mathfrak{F} \rightarrow 0 \) is a short exact sequence of coherent sheaves and \( \mathfrak{E}_0 \) is locally free, the so is \( \mathfrak{E}_1 \).

Proof: \( \mathfrak{E}_1 \) is a sub-sheaf of \( \mathfrak{E}_0 \). Now choose small enough affines on which \( \mathfrak{E}_0 \) is free and use the sub-lemma.

Therefore, it suffices to show that \( \mathfrak{F} \) receives a surjection from a locally free coherent sheaf - i.e. a torsion-free coherent sheaf.

Choose an affine \( U \). Over \( U \), the assertion clearly holds, so there is a sheaf \( \mathfrak{F}_1 \) over \( U \) surjecting onto \( \mathfrak{F} |_U \). The sheaf \( j_*\mathfrak{F}_1 \) is torsion-free sections over any open are just sections of \( \mathfrak{F}_1 \) over some other open, and \( \mathfrak{F}_1 \) is torsion-free. Let \( s_1, \ldots, s_k \) generate \( \Gamma(\mathfrak{F}_1, U) \) over \( \Gamma(X, U) \) then they lift to global sections of \( j_*\mathfrak{F}_1 \). Consider now the map \( \mathcal{O}^k_X \rightarrow \mathfrak{F} \) defined as follows: for the \( i \)-th copy of \( \mathcal{O}_X \), on every open set \( V \) we send the section \( a \in \Gamma(\mathcal{O}_X, U) \) to \( a \cdot s_i |_V \) (where \( s_i \) are regarded as global sections of \( \Gamma(j_*\mathfrak{F}_1, U) \)). This induces a map \( \mathcal{O}^k_X \rightarrow j_*\mathfrak{F}_1 \) so a composite map \( \mathcal{O}^k_X \rightarrow \mathfrak{F} \), and it is easy to see that this map becomes surjective when restricted to \( U \). Now cover \( X \) by finitely many sets \( U \) and take direct sums, to obtain the desired map.

Further, we define the determinant map be \( det \mathfrak{F} = (\mathcal{L}^{r_0} \mathfrak{E}_0) \otimes (\mathcal{L}^{r_1} \mathfrak{E}_1)^{-1} \) where \( r_0, r_1 \) are the ranks of \( \mathfrak{E}_0 \) and \( \mathfrak{E}_1 \). First we prove that it is independent of the chosen resolution. Let’s first do it locally on affines, i.e. assume we are dealing with a short exact sequence of modules \( 0 \rightarrow M_1 \rightarrow M_0 \rightarrow M \rightarrow 0 \) of \( A \)-modules where \( M_i \) are free of rank \( r_i \) and \( A \) is an integral domain.

In this context, the inverse of a free \( A \)-module of rank 1 is just its dual, so we want to show that \( \mathcal{L}^{r_0} M_0 \otimes \text{Hom}(\mathcal{L}^{r_1} M_1, A) \) is canonical.

Note that \( r_0 - r_1 = r = dim_K(M \otimes_A K) \) which is obtained by tensoring up the short exact sequence with the flat \( A \)-module \( K \) (the field of fractions). Then \( M \) contains a free submodule \( M' \) of rank \( r \): just lift a basis of \( M \otimes_A K \), first correcting the elements of the basis so they come from something in \( M \).

We then claim that \( \mathcal{L}^{r_0} M_0 \otimes \text{Hom}(\mathcal{L}^{r_1} M_1, A) \cong a \mathcal{L}^{r'} M' \) and the isomorphism is canonical - where \( a \) is in \( I \) to be specified later.

This is the same as showing a canonical isomorphism between \( \mathcal{L}^r M' \otimes \mathcal{L}^{r_1} M_1 \) and \( a \mathcal{L}^{r_0} M_0 \). There is a natural map \( \mathcal{L}^r M' \otimes \mathcal{L}^{r_1} M_1 \rightarrow \mathcal{L}^{r_0} M_0 \) obtained as follows: choose a basis \( v_1, \ldots, v_{r_1} \) of \( M_1 \) and \( w_1, \ldots, w_r \) lifts of vectors in \( M \) to \( M_0 \). Then \( (v_j) \) together form a set of vectors in \( M_0 \). We then send \( v \) to \( (c'v_1 \wedge \ldots \wedge c_{r_1}) \wedge v_1 \wedge \ldots \wedge w_r \). This map is canonical because it does not depend on the choice of the lifts of the vectors of \( M \) as they differ by vectors in \( M_1 \) which are killed when wedged with \( det(M_1) \). The map is also clearly
injective. Let’s show it is surjective. Indeed, if we choose an actual basis of \( M_1, u_1, \ldots, u_{r_1} \), then we would have \( v_1 \wedge \ldots \wedge v_{r_1} \wedge w_1 \wedge \ldots \wedge w_r = au_1 \wedge \ldots \wedge u_r \) for some \( a \in A \), hence we deduce the claim.

Finally, we need to show that the element \( a \in A \) (up to units) is intrinsic to \( M' \) and not to the resolution chosen. More precisely, \( M/M' \) is torsion and thus is an extension of modules \( A/I \). We take the product of all such ideals \( I \) - call it \( [M/M'] \) and claim that it equals exactly the ideal generated by \( a \).

Let \( M_2 \) be the preimage of \( M' \) in \( M_0 \), it free with generators \( (v_i), (w_j) \). Then \( M_0/M_2 \cong M/M' \). On the other hand \( a \) is just the determinant of the matrix of the basis of \( M_2 \) in terms of the basis of \( M_0 \) - i.e. \( a \cdot \det(M_0) = \det(M_2) \). So we want to show that in fact \( [M_0/M_2] \cdot \det(M_0) = \det(M_2) \).

This holds when \( A \) is a DVR: indeed, in particular it is a PID and then the proof for \( \mathbb{Z} \)-lattices from number theory applies: namely, \( M_2 \) has a basis which is upper-diagonal with respect to the basis of \( M_1 \) and then the proof is simple as both required quantities equal the product of the diagonal entries.

Since locally every point is a DVR, for fixed \( M_1, A \) this hold on all sufficiently small affines.

In particular, if we represent \( \mathcal{F} \) as \( \mathcal{E}_0/\mathcal{E}_1 \) and \( \mathcal{E}_0'/\mathcal{E}_1' \), then on sufficiently small affines the above claim applies, hence we can construct a canonical isomorphism between \( (\mathcal{L}_r \mathcal{E}_0) \otimes (\mathcal{L}_r \mathcal{E}_1)^{-1} \) and \( (\mathcal{L}_r \mathcal{E}_0') \otimes (\mathcal{L}_r \mathcal{E}_1')^{-1} \) - at least locally. As they are canonical, they glue to a global isomorphism of line bundles.

There is still one caveat to consider: the choice of \( M' \) is non-canonical. However if we pick two different choices for it, the isomorphisms will be the same for the following reason: as isomorphism of modules which are isomorphic to \( A \), it is enough to show that they are isomorphic when tensored up with \( K \). However in that case the maps are clearly the same because \( \mathcal{L}^r M' \otimes K \) is \( \mathcal{L}^r (M \otimes K) \).

This shows that the map is well-defined.

To show that it factors through \( K(X) \), we need to show that for a short exact sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \), \( \det(\mathcal{F}_2) = \det(\mathcal{F}_1) + \det(\mathcal{F}_3) \).

First, assume that \( \mathcal{F}_1, \mathcal{F}_2 \) are locally free, of rank \( r_1, r_2 \). The this follows immediately from the definition of \( \det \) and using the obvious resolutions.

For the general case, we claim we can complete the short exact sequence to a diagram as follows:

\[
\begin{array}{ccc}
0 & \to & \mathcal{E}_{0,1} \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{E}_{1,1} \\
\downarrow & & \downarrow \\
\mathcal{F}_1 & \to & \mathcal{F}_2 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{E}_{0,3} \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{E}_{1,3} \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Once we do this, the result will follow from applying the simple case to all columns and all but one row.

To prove it, we need a short lemma:

If \( A \to C, B \to C \) then \( \text{Ker}(A \oplus B \to C) \to A, \text{Ker}(A \oplus B \to C) \to B \) where the map \( A \oplus B \to C \) is the anti-diagonal one.

Indeed, if \( a \in A \) then as \( B \to C \) we can select \( b \in B \) such that \( b \) projects to the same element in \( c \) as \( a \) and then \( (a, -b) \) will be in the kernel and project to \( a \). Similarly for \( B \).

The lemma immediately generalizes for sheaves.

Now take a locally free sheaf \( \mathcal{E}_{0,3} \) surjecting onto \( \mathcal{F}_3 \), and consider the sheaf \( \text{Ker}(\mathcal{F}_2 \oplus \mathcal{E}_{0,3} \to \mathcal{F}_3) \). It surjects onto both \( \mathcal{E}_{0,3} \) and \( \mathcal{F}_2 \), and we take \( \mathcal{E}_{0,2} \) surjecting onto it. It remains to take \( \mathcal{E}_{0,1} \) be the kernel of the map \( \mathcal{E}_{0,2} \to \mathcal{E}_{0,3} \).
We complete the diagram by taking the kernels of the columns. Note that all sheaves are locally free, using the fact that a subsheaf of a locally free sheaf is locally free, in this case.

The last thing to prove is that $\text{det}(\psi(D)) = \mathcal{L}(D)$. It is enough to show it when $D = (p)$. We know that $\text{det}$ is defined locally therefore since $\psi(D)$ has support only at $p$, so will $\text{det}(\psi(D))$ will be isomorphic to $\mathcal{O}_X$ away from $p$ and it is enough to look at a small affine around $p$. We can assume that $p$ is principal there - and in that case $k = A/p = A/fA$ so the required sequence is simply $0 \to A \xrightarrow{f} A \to A/fA \to 0$. In that case the determinant is naturally isomorphic to the module itself so we get $A \otimes (fA)^{-1} \cong f^{-1}A$ - this is just the definition of $\mathcal{L}(D)$.

c) First, observe that in a short exact sequence of sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F}_2 \to 0$, $\mathcal{F}_2$ is torsion if and only if $\mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism at the generic point, because torsion is the same as being zero at the generic point. It is enough to show that $0 \to \mathcal{L}(D_1) \oplus \cdots \oplus \mathcal{L}(D_r) \to \mathcal{F} \to \mathcal{F}_{\mathcal{I}} \to 0$ where $\mathcal{F}$ is torsion. Indeed, if $D$ is an effective divisor then $\mathcal{L}(-D) \to \mathcal{O}_X$ (see 6.18. from Hartshorne) and the quotient is torsion hence $\mathcal{L}(D' - D) \to \mathcal{L}(D')$ and so choosing a divisor $D$ such that $D_1 - D$ are all torsion, we can map $\mathcal{L}(D)^r$ into $\mathcal{L}(D_1) \oplus \cdots \oplus \mathcal{L}(D_r)$ and it will be an isomorphism at the generic point and we compose to get the desired short exact sequence.

Before we do this, let’s prove a lemma:

Lemma: if $p, q$ are distinct points in $X$ and $q$ is not the generic point then there is an open neighborhood of $p$ not containing $q$.

Proof: there will exist such an affine unless $p$ is contained in the closure of $q$. However since $q$ has codimension 1, this is impossible (actually all points are closed in this case).

Now let $M_0$ be the stalk of $\mathcal{F}$ at the generic point, and choose $m_1, \ldots, m_r$ a basis for $M_0$. For each closed point $p$ we can define the valuation of $m_1$ at $p$ as follows: as the local ring of $p$ is a DVR, the smallest integer $k$ such that $x_p^km_1 \in \mathcal{F}_p$ exists and is unique ($x_p$ is the uniformizer). We claim that the valuation is negative for only finitely many points $p$. It is enough to work with affines, and in that case $m_1 = am$ where $m \in \mathcal{F}_p$ and $a \in K$. There are only finitely many points $p$ where $a$ has negative valuation and $m_1$ can have negative valuation only there. Call these points ”poles” (of $m_1$).

As a corollary to the lemma, we deduce that every closed point $p$ has a neighborhood which contains no poles except maybe $p$ itself. Further, we can restrict the neighborhood so that it is an affine where $p$ is principal. Now let’s cover $X$ with finitely many such affines, and define the Cartier divisor $D_1$ in the following way: over an affine $U_i$ with no pole take $f_i = 1$, and over an affine $U_i$ with a pole take $f_i = x^k$ where $x$ is the generator of the pole and $k$ is the valuation. Note that on intersection the quotient $f_i / f_j$ is always invertible.

It is now fairly easy to see that $\mathcal{L}(-D) \otimes m_1 \to \mathcal{L}(-D) \otimes \mathcal{F} \otimes K$ actually lands inside $\mathcal{F}$. Moreover over the generic point this injection is actually the same as the map $< m_1 > \to < m_1, \ldots, m_k >$ doing this for all other $m_i$ yields the required representation.

To move on, let’s first prove that a torsion sheaf $\mathcal{F}$ is equivalent to something in the image of $\psi$ (in $K_X$) - which is the same as saying that there exists a filtration $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_k = 0$ such that $\mathcal{F}_i / \mathcal{F}_{i+1} \cong k(D_i)$. Indeed, let’s concentrate on an affine $\text{Spec}(A)$. Since all associated primes have height 1, we know that $\Gamma(\mathcal{F}, \text{Spec}(A))$ is a successive extension of modules $A/p_i = k(p_i)$, and this tells that $\mathcal{F} |_{\text{Spec}(A)}$ is a successive extension of $k(p_i) |_{U_i}$. Note that any such skyscraper sheaf is 0 everywhere outside the point $p_i$; therefore taking quotient by it does not affect stalks at point different from $p_i$, and ”reduces” the stalk at $p_i$. It is now clear what we have to do: cover $X$ by finitely many affines, quotient by skyscraper sheaves like that to make the sheaf 0 over $U_1$, then proceed to $U_2$ and so on. This will not make the sheaf non-zero over $U_1$, because the stalks will remain 0, so at the end of the day we will get 0 which is what we wanted.

Particularly, we see that $[\mathcal{F}] - r[\mathcal{L}(D)] \in \text{Im}(\psi)$. It remains to check that $[\mathcal{L}(D)] - [\mathcal{O}_X] \in \text{Im}(\psi)$. Observe that $\mathcal{L}(D)$ injects into $a \otimes \mathcal{O}_X$ for some $a \in K$. Moreover it’s pretty clear that his map must be an isomorphism at the generic point tofor $[\mathcal{L}(D)] - [a \otimes \mathcal{O}_X] \in \text{Im}(\psi)$. It remains to see that $[a \otimes \mathcal{O}_X] - [\mathcal{O}_X] \in \text{Im}(\psi)$ because they are isomorphic.

d) part a) gives a map $\psi : \text{Pic} X \to K(X)$ and the association $1 \to [\mathcal{O}_X]$ gives a map $Z \to K(X)$. Hence we get a map $\text{Pic} X \oplus Z \to K(X)$ and part c) implies it is surjective. Assume it were not injective, so that $[\psi(D)] = [\mathcal{O}_X]$.
Applying the map \( det \) to it we get \( det(\psi(D)) = L(D) \cong L^k \mathcal{O}_X^k = \mathcal{O}_X \) (the equality takes place in \( Pic(X) \)). But this means that \( D \) is principal so that \( [\psi(D)] = 0 \). It follows that \( [\mathcal{O}_X^k] = 0 \) as well so \( k[\mathcal{O}_X] = 0 \). This is impossible because \( rank(\mathcal{O}_X) = 1 \) (we use c) from problem 6.10.) So we can only have \( k = 0 \). But \( D \) principal and \( k = 0 \) means that the map is injective. □

**Proposition. (problem Hartshorne, II.6.12)** Let \( X \) be a complete nonsingular curve over an algebraically closed field (complete means proper over the spectrum of the field, it is equivalent to being projective -see Hartshorne II.6). There is a unique way to define the degree of any coherent sheaf on \( X \), \( deg(\mathcal{F}) \in \mathbb{Z} \), such that:

i) If \( D \) is a divisor, \( deg(L(D)) = deg(D) \)

ii) If \( \mathcal{F} \) is a torsion sheaf (meaning a sheaf whose stalk at the generic point is zero) then

\[
deg(\mathcal{F}) = \sum_{p \in X} \text{length}(\mathcal{F}_p)
\]

(note that sum is finite, recall the notion of the length of a module as the length of the largest chain of submodules)

iii) If \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is an exact sequence, then \( deg(\mathcal{F}) = deg(\mathcal{F}') + deg(\mathcal{F}'') \).

Proof: Part iii) implies that the map factors through \( K[X] \) (note that \( Coh(X) \) surjects onto \( K(X) \)). Part ii) determines the map on the image of \( \psi \), because anything in the image of \( \psi \) is clearly torsion (and actually in the end of II.6.11.c) we showed the converse). Then as condition i) determines it on \( [\mathcal{O}_X] \) and \( [\mathcal{O}_X] \) together with the image of \( \psi \) generate \( K(X) \), we conclude that the map, if it exists, is unique.

Now we prove that it actually exists, i.e. is well-defined.

We first build a map that satisfies i) and iii). As \( K[X] \cong \mathbb{Z} \oplus Pic(X) \) is is enough to construct a map from \( Pic(X) \) and from \( \mathcal{O}_X \). The first map is the degree - which is well-defined by Hartshorne 6.10 (proven before). The second map is 0.

It remains to show that the map now satisfies (2). (Remark: although it may seem intuitively wrong, we already know the map is a map from \( K_1(X) \) because \( K(X) \) is a direct sum of \( \mathbb{Z} \) and \( Pic(X) \).)

Since every torsion is an extension of skyscraper sheaves as proven before, it is enough to prove it for the skyscraper sheaf. There it is true because the skyscraper sheaf corresponds to \( \mathfrak{p} \) so gets sent to 1 by (1), but this is also the length of the unique non-zero stalk of the sheaf. □

**Proposition. (problem Hartshorne III.5.1.)** Let \( X \) be a projective scheme over a field \( k \), and let \( \mathcal{F} \) be a coherent sheaf on \( X \). Define the Euler characteristic of \( \mathcal{F} \) by

\[
\chi(\mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F})
\]

Then if \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is a short exact sequence of coherent sheaves on \( X \), then \( \chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'') \).

Proof: If \( 0 \to V_1 \to V_2 \to \ldots \) is long exact sequence of finite-dimensional vector spaces which terminates, then \( \sum(-1)^i \dim_k V_i = 0 \). This is done by replacing \( V_1 \) and \( V_2 \) with \( V_2/V_1 \) which has dimension \( \dim_k(V_2) - \dim_k(V_1) \) as the short exact sequence \( 0 \to V_1 \to V_2 \to V_2/V_1 \to 0 \) splits, and by repeating the procedure.

Therefore if \( 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \) is a short exact sequence we get the long exact sequence of cohomology

\[
H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}') \to \ldots
\]

It consists of finitely generate \( k \)-vector spaces and it terminates, both according to Serre’s theorem, so now we apply the above fact to conclude. □

**Proposition. (problem Hartshorne III.5.2.)** a) Let \( X \) be a projective scheme over a field \( k \), and consider the very ample sheaf \( \mathcal{O}_X(1) \) given by some embedding into projective space, and let \( \mathcal{F} \) be a coherent sheaf on \( X \).
Then there is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $\chi(F(n)) = P(n)$ for all $n \in \mathbb{Z}$. This is the Hilbert polynomial of $F$ with respect to the sheaf $\mathcal{O}_X(1)$.

b) Now let $X = \mathbb{P}^r_k$ and let $M = \Gamma_*(\mathcal{F})$ considered as a graded $S = k[x_0, \ldots, x_r]$-module. The Hilbert polynomial of $\mathcal{F}$ defined in a) is the same as the Hilbert polynomial as a graded module, defined in commutative algebra.

Proof: a) This has been done in Arnav’s notes. Here are just the basic steps:

First, Arnav’s project discusses only the “normal” case of $\mathcal{O}(1)$. However any very ample sheaf could work just as well, as done in Hartshorne chapter III - we have a version of Serre’s theorem etc.

Later, as $i: X \to \mathbb{P}^n$ is a closed embedding so is affine, the cohomologies of $\mathcal{F}$ over $X$ are the same as the cohomologies of $i_*\mathcal{F}$ over $\mathbb{P}^n$, so we may assume we are over $\mathbb{P}^n$.

After that, as $k$ is integrally closed so infinite, we can find a linear functional that acts injectively on $\mathcal{F}(-1)$: it is enough to do so on affines and there we look for a hyperplane that does not contain any of the finitely many associated primes. Every hyperplane is given by a vector in $k^n = k \oplus kx_1 \oplus \ldots kx_{n-1}$ (modulo scalars) and by fixing all but the constant coordinate we get infinitely many hyperplanes and if two of them contain a prime than the difference between the two generators must be in the prime but it is actually constant.

Using that linear functional $x$ we construct the short-exact sequence $0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{G} \to 0$ so $0 \to \mathcal{F}(m-1) \to \mathcal{F}(m) \to \mathcal{G}(m) \to 0$

Now by using the previous problem, we conclude that $\chi(\mathcal{F}) - \chi(\mathcal{F}(-1)) = \chi(\mathcal{G})$.

To prove something is a polynomial over integers it suffices to prove the difference $f(x) - f(x-1)$ is a polynomial, so we apply induction on the support as $\Gamma$ is supported on something of strictly smaller dimension: as $\mathcal{G}_p = \mathcal{F}_p/x\mathcal{F}_p$ the support of $\mathcal{G}$ is contained in the support of $\mathcal{F}$ intersected with $V(x)$. This has smaller dimension by Krull’s Hauptidealsatz: we may assume we are on an affine $A$ and the support of $\mathcal{F}$ is $A/I$ then the support of $\Gamma$ is a quotient of $A/(I,x) = (A/I)/x$ (note $x$ is not in $I$ and $x$ is not invertible in $A$) and the latter has dimension strictly smaller by the theorem.

b) The Hilbert polynomial of $\mathcal{F}$ is eventually $\Gamma(X, \mathcal{F}(n))$ as by Serre’s theorem all higher cohomologies eventually vanish, and this is by definition $(\Gamma_*, \mathcal{F})_n$ whose dimension is the ”old” Hilbert polynomial. Two polynomials coinciding on infinitely many values must coincide. □

03/23/2010

We have encountered the determinant construction for curves before. This approach can be generalized to schemes of bounded cohomological dimension.

**Theorem-Construction.** There exists a map $K(X) \xrightarrow{\text{det}} \text{Pic}(X)$ for $X$ a Noetherian scheme of bounded cohomological dimension.

Proof: the construction we perform is local, so we assume $X = \text{Spec}(A)$, in which case any module has finite projective dimension, so any $A$-module $M$ has a projective resolution $0 \to P^n \to \ldots \to P^1 \to P^0 \to M$. We associate to $M$ the line bundle $\text{det}(P^*) := \bigotimes_{i=0, \ldots, n} \text{det}(P^i)^{\otimes(-1)^i}$. Note that $\text{det}$ can be defined for projective modules (of finite dimension) because they are locally free, and the determinant of a free module $N$ of rank $r$ is $\Lambda^r N$ - alternatively we shrink $\text{Spec}(A)$ until the projective resolution becomes a free resolution. So it suffices to assume that the resolution consists of free modules. We need to show this is well-defined i.e. does not depend on the choice of the resolution - this also will ensure that the determinants ”glue” to create a global determinant map.

Assume $(Q^i)$ form another projective/free resolution of $M$ (of the same length) - then there is a map from $(Q^i)$ to $(P^i)$ that is a quasi-isomorphism. In particular its cone $R^*$ (whose objects are $Q^i \oplus P^{i+1}$) is acyclic. We will show that $\text{det}(Q^*) \otimes \text{det}(R^*)^{-1} \cong \text{det}(P^*)$ and that the determinant of an acyclic complex of projectives/free modules is trivial (i.e. $\mathcal{O}$). The first part is easy using the identity $\Lambda^{r_1+r_2}(R_1 \oplus R_2) \cong \Lambda^{r_1}R_1 \otimes \Lambda^{r_2}R_2$ where $R_1, R_2$ are free modules of rank $r_1, r_2$. Essentially, there is a natural map from the right hand-side to the left hand-side and this map is injective. It is also surjective as seen by lifting bases of $R_1$ and $R_2$ to a basis of $R_1 \oplus R_2$.

For the second part, we break an acyclic complex $R^*$ into short exact sequence $0 \to K^1 \to R^1 \to R^0 \to 0$, $0 \to K^2 \to P^2 \to K^1 \to 0$ etc. - note that the long exact sequence of Ext ensures that the kernel of a surjection of
projectives is projective, so \( K^i \) are all projective (free). It remains to observe that in a short exact sequence of free modules \( 0 \to R_1 \to R_2 \to R_3 \) or ranks \( r_1, r_2 = r_1 + r_3, r_3 \), we have \( \text{det}(R_1) \otimes \text{det}(R_3) \cong \text{det}(R_2) \). Indeed, we lift a basis of \( R_3 \) to a set of linearly independent vectors in \( R_2 \), and together with a basis of \( R_1 \) this forms a basis of \( R_2 \). It is now easy to map \( \text{det}(R_1) \otimes \text{det}(R_3) \) into \( \text{det}(R_2) \) and vice-versa (the maps are well-defined and these two maps are inverse to each other.

In particular, tensoring up the relations \( \text{det}(K^1) \otimes \text{det}(R^1) \cong \mathcal{O}, \text{det}(K^2) \otimes \text{det}(P^2) \otimes \text{det}(K^1) \cong \mathcal{O} \) etc. will yield the required relation \( \text{det}(R^*) \cong \mathcal{O} \) etc. We will therefore send the class of \( [\mathcal{F}] \) to the line bundle \( \text{det}(\mathcal{F}) \). It will factor through \( K(X) \) because if \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) is a short exact sequence, locally this short exact sequence induces a cone sequence between free resolutions of modules, and we’ve seen that this implies \( \text{det}(\mathcal{F}_1) \otimes \text{det}(\mathcal{F}_2) \otimes \text{det}(\mathcal{F}_3) \cong \mathcal{O} \).

Remark: the determinant allows us to show in another way that \( X \) is locally factorial (thus of bounded cohomological dimension) and \( Y \to X \xrightarrow{j} U \) with \( \text{codim}(Y) \geq 2 \) (in fact the codimension clause is not needed), then any line bundle \( \mathcal{L}_U \) on \( U \) admits an extension \( \mathcal{L} \) to \( X \). Indeed, choose a coherent extension \( \mathcal{F} \) of \( \mathcal{L}_U \) to \( X \) (for example \( j_* \mathcal{L}_U \) which is coherent by Jonathan’s project from the first semester) and then the required line bundle is \( \text{det}(\mathcal{F}) \) which on \( U \) is \( \text{det}(\mathcal{L}_U) = \mathcal{L}_U \).

Note: being of bounded cohomological dimension is local for Noetherian schemes. This is because the Ext and Tor functors commute with restrictions - to verify that, it suffices to work with injective resolutions (which exist) that restrict to injective resolutions.

Let \( X \) now have bounded Tor dimension. Then the (abelian) Grothendieck group \( K(X) \) has the structure of a (commutative) ring, with multiplication defined by

\[
[f_1] \cdot [f_2] = \sum_{i=0}^{\infty} (-1)^i [\text{Tor}_i(f_1, f_2)]
\]

The distributivity follows immediately from the (finite) long exact sequence of Tor. The associativity can be verified with the help of injective resolutions - instead of \( f \) we consider their injective resolutions, and Tor can be computed by tensoring up complexes (recall that the tensor product of complexes is not term by term, but like multiplication of polynomials).

If \( X \) is regular and connected, we define a decreasing filtration of the Grothendieck group defining \( K_i(X) \) be the span of \( [\mathcal{F}] \) where \( \text{Supp}(\mathcal{F}) \) is of codimension \( \geq i \).

The quotient \( K/K_2 \) is isomorphic to \( \mathbb{Z} \oplus \text{Pic}(X) \) via \( [\mathcal{F}] \to (rk_\xi(\mathcal{F}), \text{det}(\mathcal{F})) \) being the map in one direction, and \( 1 \to \mathcal{O}, \mathcal{L} \to [\mathcal{L}] - [\mathcal{O}] \). This is a generalization of a previous proposition that states that \( K(X) \cong \mathbb{Z} \oplus \text{Pic}(X) \) for \( X \) a projective curve, as then \( K_2 = 0 \).

Note that it is important to consider coherent sheaves instead of quasi-coherent ones: the free group on quasi-coherent sheaves modulo short exact sequences in zero, as evidenced by the short exact sequence \( 0 \to M \to M^\oplus \mathbb{Z} \to M^\oplus \mathbb{Z} \to 0 \).

Let \( X \xrightarrow{j} Y \) be a proper map of Noetherian schemes. We construct the map \( f_* : K(X) \to K(Y) \) by sending \( \mathcal{F} \in \text{Coh}(X) \) to \( \sum_i (-1)^i [R^if_*(\mathcal{F})] \) - it turns out that these sheaves are coherent and the sum is finite.

Recall now that if \( S \) is regular, then \( \text{Pic}(S) \xrightarrow{\pi^*} \text{Pic}(S \times \mathbb{A}^1) \). There is an analog for the Grothendieck group.

**Theorem.** If \( S \) is regular, then

\[
K(S) \xrightarrow{\pi^*} K(S \times \mathbb{A}^1)
\]

**Proof:** first we need to define the pullback functor on \( K \). Assuming \( f : X \to Y \) is a map with \( Y \) regular (so pullback exists??), we define \( f^* \mathcal{F} = \sum (-1)^i [L^if^*(\mathcal{F})] \)
Also, we call two maps $f_1, f_2: X \to Y$ $\mathbb{A}^1$-homotopic if there exists a map $X \to \mathbb{A}^1 \to Y$ that when restricted to two points gives $f_1$ and $f_2$ (points give maps $X \to X \times \mathbb{A}^1$ by base change).

We claim that $\mathbb{A}^1$-homotopic maps lead to the same map on $K$-theory. This is a direct consequence of the theorem as then become equal when composed with the isomorphism $K(X \times \mathbb{A}^1) \to K(X)$.

In our case, the map $\pi$ is flat and $\pi^*$ is exact, so the above definition produces a simpler map $K(S) \to K(S \times \mathbb{A}^1)$. The inverse map $K(S \times \mathbb{A}^1) \to K(S)$ is constructed as follows: choose a point i.e. a map $i: pt \to \mathbb{A}^1$ and then he claim that the map $L(id_S \times i)^*$ is the required inverse - note that this is independent of the choice of $i$ according to the claim.

We will show how to deduce the statement of this theorem from the analogue for $\mathbb{P}^1$:

**Theorem.** The maps $[\mathcal{F}] \to [\pi^*(\mathcal{F})], [\mathcal{F}] \to [\pi^*(\mathcal{F}) \times \mathcal{O}(−1)]$ give an isomorphism

$$K(S) \oplus K(S) \xrightarrow{\sim} K(S \times \mathbb{P}^1)$$

The inverse is given by $[\mathcal{F}'] \to ([R\pi_*([\mathcal{F}'])], [R\pi_*([\mathcal{F}'] \otimes \mathcal{O}(1))])$.

Before we show this theorem, here is how it implies the previous one.

Consider the "skyscraper" sheaf $\delta = i_*\mathcal{O}_S$, then there exists a short exact sequence $0 \to \mathcal{O}(−1) \to \mathcal{O} \to \delta \to 0$ - the base change of the familiar sequence $0 \to \mathcal{O}(i−1) \xrightarrow{\delta} \mathcal{O}(i) \to i_*\mathcal{O}(i)$.

Thus $[\delta] = [\mathcal{O}] - [\mathcal{O}(−1)]$ so the maps $\mathcal{F} \to \pi^*\mathcal{F}, \mathcal{F} \to \pi^*\mathcal{F} \otimes \delta$ induce an isomorphism $K(S) \oplus K(S) \to K(S \times \mathbb{P}^1)$. However note that $\mathcal{F} \to \pi^*\mathcal{F} \otimes \delta$ is simply the functor $i_*$ - so that the isomorphism is simply the pair of maps $(i_*, \pi^*)$.

We then recall the exact sequence corresponding to an exact pair: $K(S) \xrightarrow{\sim} K(S \times \mathbb{P}^1) \to K(S \times \mathbb{A}^1) \to 0$. Of course in our case this is actually a short exact sequence, as $i_*$ is an injection that splits - therefore $K(S \times \mathbb{A}^1)$ is identified with the copy of $K(S)$ inside $K(S \times \mathbb{P}^1)$ given by $\pi^*$ - and this proves the previous theorem (there are actually, by abuse of notation, two version of $\pi^*$ - one for $S \times \mathbb{P}^1$ and one for $S \times \mathbb{A}^1$, but they one is just the restriction of the other).

Note that once we know that $\pi^*$ is an isomorphism, it is easy to find it’s inverse: if we choose $i: pt \to \mathbb{A}^1$ then $L(id_S \times i)^*$ is the inverse. It is enough to observe that the composition $\pi \circ (id_S \times i)$ is the identity and then to conclude that $L(id_S \times i)^* \circ \pi^*$ is the identity on $K(X)$.

It remains to prove the theorem for $\mathbb{P}^1$.

In the first direction, we want the composite map $K(S) \oplus K(S) \to K(S \times \mathbb{P}^1) \to K(S) \oplus K(S)$ be the identity. Let’ look where $(\mathcal{F}, 0)$ will go under this composition. We claim that for a sheaf $\mathcal{F}$ we have $R\pi_*([\pi^*(\mathcal{F})]) \cong \mathcal{F}$. This is a generalization of the assertion that $H^0(\mathbb{P}^1, \mathcal{O}) = k, H^i(\mathbb{P}^1, \mathcal{O}) = 0, i > 0$ - this is a particular case for the map $\pi: \mathbb{P}^1 \to Spec(k)$.

It follows from a more general lemma (whose proof will be deferred for later):

**Lemma.** (Projection formula) Let $X \xrightarrow{\Phi} Y$ be a separated quasi-compact map of schemes, and say $\Phi$ is flat. Then for any $\mathcal{F}_1 \in D_{QCoh}(X), \mathcal{F}_2 \in D_{QCoh}(Y)$ the following isomorphism takes place:

$$R\Phi_*([\mathcal{F}_1]) \otimes L \mathcal{F}_2 \xrightarrow{\sim} R\Phi_*([\mathcal{F}_1 \otimes \Phi^*(\mathcal{F}_2)])$$

Note that the claim is a direct consequence of the projection formula, for $\mathcal{F}_1 = \mathcal{O}_X, \mathcal{F}_2 = \mathcal{F}, \Phi = \pi$.

For the second coordinate, we have to show $R\pi_*([\pi^*(\mathcal{F}) \otimes \mathcal{O}(−1)]) = 0$. According to the projection formula, it is $R\pi_*([\pi^*(\mathcal{F}) \otimes \mathcal{O}(−1)])$ and $R\pi_*([\mathcal{O}(−1)]) = 0$. To prove the last assertion, we can assume $S = Spec(A)$ is affine and then $\pi_* = \Gamma$ so $R\pi_*$ is $R\Gamma$ but we know that $\mathcal{O}(−1)$ has all cohomologies zero.

Let’s look where $(0, \mathcal{F})$ goes. On the first coordinate, the map is zero just like before. On the second coordinate, $\mathcal{F}$ goes to $\mathcal{O} \otimes R\pi_*(\mathcal{O}(−2))$. But $R\pi_*(\mathcal{O}(−2))$ is quasi-isomorphic to $\mathcal{O}_S[−1]$ because we know that $H^i(\mathbb{P}^1, \mathcal{O}(−n − 1))$ is one-dimensional and all other cohomologies are 0. $[\mathcal{F} \otimes \mathcal{O}_S[−1]]$ in $K(X)$ is just $−[\mathcal{F}]$ and then we obtain the conclusion.

It remains to check that the map $K(S) \oplus K(S) \to K(S \times \mathbb{P}^1)$ is surjective.
Consider the scheme $\mathbb{P}^1_S \times \mathbb{P}^1_S$. Recall the "tensor product" $\boxtimes$ defined by $\mathcal{F}_1 \boxtimes \mathcal{F}_2 = p_1^*(\mathcal{F}_1) \otimes p_2^*(\mathcal{F}_2)$ where $p_1, p_2$ are the two projections from $\mathbb{P}^1_S \times \mathbb{P}^1_S$ to $\mathbb{P}^1_S$. Consider also the diagonal embedding $\Delta: \mathbb{P}^1_S \to \mathbb{P}^1_S \times \mathbb{P}^1_S$.

We claim that there exists a short exact sequence

$$0 \to \mathcal{O}(\mathbb{P}^1_S) \boxtimes \mathcal{O}(\mathbb{P}^1_S) \to \mathcal{O}(\mathbb{P}^1_S \times \mathbb{P}^1_S) \to \Delta_*(\mathcal{O}(\mathbb{P}^1_S)) \to 0$$

This is easy to check explicitly: if $S = \text{Spec}(A)$, on a suitable affine this sequence looks like $0 \to \frac{1}{s_0t_0} A[\frac{s_1}{s_0}, \frac{t_1}{t_0}] \to A[\frac{s_1}{s_0}, \frac{t_1}{t_0}] \to A[\frac{s_1}{s_0}] \to 0$.

The maps go as follows - the last map is produced by sending $s_1, t_1$ to $x_1$ and $s_0, t_0$ to $x_0$; the first map is obtained by multiplication by $s_1 t_0 - s_0 t_1$. It is easy to see that this creates a short sequence, and the maps correspond to the analogous maps for the other affines.

Therefore, in $K(\mathbb{P}^1 \times \mathbb{P}^1)$, $[\Delta_*(\mathcal{O}(\mathbb{P}^1_S))] = [p_1^*(\mathcal{O}(\mathbb{P}^1_S)) \otimes p_2^*(\mathcal{O}(\mathbb{P}^1_S))] - [p_1^*(\mathcal{O}(\mathbb{P}^1_S)) \otimes p_2^*(\mathcal{O}(\mathbb{P}^1_S))]$.

Surjectivity (and therefore the theorem) will follows from the following proposition:

**Proposition.** Let $X$ be a regular proper scheme (e.g. $\mathbb{P}^1$). Consider the diagonal embedding $\Delta: X \to X \times X$. Suppose that, in $K(X \times X)$,

$$[\Delta^*(\mathcal{O}_X)] = \sum_{i=1}^r [a_i \boxtimes b_i]$$

Then the set $\{b_i\}$ (or $\{a_i\}$) span $K(X)$ in the following sense: the map $K(S) \oplus K(S) \oplus \ldots \oplus K(S) \to K(S \times X)$ whose $i$-th coordinate is $\mathcal{F} \to \mathcal{F} \boxtimes b_i$ (why is this a "map" i.e. factors through $K(S)$?) is surjective.

**Proof:** a slightly stronger version of the projection formula (without the flatness requirement), states that if $\Phi: S \to Y$ is a quasi-compact and quasi-separated map of schemes, and $\mathcal{F}_X \in D_{Qcoh}(\mathcal{O}_X - \text{mod})$, $\mathcal{F}_Y \in D_{Qcoh}(\mathcal{O}_Y - \text{mod})$, then

$$R\Phi_*((\mathcal{F}_X) \boxtimes \mathcal{F}_Y) \cong R\Phi_*((\mathcal{F}_X \boxtimes \mathcal{F}_Y))$$

We apply the lemma for the projection $p_2: X \times X \times S \to X \times S$ and the sheaves $\Delta_*\mathcal{O}_{X \times S}$ (where $\Delta$ is the diagonal embedding which has no higher direct images as it is a closed embedding, by properness) and $\mathcal{F}$. We deduce

$$R(p_2)_*(\Delta_*\mathcal{O}_{X \times S}) \boxtimes \mathcal{F} \cong R(p_2)_*((\Delta_*\mathcal{O}_{X \times S}) \boxtimes \mathcal{F})$$

Since $p_2 \circ \Delta = \text{id}$ this simplifies to $R(p_2)_*((\Delta_*\mathcal{O}_{X \times S}) \boxtimes \mathcal{F}) \cong \mathcal{F}$

Therefore writing $[\Delta_*\mathcal{O}_{X \times S}] = [\sum a_i \boxtimes b_i]$ we obtain $[\mathcal{F}] = [\sum a_i \boxtimes b_i]Lp_2^*(\mathcal{F})$ - which can be written as $[\sum a_i \boxtimes (\mathcal{F} \boxtimes p_1^*(b_i))]$ - and this proves the proposition. $\square$

**Proposition.** (PSET) Let $S$ be a Noetherian scheme. Then $K(S)^{\oplus(n+1)} \cong K(S \times \mathbb{A}^n)$ where the isomorphism is given by the map whose $i$-th coordinate is $[\mathcal{F}] \to [\mathcal{F} \boxtimes \mathcal{O}(1-i)]$.

**Proof:** We use induction on $n$. The base was already shown shown in class. Choose $x_0, \ldots, x_n$ as usual.

First, we know that $K(X) \cong K(X \times \mathbb{A}^n)$ via the map $\pi^*$, by induction as we have shown $K(X) \cong K(X \times \mathbb{A}^1)$. The induction step follows from $\mathbb{A}^{n-1} \times \mathbb{A}^1 \cong \mathbb{A}^n$.

Further, choose the closed embedding and the open embedding of the complement corresponding to $x_0: \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{A}^n$. We "multiply" it by $S$ to get $\mathbb{P}^{n-1} \times S \hookrightarrow \mathbb{P}^n \times S \hookrightarrow \mathbb{A}^n \times S$.

Recall that $K(\mathbb{P}^{n-1} \times S) \to K(\mathbb{P}^n \times S) \to K(\mathbb{X}) \to 0$ - where the first map is given by $i_*$ and the second by restriction $j^*$. We will prove that the first map is injective, but we defer the proof for now.

Recall the short exact sequence $0 \to \mathcal{O}(m-1) \to \mathcal{O}(m) \to i_*\mathcal{O}(m) \to 0$. We claim that by tensoring with a sheaf $\mathcal{F} \in \text{Coh}(S)$ it will still be exact: $0 \to \mathcal{O}(m-1) \boxtimes \mathcal{F} \to \mathcal{O}(m) \boxtimes \mathcal{F} \to i_*\mathcal{O}(m) \boxtimes \mathcal{F} \to 0$.

Indeed on the affine $\text{Spec}(A) \times \mathbb{A}^n$ this is just $M \otimes k[y_0, \ldots, y_{n-1}]_{i=1} \to M \otimes k[y_0, \ldots, y_{n-1}]_{i=1} \to 0$ which is obviously exact.
Further, it is also pretty clear using the same explicit verification on affines that $i_*\mathcal{O}(m) \boxtimes \mathcal{F} \cong i_*(\mathcal{O}(m) \boxtimes \mathcal{F})$ where $i$ is the embedding $\mathbb{P}^{n-1} \times S \to \mathbb{P}^n \times S$ produced from $i$ by base change.

From here, we deduce that $[i_*\mathcal{O}(m) \boxtimes \mathcal{F}] = [\mathcal{O}(m) \boxtimes \mathcal{F}] - [\mathcal{O}(m-1) \boxtimes \mathcal{F}]$.

Now, to prove that the map $K(S)^n \to K(\mathbb{P}^n \times S)$ given by $(\mathcal{F})_i \to \mathcal{F} \boxtimes \mathcal{O}(i)$ is an isomorphism is is enough to prove that the map $f$ given by $(\mathcal{F})_0 \to \mathcal{F} \boxtimes \mathcal{O}, (\mathcal{F})_1 \to \mathcal{F} \boxtimes \mathcal{O}(i) - \mathcal{F} \boxtimes \mathcal{O}$ is an isomorphism.

Observe that the map $(\mathcal{F})_i \to \mathcal{F} \boxtimes \mathcal{O}(i) - \mathcal{F} \boxtimes \mathcal{O}$ becomes 0 when mapped to $S \times \mathbb{A}_n$ becomes 0 as the two sheaves become isomorphic (because $\mathcal{O}(i), \mathcal{O}$ both become free).

Now let’s consider the split short exact sequence $0 \to K(S)^{n-1} \to K(S)^n \to K(S) \to 0$ given by mapping the last $n-1$ coordinates and the first coordinate, and let’s look at the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & K(S)^{n-1} & \rightarrow & K(S)^n & \rightarrow & K(S) & \rightarrow & 0 \\
& & \sim & f & & \sim & & & \\
& & K(S \times \mathbb{P}^{n-1}) & \rightarrow & K(S \times \mathbb{P}^n) & \rightarrow & K(S \times \mathbb{A}_n) & \rightarrow & 0 \\
\end{array}
\]

The first horizontal map is given by $(\mathcal{F})_0 \to \mathcal{F} \boxtimes \mathcal{O}, (\mathcal{F})_i = \mathcal{F} \boxtimes (\mathcal{O}(i)) + \mathcal{F} \boxtimes \mathcal{O}(i-1) + \ldots \mathcal{F} \boxtimes \mathcal{O}(1)$ which is an isomorphism according to the induction step (it is related to the original map by an inner upper-triangular isomorphism), the last map is given by the isomorphism considered above.

We need to show that the diagram is commutative. Indeed, the commutativity of the right square is equivalent to $f|_{\mathbb{A} \times S}$ factoring through $K(S)^{n-1}$, which was shown before. The commutativity of the left square is verified directly as $i_*(\mathcal{F} \boxtimes \mathcal{O}(i)) = [\mathcal{F} \boxtimes \mathcal{O}(i)] - [\mathcal{F} \boxtimes \mathcal{O}(i-1)]$ and it telescopes.

From here we immediately deduce that $f$ is surjective. For the injectivity, we return to the original map and construct a “left inverse” in the sense that the composition will be an injective map $K(S)^{n+1} \to K(S)^{n+1}$. This map $g$ is $\mathcal{F} \to [R\pi_*(\mathcal{F} \boxtimes \mathcal{O}(i))]$ (starting from $i = 0$). Remark: for both the map $f$ and the inverse map it would be better to consider the left derived functor of $\operatorname{Ext}$ i.e $\mathcal{L}Tor(\pi^*\mathcal{F}, \pi_2^*\mathcal{O}(i))$ but because $\pi_2^*\mathcal{O}(i)$ is locally free, it is the same as $\boxtimes$.

Now consider $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n)$ in the kernel of $g \circ f \ (\mathcal{F}_1 \in K(S))$.

The $i$-th coordinate will be $R\pi_*(\mathcal{F}_0 \boxtimes \mathcal{O}(i) + \mathcal{F}_1 \boxtimes \mathcal{O}(i-1) + \ldots + \mathcal{F}_n \boxtimes \mathcal{O}(i-n))$. Now observe that for $j > i$ we have $R\pi_*(\mathcal{F}_i \boxtimes \mathcal{O}(i-j)) = 0$ - because according to the projection formula it equals $R\pi_*\pi^*\mathcal{F}_i \boxtimes R\pi_*\pi_2^*\mathcal{O}(i-j)$ and $R\pi_*\pi_2^*\mathcal{O}(i-j) = 0$ because $\mathcal{O}(i-j)$ has all cohomologies zero so is zero in the derived category. Also, remember that $R\pi_*\pi^*\mathcal{F} = \mathcal{F}$.

Therefore for $i = 0$ all terms become 0 except $R\pi_*(\mathcal{F}_0 \boxtimes \mathcal{O}) = R\pi_*\pi^*\mathcal{F}_0 \boxtimes \mathcal{O} = \mathcal{F}_0$ so $\mathcal{F}_0 = 0$. Then for $i = 1$, all terms $\mathcal{F}_j$ with $j > 1$ become 0 and for $j = 0$ we just deduced $\mathcal{F}_0 = 0$ so we get $R\pi_*(\mathcal{F}_i \boxtimes \mathcal{O}(i-n) \boxtimes \mathcal{O}(1)) = 0$ i.e. $\mathcal{F}_i = 0$. Continuing like that we get $\mathcal{F}_i = 0$ which means the map $g \circ f$ is injective and hence so is $f$. □

**Proposition. (PSET)** Let $X$ be a regular connected scheme. Let $K_i(X)$ be the decreasing filtration on $K(X)$, where $K_i(X)$ is spanned by classes of coherent sheaves $F$, such that $\operatorname{codim}(\operatorname{supp}(\mathcal{F})) \geq i$.

a) $det$ is a well-defined map $K(X)/K_2(X) \to \operatorname{Pic}(X)$

b) The map $K(X)/K_2(X) \to \mathbb{Z} \oplus \operatorname{Pic}(X)$ is an isomorphism, where the first map assigns to a coherent sheaf its rank at the generic point, and the second map is the determinant map.

c) In $K(X)/K_2(X)$ we have the following relation: for two line bundles $\mathcal{L}_1, \mathcal{L}_2$ we have

$$[\mathcal{L}_1] + [\mathcal{L}_2] + [\mathcal{L}_1 \otimes \mathcal{L}_2] + [\mathcal{O}]$$

Proof: Since $X$ is regular it is locally factorial hence normal, and since it is connected, it is integral according what was done before (last problem set).

a) We have shown before that the determinant is a well-defined map from $K(X)$ to $\operatorname{Pic}(X)$ and are left to show it factors through $K_2(S)$. Let $\mathcal{F}$ be in $\mathcal{Coh}(X)$ with support of codimension $\geq 2$. Let $U$ be the complement of the support. We claim that $det(\mathcal{F})|_U \cong \mathcal{O}|_U$ which will imply the conclusion because by normality every line bundle on
U extends uniquely to a line bundle on X, and O|_U extends to O. Indeed, the determinant is determined locally, and locally \( \mathcal{F} \) is 0. Therefore \( det(\mathcal{F}) \) is locally O - this is not enough to conclude it is O on the entire U of course, but we claim that there is actually a canonical map \( det(\mathcal{F}) |_U \to O |_U \) which realizes the local isomorphism (and hence this is an isomorphism on the entire U).

Indeed, this will follow from the fact that if \( 0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to 0 \) is an exact sequence of projectives, then \( (\Lambda^{k_2-k_1}K_1 \otimes (\Lambda^{k_2}P_2)^{-1} \otimes (\Lambda^{k_1}P_1)) \sim \sim O \) and there is a canonical map providing this isomorphism (and the right-hand side is equal to the determinant).

Indeed, recall how we did it: we split complex into short exact sequences \( 0 \to K_1 \to P_2 \to P_1 \to P_0 \) etc. and we showed that there is a canonical map \( \Lambda^{k_2-k_1}K_1 \to (\Lambda^{k_2}P_2)^{-1} \otimes (\Lambda^{k_1}P_1) \sim \sim O \) which is the same as constructing a canonical map \( \Lambda^{k_2-k_1}K_1 \otimes (\Lambda^{k_1}P_1) \sim \sim \Lambda^{k_2}P_2 \). We have constructed this map (locally) and the construction of this map was canonical: recall that it involved lifting a basis of \( P_1 \) (which locally is free so has a basis) to \( P_2 \) and this lifting is canonical up to terms from \( K_1 \) which are killed by tensoring with \( \Lambda^{k_2-k_1}K_1 \).

b) I will refer to the proof of 6.10. and 6.11. from Hartshorne which were done before. We define the map \( \psi: Cl^W(X) \to K(X) \) (of course \( Cl^W \cong Cl^C \) because the scheme is locally factorial) like in 6.11.a) - except \( k(p) \) will denote the sheaf \( i_*O_{Y_p} = O_p \) where \( Y_p \) is the closed subscheme corresponding to \( p \) and like there it factors through a map \( Cl^W(X) \to K(X) \) i.e. \( Pic(X) \to K(X) \).

To prove that a) adapts, we use the fact that 6.18. from Hartshorne adapts to curves as well. We also need to show that there is a short exact sequence \( 0 \to O_p \to O_D \to O_{D-p} \) for \( D - p \) effective and then the method of a) will follows. Indeed, we have the map \( O_D \to O_{D-p} \) and we claim it’s kernel is isomorphic to \( O_p \). On a small affine containing \( p \) where everything is principal we simple get the map \( A/ax \to A/a \) with kernel \( axA/aA \cong A/axA \) because \( A \) is integral, where \( (a) = D - p, (x) = p \). On other small affines not containing \( p \) we get a similar map which is an isomorphism so whose kernel is 0. Therefore the kernel sheaf if \( L(p-D) \otimes O_p \). It remains to show its class is equal to the class of \( O_p \). Because the sheaf \( L(p-D) \) is locally free, we have \( [L(p-D) \otimes O_p] = [L(p-D) \otimes (O - L(-p))] = [L(p-D) \otimes O] - [L(p-D) \otimes L(-p)] = [L(p-D)] - [L(p-D)] - [L(-p)] + [O] = [O] - [L(-p)] = [O_p] \).

Remark: this computation actually used part c). Here is another way to show that \( [L(p-D) \otimes O_p] \cong [O_p] \). These sheaves are isomorphic in a neighborhood of \( p \) (i.e. their restrictions are). Moreover, at any open set not meeting \( p \) their restrictions are isomorphic too because they are zero. As any point of codimension 1 has a neighborhood not meeting \( p \) we can conclude that there is a set \( U \) whose complement as codimension 1 such that the two sheaves restricted to \( U \) are isomorphic. We claim that they are then isomorphic modulo \( K_2(X) \) - indeed that’s just the content of lemma 1 from the addition to problem 6.10.d) on Pset 7 (last page) - and this shows the map is well-defined modulo \( K_2(X) \).

Next step is to show the analogue of c): namely that for any coherent sheaf \( \mathcal{F} \) of rank \( r \) we can construct an exact sequence \( 0 \to L(D)^{\oplus r} \to \mathcal{F} \to \mathcal{F} \to 0 \) where \( \mathcal{F} \) is a torsion sheaf. The proof adapts from there: indeed recall that the very same method allows us to construct a map \( L(D)^{\oplus r} \sim \sim \mathcal{F} \otimes K \) such that the image lands inside the stalk of \( \mathcal{F} \) at points of codimension 1 and is an isomorphism at the generic point. This is enough to conclude that the image lands inside \( \mathcal{F} \): on an affine \( Spec(A) \), everything in the image is in \( M \otimes_A K \) that belongs to all \( M \otimes_A A_p \) for \( p \) of height 1, and as we know this implies that the element is actually in \( M \).

Now because \( \mathcal{F} \) is torsion, just like in the original problem we deduce that it is in \( Im(\psi) \) - except this time modulo \( K_2(X) \) because the filtration terminates in a sheaf supported on codimension \( \geq 2 \).

So just like there, \( [\mathcal{F}] - r[O] \in Im(\psi) + K_2(X) \).

We now construct an inverse for the map: the map that sends 1 to \( [O_X] + K_2(X) \) and that sends \( D \) to \( \psi(D) \). This map is immediately surjective, by what we’ve just proved, and therefore it is enough to show that the map is a left inverse. Indeed, if we start with 1 we send it first to \( [O_X] \) and then back to 1, as the determinant of \( O_X \) is 0. If we start with \( D \) we send it to \( \psi(D) \) and then back to \( L(D) \) (which is identified with \( D \)) as \( det(\psi(D)) = L(D) \).

c) Follows immediately from b), because the projection of both sides to \( Z \oplus Pic(X) \) is the same: the total rank is 2 for both sides, and the determinant of a line bundle is itself, with addition in \( Pic(X) \) given by the tensor product.
Proposition. (PSET) Let $\mathbb{P}^n$ be the $n$-dimensional projective space over a field. Recall the Hilbert polynomial construction, which associates to every coherent sheaf $\mathcal{F}$ a polynomial $p_\mathcal{F}(t)$, such that $p_\mathcal{F}(n) = \chi(\mathcal{F}(n))$.

a) The assignment $\mathcal{F} \mapsto p_\mathcal{F}$ defines a homomorphism $K(\mathbb{P}^n) \to \mathbb{Z}^{\oplus(n+1)}$, if we send a $p_\mathcal{F}$ to its coordinates in the basis $(\binom{x_0}{n}, \binom{x_1}{n}, \ldots, \binom{x_n}{n})$ of polynomials of degree $\leq n$ in $x$ (those coordinates happen to be integers).

b) For $\mathcal{F} \in K_1(\mathbb{P}^n)$, the degree of $p_\mathcal{F}$ is $\leq i$.

c) The map in a) is actually an isomorphism.

Proof: a) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ is short exact then so is $0 \to \mathcal{F}_1(n) \to \mathcal{F}_2(n) \to \mathcal{F}_3(n) \to 0$ because it is obtained by tensoring with the locally free (flat) sheaf $\mathcal{O}(n)$. It follows that $\chi(\mathcal{F}_1(n)) - \chi(\mathcal{F}_2(n)) + \chi(\mathcal{F}_3(n)) = 0$ from the long exact sequence of cohomology (that terminates). Hence $p_\mathcal{F}_1 - p_\mathcal{F}_2 + p_\mathcal{F}_3 = 0$ as this polynomial is $0$ when evaluated at any non-negative integer. This implies that the assignment induces a map $K(X) \to \mathbb{Q}[X]$ and by projecting $\mathbb{Q}[X]$ to $\mathbb{Q}^{n+1}$ via the coordinates in the basis $((\binom{x_i}{n}))$ we get the required map, once we can prove that the image of the map lands inside $\mathbb{Z}^{n+1} \subset \mathbb{Q}^{n+1}$. Indeed, the polynomial $p_\mathcal{F}$ clearly sends $\mathbb{Z}$ to $\mathbb{Z}$, and we know that all such polynomials are $\mathbb{Z}$-combinations of the polynomials $(\binom{x_i}{n})$ which satisfy the condition. (This theorem is really well-known, and is done by induction using $p(x + 1) - p(x)$).

b) We use induction on $i$. For $i = 0$ we are dealing with sheaves supported at the generic point $p$. Using the previous problem set, all such sheaves are generated in $K(X)$ by sheaves that are actually induced from the closed embedding $p \to \mathbb{P}^n k$. All such sheaves are simply finite-dimensional vector spaces over $k$, and because the map is affine the cohomology of this sheaf is the same as the cohomology of the vector space over $p$ - which equals the dimension of the vector space, and when tensoring with $\mathcal{O}(n)$ the vector space does not change up to isomorphism, which shows that $p_\mathcal{F}$ is constant.

For the induction step, we use the fact that for $\mathcal{F}$ supported on codimension $i$, $i, \mathcal{F}$ is supported in codimension $\leq i - 1$ - once we assume we are over an infinite field which we can do by tensoring (the Hilbert polynomial stays the same), for some closed embedding $1: \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ - details in Arnav’s notes. The short exact sequence $0 \to \mathcal{F}(n - 1) \to \mathcal{F}(n) \to \mathcal{O}(n) \to 0$ implies $p(\mathcal{F}(n)) - p(\mathcal{F}(n - 1)) = p_\mathcal{F}(n)$. The conclusion follows from the induction hypothesis and the easy assertion that for a non-constant polynomial $f$ over a ring of characteristic zero, the degree of $f$ is $1$ more than the degree of $f(x) - f(x - 1)$.

Indeed, the polynomial clearly sends $\mathbb{Z}$ to $\mathbb{Z}$, and we know that all such polynomials are $\mathbb{Z}$-combinations of the polynomials $(\binom{x_i}{n})$ which satisfy the condition. (This theorem is really well-known, and is done by induction using $p(x + 1) - p(x)$)

c) We already know that $K(\mathbb{P}^n)$ is isomorphic to $\mathbb{Z}^n$. Therefore the induced map is a map from $\mathbb{Z}^n$ to $\mathbb{Z}^n$. It is enough to prove that this map is surjective - because then it is induced by a matrix $M \in M_n(\mathbb{Z})$ and surjectivity implies that $M$ is non-singular as well if we tensor with any field containing $\mathbb{Z}$ we get a surjective map of finite-dimensional vector fields of the same dimension. In this case, constructing the preimages of the basis vectors in $\mathbb{Z}^n$ produces an inverse for $M$ which shows that $M$ is an isomorphism.

So it remains to prove surjectivity of the map $K(\mathbb{P}^n) \to \mathbb{Z}^n$. We can identify the latter with the set of polynomials of degree at most $n$ - and in this setting, the map is simply $\mathcal{F} \mapsto \chi_\mathcal{F}$ because $\chi_\mathcal{F}$ has degree at most $n$. To prove surjectivity, it is enough to show that it contains a monic polynomial of every degree.

But first, we need to observe that the map is actually twisted by multiplying the coefficient of $x^i$ by $i!$. Therefore we need to find a polynomial in the image that has leading coefficient $\binom{x}{n}$. Indeed, its Hilbert polynomial is $(\binom{x}{n+1})$ - this is shown by induction on $i$ and the Pascal’s identity (we only need to regulate the free coefficient - which is actually not needed for this problem - but it is one as $\chi(\mathcal{O}) = 1$) which is exactly what we need. □

Proposition. (Riemann-Roch for regular curves - PSET) Let $X$ be a non-singular complete curve over a field $k$. Let $deg$ de the map $K(X) \to \mathbb{Z}$ that sends $\mathcal{F}$ to $\chi(\mathcal{F}) - rk(\mathcal{F}) \cdot \chi(\mathcal{O})$ where $rk(\mathcal{F})$ denotes the generic rank.
of $\mathcal{F}$, and $\chi$ is the Euler characteristic of cohomology i.e.

$$\chi = \dim_k(H^0(X, \mathcal{O}) - \dim_k(H^1(X, \mathcal{O}))$$

a) $\deg$ defines a homomorphism $Pic(X) \to \mathbb{Z}$

b) Consider the line bundle $\mathcal{O}(D)$ where $D$ is a divisor on $X$, then

$$\deg(\mathcal{O}(D)) = \deg(D)$$

where $\deg(D)$ is defined as follows: for $D = \sum d_i \cdot x_i$,

$$\deg(D) = \sum d_i \cdot \dim_k(k_{x_i})$$

c) If two divisors $D_1$ and $D_2$ are rationally equivalent, they have the same degree.

d) Define the genus of $X$ as $\dim_k(H^1(X, \mathcal{O}))$. Then the Riemann-Roch theorem holds:

$$\dim(H^0(X, \mathcal{O}(D))) - \dim(H^1(X, \mathcal{O}(D))) = \deg(D) + 1 - g$$

Proof: a) It has been basically proven before. The map $\chi$ factors through short exact sequences using the long exact sequence of cohomology- this has basically been done in 3a), and so does the rank because taking the stalk at the generic point is exact and dimension of a vector space factors through short exact sequences (which split).

Here, we used that $Pic(X)$ injects into $K(X)$ which is the content of a problem on the previous problem set. Alternatively, the map factors through $Pic(X)$ because two isomorphic line bundles get obviously map to the same thing as their cohomologies and ranks are isomorphic. Also by 2c) we have $\dim_k([\mathcal{L}_1] + [\mathcal{L}_2] = [\mathcal{L}_1 + \mathcal{L}_2] + [\mathcal{O}]$ and since $\deg(O) = 0$ we obtain that $\deg$ is actually a homomorphism.

b) It is enough to prove it for the skyscraper sheaf $k_x$ where $x$ is a closed point, as in $K(X)$ such points together with $\mathcal{O}$ obviously generate each divisor $D$ (also see previous problem set for that). This sheaf is actually a direct image from the closed embedding $x \to X$ and so its cohomologies are the same as the cohomologies of $k_x$ over the point $Spec(k_x)$ - that is $k_x$ and 0. The result immediately follows.

c) The line bundles $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are isomorphic hence they have the same degree which coincides with the degree of $D_1$ and $D_2$.

d) As $\dim_k(H^0(X, \mathcal{O})) = 1$, Riemann-Roch can be rewritten as $\chi(\mathcal{O}(D)) = \deg(D) + \chi(\mathcal{O})$ i.e. $\chi(\mathcal{O}(D)) - \chi(\mathcal{O}) = \deg(D)$. But since $\mathcal{O}(D)$ has generic rank 1, the left-hand side equals $\deg(\mathcal{O}(D))$ which equals $\deg(D)$ by part b). \]

**Proposition (Bezout’s theorem - PSET)** Recall that the Grothendieck group of a regular scheme has a natural ring structure.

a) Each $K_i(\mathbb{P}^n)$ is an ideal, generated by the $i$-th power of $[\mathcal{O}] - [\mathcal{O}(-1)]$.

b) Multiplication on $K(\mathbb{P}^n)$ sends $K_i \otimes K_j$ to $K_{i+j}$.

c) Let $a_i \in K_i(\mathbb{P}^n), a_j \in K_j(\mathbb{P}^n)$ be classes of degrees $d_i$ and $d_j$ respectively. (The degree of a class $a_i \in K_i(\mathbb{P}^n)$ is the $n - i$-coefficient of its Hilbert polynomial, up to $(n - i)!$). The degree of $a_i \cdot a_j$ is $d_i \cdot d_j$.

d) Let $\mathcal{F}_i, \mathcal{F}_j$ be coherent sheaves supported in codimensions $i, j$ respectively. Assume that $\mathcal{F}_i \otimes \mathcal{F}_j$ is supported in codimension $i_j$, and that the higher Tors between $\mathcal{F}_i$ and $\mathcal{F}_j$ are supported in codimension $> (i + j)$. Then

$$\deg(\mathcal{F}_i) \cdot \deg(\mathcal{F}_j) = \deg(\mathcal{F}_i \otimes \mathcal{F}_j)$$

e) Let $Y_i, Y_j \subseteq \mathbb{P}^n$ be two reduced irreducible subschemes, of codimensions $i$ and $j$ respectively. Let $Z$ be an irreducible component of their intersection, such that $codim(Z) = i + j$. Then the higher Tors $Tor^k(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})$ vanish when localized at the generic point at $Z$.

f) Let $Y_i, Y_j$ be as above. Assume that $Y_i \cap Y_j$ is of codimension $i + j$ (a priori, the codimensions is $\leq$ than that). Then all higher Tors $Tor^k(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})$ vanish.
f) (The general form of Bezout’s theorem). Let $Y_i$ and $Y_j$ be as in part e’). Then

$$deg(Y_i) \cdot deg(Y_j) = \sum_Z deg(Z)n_Z(Y_i,Y_j)$$

where $n_Z(Y_i,Y_j)$ is the length of the Artinian sheaf obtained by localization at the generic point of $Z$ of the coherent sheaf $\mathcal{O}_{Y_1} \otimes \mathcal{O}_{Y_2}$

03/25/2010, Inverse Images handout

The contents of the previous lecture have been done with the assumption that the left derived functor of the inverse image functor exists. Similarly, we want to construct the left derived functor of the tensor product: denoted by $L \otimes$. We can also work with a slightly larger category of $X$ a ringed space with a sheaf of rings $R_X$ - it turns out that we need to use the larger category of $\mathcal{O}_X$-modules instead of quasi-coherent sheaves, the reason for that is that the extension by zero functor $j_!$ we will use does not send quasi-coherent sheaves to quasi-coherent sheaves. The problem with constructing these functors in the usual way is that the category $\text{Sh}(R_X - \text{mod})$ does not have enough projectives, i.e. the projection $K(D(\text{Sh}(R_X - \text{mod}))) \to D(\text{Sh}(R_X - \text{mod}))$ does not have a left adjoint.

The idea is to find another intermediate subcategory instead of the category of projectives, called adapted.

**Lemma 1.** Let $D'$ be a triangulated category and $D', D' \subset D$ be full triangulated subcategories. Assume that for every object $d \in D$ there exists an arrow $d \to d'$ with $d' \in D'$, which is a $D'$-quasi-isomorphism. Then the natural functor

$$D/D' \to D/D'$$

is an equivalence, where $D' = D' \cap D$.

The proof of the lemma is easy.

In the setting of Lemma 1, let $F: D \to C$ be a functor between triangulated categories. If we want to show that its left derived functor with respect to $D'$, $LF: D/D' \to C$ exists, we can use $D$ as $D'$. Namely, assume that the restriction to $D'$ vanishes, to that $F': D' \to C$ descends to a functor $F': D/D' \to C$.

**Lemma 2.** Under the above circumstance, the functor

$$DD/D' \to D/D'$$

canonically identifies with $LF$.

The proof of this lemma is also easy, using the universal property.

Now we let $D = K(D(\text{Sh}(R_X - \text{mod})))$ and $D'$ be the category of acyclic complexes. We will construct the required category $D'$, whose objects we will call "adapted".

**Definition.** We say that $\mathcal{F}$ is adapted is it admits an increasing filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \ldots$ with $\cup \mathcal{F}_i = \mathcal{F}$, such that each subquotient $\mathcal{F}_i / \mathcal{F}_{i-1}$ is isomorphic to a direct sum of shifts of sheaves of the form $j_!(R_X |_U)$ for some open $U \subset X$.

The proof that every object is quasi-isomorphic to an object consisting of adapted sheaves is tedious, but it is absolutely similar to the proof that every object is quasi-isomorphic to a $K$-projective object (problem 2, PS 2). The only difference is in the construction of sheaves surjecting on cohomologies: for this, it suffices for any open $U$ to construct a surjection of a free $\Gamma(R_X |_U)$-module onto $\Gamma(\mathcal{F}, U)$ then use $j_!$ and take direct sum over all $U$.

We want to construct the left derived functor of tensoring up with $\mathcal{F}^\tau$ - a complex of right $R_X$-modules (we do not need our rings to be commutative). To apply lemma 2 and construct $\text{LTor}$ we need to verify the condition of $F$ vanishing on $D'$ - i.e. if an adapted complex $\mathcal{F}$ is acyclic, then $\mathcal{F}^\tau \otimes_{R_X, \mathcal{F}}$ is acyclic too.
Lemma 3. Let $\mathcal{F}$ be an adapted complex of sheaves of left $R_X$-modules. Let $\mathcal{F}^r$ be a complex of sheaves of right $R_X$-modules. Consider the tensor product $\mathcal{F}^r \otimes \mathcal{F}$. We have

i) If $\mathcal{F}^r$ is acyclic, then $\mathcal{F}^r \otimes \mathcal{F}$ is acyclic.

ii) If $\mathcal{F}$ is acyclic, then $\mathcal{F}^r \otimes \mathcal{F}$ is acyclic.

Proof: For i), tensor products commute with direct limits, and thus we can replace $\mathcal{F}$ by $\mathcal{F}_i$. From here, it is enough to consider $\mathcal{F}_i / \mathcal{F}_{i-1}$ because tensoring up is left exact, and thus we can only consider $j_!(R_X | U)$. Then

$$\mathcal{F}_1 \otimes j_!(R_X | U) \cong j_!(\mathcal{F}_1 | U)$$

and hence is acyclic.

For the second part, choose a quasi-isomorphism $\mathcal{F}^r_{\text{ad}} \to \mathcal{F}^r$, where $\mathcal{F}^r_{\text{ad}}$ is adapted. By i), the map

$$\mathcal{F}^r_{\text{ad}} \otimes \mathcal{F} \to \mathcal{F}^r \otimes \mathcal{F}$$

is a quasi-isomorphism because its cone will stay acyclic after tensoring it up with $\mathcal{F}$. But $\mathcal{F}^r_{\text{ad}} \otimes \mathcal{F}$ is acyclic by part i) with left modules exchanged with right modules. $\square$

Therefore we have constructed the desired left derived functor

$$\mathcal{F}^r \otimes \left. \begin{array}{c} L \vspace{1pt} \\
R_X \end{array} \right\} \rightarrow D(\text{Sh}(R_X - \text{mod}^d)) \rightarrow D(\text{Sh}(Ab))$$

By construction, the dependence of it on $\mathcal{F}^r$ is functorial, so in fact we have a functor

$$K(\text{Sh}(R_X - \text{mod}^d)) \times D(\text{Sh}(R_X - \text{mod}^d)) \rightarrow D(\text{Sh}(Ab))$$

The previous lemma implies that the latter functor in fact factors through a functor

$$D(\text{Sh}(R_X - \text{mod}^d)) \rightarrow D(\text{Sh}(Ab))$$

Moreover, this construction is symmetric: if we were to fix an object $\mathcal{F}^l \in K(\text{Sh}(R_X - \text{mod}^d))$ and derive with respect to $\mathcal{F}^r$, we’d get the same result. Indeed, both are isomorphic to $\mathcal{F}^r_{\text{ad}} \otimes \mathcal{F}^l_{\text{ad}}$, where $\mathcal{F}^r_{\text{ad}}, \mathcal{F}^l_{\text{ad}}$ are adapted replacements of $\mathcal{F}^r$ and $\mathcal{F}^l$ respectively.

Let now $R_X'$ be another sheaf of rings and $R_X \to R_X'$ a homomorphism. We obtain a functor

$$R_X' \otimes \left. \begin{array}{c} L \vspace{1pt} \\
R_X \end{array} \right\} \rightarrow D(\text{Sh}(R_X - \text{mod}))$$

Proposition 1. The above functor is the left adjoint to the forgetful functor

$$D(\text{Sh}(R_X' - \text{mod})) \rightarrow D(\text{Sh}(R_X - \text{mod}))$$

Proof: We need to show that if $\mathcal{F}$ is an adapted complex of sheaves of $R_X$-modules, then

$$\text{Hom}_{D(\text{Sh}(R_X - \text{mod}))}(R_X' \otimes \mathcal{F}, \mathcal{F}') \cong \text{Hom}_{D(\text{Sh}(R_X - \text{mod}))}(\mathcal{F}, \mathcal{F}')$$

for any $\mathcal{F}' \in D(\text{Sh}(R_X' - \text{mod}))$. Note that $R_X' \otimes \mathcal{F}$ is automatically an adapted complex of sheaves of $R_X'$-modules.

By problem 4 on PS 3, we reduce the verification to the case when $\mathcal{F}$ is replaced by $j_!(R_X | U)$ for some $U$, and therefore $R_X' \otimes \mathcal{F} \cong j_!(R_X' | U)$. In this case, the LHS computes the cohomology of $\mathcal{F}' | U$ as a complex of sheaves of $R_X$-modules and the RHS the cohomology of $\mathcal{F}' | U$ as a complex of sheaves of $R_X$-modules, but we know that both are isomorphic to the cohomology of $\mathcal{F}'$ as a complex of sheaves of abelian groups. $\square$
Let $\Phi: X \to Y$ be a map between ringed topological spaces. We wish to define a functor $L\Phi^*: D(Sh(R_Y - \text{mod})) \to D(Sh(R_X - \text{mod}))$

We'll first define it as a left derived functor of $\Phi^*: D(Sh(R_Y - \text{mod})) \to D(Sh(R_X - \text{mod}))$ but we have to show that the latter exists. We'll use lemmas 1 and 2 with respect to adapted complexes of sheaves of $R_Y$-modules. Note that

$$\Phi^*(\mathcal{F}) \cong R_X \otimes_{\Phi^*(R_Y)} \Phi^*(\mathcal{F})$$

while $\Phi^*$ is exact. Hence the conditions of lemma 2 are satisfied by the same argument as above. This gives rise to the existence (and construction) of $L\Phi^*$. Moreover, the above shows that $L\Phi^*$ is canonically isomorphic to the composition

$$D(Sh(R_Y - \text{mod})) \xrightarrow{\Phi^*} D(Sh(\Phi^*(R_Y) - \text{mod})) \xrightarrow{LTor^\bullet(\Phi^*(R_Y), R_X \cdot \cdot \cdot)} D(Sh(R_X - \text{mod}))$$

Now we wish to show that the functor $L\Phi^*$ constructed above is the left adjoint to the functor

$$R\Phi_*: D(Sh(R_X - \text{mod})) \to D(Sh(R_Y - \text{mod}))$$

This follows from (1) and Proposition 1, since $R\Phi_*$ is itself the composition

$$D(Sh(R_X - \text{mod})) \to D(Sh(\Phi^*(R_Y) - \text{mod})) \xrightarrow{R\Phi_*} D(Sh(R_Y - \text{mod}))$$

-recall that $R\Phi_*$ is right adjoint to $\Phi^*$ as proven previously.

Finally, we need to discuss the projection formula in its full generality.

Let $X, Y$ be as above. Let $\mathcal{F}_X \in D(Sh(R_X - \text{mod}^*))$ and $\mathcal{F}_Y \in D(Sh(R_X - \text{mod}^*))$. Consider the following two objects in the category $D(Sh_X(\mathbb{A}^1))$:

$$R\Phi_*(\mathcal{F}_X) \xrightarrow{L} \mathcal{F}_Y$$

and

$$R\Phi_*(\mathcal{F}_X \otimes_{R_X} L\Phi^*(\mathcal{F}_Y))$$

There exists a natural map $\to$. Indeed, by the $(\Phi^*, R\Phi_*)$ adjunction, to construct such a map, it suffices to construct a map $\Phi^*(R\Phi_*(\mathcal{F}_X) \otimes_{R_X} L\Phi^*(\mathcal{F}_Y)) \to \mathcal{F}_X \otimes_{R_X} L\Phi^*(\mathcal{F}_Y)$.

It is easy to see that $\Phi^*(j_!(O_U)) = j_!(O_{\Phi^{-1}(U)})$ (the $j$ are different). Using also exactness of $\Phi^*$, it follows that the LHS is isomorphic to

$$\Phi^*(R\Phi_*(\mathcal{F}_X)) \otimes_{\Phi^*(R_X)} \Phi^*(\mathcal{F}_Y)$$

which naturally maps to the RHS.

However, this morphism doesn't have to be an isomorphism. Indeed, let $\Phi$ be an open embedding $j: U \to Y$, $R_X = R_Y |_U$, $\mathcal{F}_X = R_X$, $\mathcal{F}_Y = j(R_Y |_U)$ for the same $U$. Then

$$R\Phi_*(\mathcal{F}_X) \otimes_{R_X} \mathcal{F}_Y \cong j(R_Y |_U)$$

while

$$R\Phi_*(\mathcal{F}_X \otimes_{R_X} L\Phi^*(\mathcal{F}_Y)) \cong R\Phi_*(R_Y |_U) = Rj_*(R_Y |_U)$$

In a particular case, though, it is:

**Proposition 2. (Projection Formula)** Let $X, Y$ be Noetherian schemes, with $R_X, R_Y$ their respective structure sheaves, and $\Phi$ a morphism. Assume that $\mathcal{F}_Y \in D_{QCoh}(Sh(O_Y - \text{mod}))$. Then the map constructed above

$$\Phi^*(R\Phi_*(\mathcal{F}_X) \otimes_{R_X} F_Y) \to \mathcal{F}_X \otimes_{R_X} L\Phi^*(\mathcal{F}_Y)$$

is an isomorphism.
Proof: The question is local so we can assume that $Y$ is affine. The functor $\Phi^*$ and $\text{L}Tor$ by construction, commute with the formation of direct sums. By proposition III.2.9 from Hartshorne, the same is true for $R\Phi_*$. This reduces the assertion to the case when $\mathcal{F}_Y$ is a single quasi-coherent sheaf (WHY??). Moreover, $Y$ being affine, we can replace $\mathcal{F}_Y$ by a free resolution $\ldots \rightarrow P^n \rightarrow \ldots \rightarrow P^1 \rightarrow P^0$.

Again, since all functors commute with direct sums, since the free resolution is isomorphic to the holimit of its truncations, it suffices to prove the assertion for each of the truncated complexes $P^n \rightarrow \ldots \rightarrow P^1 \rightarrow P^0$ which, in turn, reduces the assertion to the case when $F_Y \cong O_Y$. In this case, the two sides we’re comparing agree immediately, and it’s straightforward to check that the constructed map is indeed the natural identity. □

Let $\Phi: X \rightarrow Y$ be a morphism between Noetherian schemes. Let $\Psi: Y' \rightarrow Y$ be another morphism. Let $X' = X \times_Y Y'$ be the Cartesian product. We’ll assume that both $Y'$ and $X'$ are Noetherian.

Let $\mathcal{F}$ be an object of $D_{\text{QCoh}}(\text{Sh}(O_X - \text{mod}))$. Consider the following two objects in $D_{\text{QCoh}}(\text{Sh}(O_{Y'} - \text{mod}))$:

$L\Psi^* R\Phi_*(\mathcal{F})$ and $R\Phi'_* L\tilde{\Psi}^*(\mathcal{F})$

By adjunction, there is a canonical map between them.

**Proposition 3.** Assume that $X$ is flat over $Y$. Then the map

$L\Psi^* R\Phi_*(\mathcal{F}) \rightarrow R\Phi'_* L\tilde{\Psi}^*(\mathcal{F})$

is an isomorphism.

Proof: The question is local, so we can assume both $Y$ and $Y'$ are affine. In this case map in $D_{\text{QCoh}}(\text{Sh}(O_{Y'} - \text{mod}))$ is an isomorphism if and only if it becomes an isomorphism after applying the functor $R\Psi_*$. Hence, it is enough to show that

$R\Psi_* L\Psi^* R\Phi_*(\mathcal{F}) \rightarrow R\Psi_* R\Phi'_* L\tilde{\Psi}^*(\mathcal{F})$

is an isomorphism. The RHS in (*) is the same as

$R\Phi_* R\tilde{\Psi}_* L\tilde{\Psi}^*(\mathcal{F})$

By the projection formula for $\tilde{\Psi}$ we have

$R\tilde{\Psi}_* L\tilde{\Psi}^*(\mathcal{F}) \cong \mathcal{F} \otimes_{\tilde{\Psi}_*(O_{Y'})} \mathcal{F}$

By the projection formula for $\Psi$, the LHS in (2) identifies with

$R\Phi_*(\mathcal{F}) \otimes_{\Psi_*(O_{Y'})} \mathcal{F}$

which by the projection formula for $\Phi$ is isomorphic to

$R\Phi_*(\mathcal{F} \otimes_{\Phi_*(O_{Y'})} \mathcal{F})$
Hence to prove the desired isomorphism, it suffices to show that the natural arrow $L\Phi^*\Psi_s(\mathcal{O}_Y) \to \Psi_s(\mathcal{O}_Y)$ is an isomorphism, assuming that $X$ is flat over $Y$. And indeed, $L\Phi^*\Psi_s(\mathcal{O}_Y)$ is given, locally on $X$, as $\text{LTor}^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y)$, with the flatness guaranteeing that all cohomologies but zero vanish.

03/30/2010.

Consider $X$ be a scheme of finite type over an algebraically closed field $k$.

**Definition.** $X$ is smooth if it’s regular.

We would like to generalize this definition for schemes over a general scheme $S$ instead of the spectrum of an algebraically closed field. For now, assume $S$ is Noetherian.

**Definition.** A scheme $X$ of finite type over a Noetherian scheme $S$ is said to be smooth if and only if $X$ is flat over $S$, and for any point $s \in S$ the scheme $X \times_S k_s$ is smooth over $k_s$ i.e. regular. (by $k_s$ we denote the algebraic closure of the residue field $k_s$).

As will be proven later, smoothness implies regularity. Here is an example of a scheme which is not smooth but is regular.

Consider $S = \text{Spec}(k)$ where $k$ is a field that is not algebraically closed, and $X = \text{Spec}(k')$ where $k'$ is a finite algebraic extension of $k$ which is not separable. We claim it’s not smooth.

**Claim.** In the above situation $\text{Spec}(k')/\text{Spec}(k)$ is smooth if and only if $k'/k$ is separable.

Proof: $k' \otimes_k \overline{k}$ is an Artinian algebra over $k$. It is regular if and only if it is a direct sum of fields - an Artinian Noetherian ring has finitely many prime ideals all of which are maximal because their residue rings are domains that are finite over $k$ i.e. fields, and from there it’s easy to deduce that such an Artinian algebra is regular if an only if it contains no nilpotents. If $k'$ over $k$ is not separable, this is not the case: take an element $t$ that is not separable over $k$ i.e. satisfies a minimal polynomial of form $g(x^p), p = \text{char}(k)$, then the intermediate algebra $k(t) \otimes_k \overline{k}$ equals $\overline{k}[x]/g(x^p)$ which has nilpotents since $g(x^p)$ splits as a $p$-th power in $\overline{k}$.

Conversely, if $k'$ is separable, it is generated by a single element whose minimal polynomial if $g$ that has distinct roots in $\overline{k}$ this $k' \otimes_k \overline{k}$ is $\overline{k}[x]/g(x)$ which splits as a direct sum of $\overline{k}$ as $g(x)$ splits as a product of distinct linear factors.

Alternatively, if $k'$ is separable over $k$, we have a canonical identification of $k' \otimes_k \overline{k}$ with $\oplus \overline{k}$ where the direct sum is taken over all isomorphisms $\sigma: k' \xrightarrow{\sim} k'$ inside $\overline{k}$ that fix $k$. This identification is obtained as follows: both sides clearly have dimension $m = [k': k]$ over $\overline{k}$, and we have a map from $k' \otimes_k \overline{k}$ into $\oplus \overline{k}$ that sends $a \to b$ to $(\sigma(a) \cdot b)_{\sigma}$. This map is an isomorphism: if $b_1, \ldots, b_m$ are a basis of $k'$ over $k$ and $\sigma_1, \ldots, \sigma_m$ are all the automorphisms of $k'$ inside $\overline{k}$ fixing $k$, then via the basis $(b_i \otimes 1)$ of $k' \otimes_k \overline{k}$ this linear transformation is given by the matrix $A = (\sigma_i(b_j))$. This matrix is invertible, because $AAT$ is the matrix $Tr_{k'/k}(b_ib_j)$ which is non-singular according to one of the alternate definitions of separability. □

**Kahler differentials**

If $A \xrightarrow{\phi} B$ is a ring homomorphism we define the $B$-module $\Omega_{B/A}$ of Kahler differentials spanned by formal expressions $db, b \in B$ modulo he relations $d\phi(a) = 0, a \in A, d(b_1 + b_2) = db_1 + db_2$ and $d(b_1b_2) = b_1db_2 + b_2db_1$.

This module satisfies the universal property

$$\text{Hom}_B(\Omega_{B/A}, M) \cong \text{der}_A(B, M)$$

where $\text{der}_A$ denotes the set of differentials i.e. maps $f: B \to M$ with the properties $f(b_1 + b_2) = f(b_1) + f(b_2), f(b_1b_2) = b_1f(b_2) + b_2f(b_1), f(\phi(a)b) = \phi(a)f(b)$. 

104
The module of Kahler differentials can also be given as follows: choose the diagonal embedding \( Spec(B) \times_{Spec(A)} Spec(B) \) and the closed diagonal embedding \( Spec(B) \to Spec(B) \times_{Spec(A)} Spec(B) \) that is given by the ideal \( I \), i.e. \( I \) is the kernel of the multiplication map \( B \otimes B \to B \). Then

\[
\frac{I}{I^2} \cong I \otimes_{B \otimes B} B \cong \Omega_{B/A}
\]

The identification of \( \frac{I}{I^2} \) with \( \Omega_{B/A} \) is given by the map \( db \to b \otimes 1 - 1 \otimes b \) (modulo \( I^2 \)). One immediately notes that this map factors through the relations defining \( \Omega_{B/A} \): the essential part is that \( (b_1b_2 \otimes 1 - 1 \otimes b_1b_2) - b_2(b_1 \otimes 1 - 1 \otimes b_1) \in I^2 \) - it can be computed as \( -1 \otimes b_1b_2 + b_2 \otimes b_2 - b_2 \otimes b_1 + b_1 \otimes b_1 = (b_1 \otimes 1 - 1 \otimes b_1)(b_2 \otimes 1 - 1 \otimes b_2) \).

(Note: \( B \) can act on \( I \) in two ways, by multiplying the left side or the right side, but both are equal modulo \( I^2 \) since \( (x \otimes a - xa \otimes 1) - (1 \otimes ax - a \otimes x) = (1 \otimes x - x \otimes 1)(a \otimes 1 - 1 \otimes a) \). This map is surjective: if \( \sum b_i b'_i = 0 \) then \( \sum b_i \otimes b'_i = \sum (b_i \otimes b'_i - b_i \otimes 1) = \sum b_i(1 \otimes b'_i - b'_i \otimes 1) \) which is in the image of the map. It will also be injective as we can construct a right inverse that sends \( \sum b_i \otimes b'_i \) to \( \sum b_i d(b'_i) \). This factors through \( I^1 \) since \( (b_1 \otimes 1 - 1 \otimes b_1)(b_2 \otimes 1 - 1 \otimes b_2) \) gets sent to \( b_1 b_2 d(1) - b_2 d(b_1) - b_1 d(b_2) + 1d(b_1 b_2) \) which is 0.

Assume now we have the following commutative diagram

\[
\begin{array}{ccc}
Spec(C) & \xrightarrow{f} & Spec(B) \\
& \searrow & \\
& Spec(A) &
\end{array}
\]

Then there exists a canonical exact sequence

\[
f^* \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0
\]

- here \( f^* \Omega_{B/A} \) is \( \Omega_{B/A} \otimes_{B} C \)

Now assuming that \( f = i \) is a closed embedding, given by an ideal \( I \) of \( B \), we have a short exact sequence

\[
I/I^2 \to i^* \Omega_{B/A} \to \Omega_{C/A} \to 0
\]

Kahler differentials commute with localization, as is easy to see. Therefore we can define the sheaf of Kahler differentials for a general map \( X \to S \): the sheaf \( \Omega_{X/S} \) is given locally, \( Spec(B) \subset X \) maps inside \( Spec(A) \subset S \) by \( loc_B \Omega_{B/A} \) - note that it does not depend on what \( Spec(A) \) we choose, and then by gluing/sheafifying.

Also, Kahler differentials commute with base change: namely if we have a cartesian diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{f} & S
\end{array}
\]

then

\[
\Omega_{X'/S'} \cong f^*(\Omega_{X/S})
\]

That’s easiest to see using the definition using \( I/I^2 \), but can also be proven directly by constructing maps in both directions.

Smoothness can be expressed in terms of properties of Kahler differentials.

**Definition.** Let \( X \) be a scheme over a base scheme \( S \). \( X \) is said to be smooth of relative dimension \( n \) over \( S \) if it is smooth over \( S \), and all geometric fibers \( X \times_S \overline{k}_s \) are varieties of dimension \( n \) (i.e. all their irreducible components have dimension \( n \)).
In what follows, we will usually implicitly assume that a smooth scheme is smooth of some relative dimension. Note that the dimension of fibers is actually a semi-continuous function, in general.

**Theorem.** Let $X \to S$ be a flat morphism of finite type. Then the following are equivalent:

i) $X$ is smooth of relative dimension $n$.

ii) $\Omega_{X/S}$ is locally free of rank $n$, and the dimension of every irreducible component of the fibers $X \times_S \overline{k}_x$ is $\geq n$.

Proof: ii) $\to$ i) By base changing with $\overline{k}_x$, since Kahler differentials commute with base change, we can assume $S = Spec(k)$ where $k$ is an algebraically closed field.

Regularity is equivalent to the condition: for any $x \in X$ closed, $dim_k m_x/m_x^2 = n$ and the local ring $\mathcal{O}_{X,x}$ having no nilpotents.

We claim that there is a natural map $m/m^2 \to \Omega_{X/x} \otimes_{\mathcal{O}_X} k_x$.

This map is an isomorphism for $k$ algebraically closed, but necessarily so otherwise. More precisely if $k_x = k$ (which happens if $k$ is algebraically closed) then it is an isomorphism.

We proceed by a Yoneda argument (although of course this can be written out explicitly):

We know $Hom_k(\Omega_{X/k} \otimes k, M) \cong Hom_{\mathcal{O}_X} (\Omega_{X/k}, M) \cong Der_k(\mathcal{O}_{X,x}, M)$ - where $M$ is acted upon by letting the maximal ideal act by zero.

We now claim $Der_k(\mathcal{O}_{X,x}, M) \cong Hom_k(m/m^2, M)$. The map $\to$ is given by restriction, and indeed it is an isomorphism: first it’s enough to restrict $\mathcal{O}_{X,x}$ to $m$ because $\mathcal{O}_{X,x}$ splits as $k \oplus m$ and $k$ must get sent to 0, and now $m^2$ will also get sent to zero by the Leibniz rule: $d(ab) = adb + bda$ but $a, b \in m$ act by 0 on $M$.

There is still a bug to be fixed, about the absence of nilpotents.

i) $\to$ ii) Since everything is local on $X$ and $S$, we can assume we can embed $X$ into $X' = k_S^n$ then $\Omega_{X'/S}$ is a locally free coherent sheaf of rank $n$: in general, it is easy to see that $\Omega_{A[x_1,\ldots,x_n]/A}$ is free on generators $dx_1, dx_2, \ldots, dx_n$.

We have a short exact sequence (locally) $0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ and recall the exact sequence $I/I^2 \to \Omega_{X'/S} \to \Omega_{X/S} \to 0$.

First, we need a lemma:

**Lemma.** Let $\mathcal{F} \to \mathcal{E}$ be a map of finitely generated modules over a Noetherian local ring. Assume $\mathcal{E}$ is (locally) free and the map of fibers $\mathcal{F}/m\mathcal{F} \to \mathcal{E}/m\mathcal{E}$ is injective. Then $\mathcal{F}$ is (locally) free, $\mathcal{F} \to \mathcal{E}$ is injective, and $\mathcal{E}/\mathcal{F}$ is (locally) free.

Proof of lemma: let $\mathcal{G}$ be the image of $\mathcal{F}$ inside $\mathcal{E}$ so that we have $\mathcal{F} \to \mathcal{G} \to \mathcal{E}$ and modulo $m$ $\mathcal{F}_x \cong \mathcal{G}_x \cong \mathcal{E}_x$.

The short exact sequence $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{E}/\mathcal{G} \to 0$ yields $Tor^1(k, \mathcal{E}) \to Tor^1(k, \mathcal{E}/\mathcal{G}) \to G_x \to E_x$. Because $\mathcal{E}$ is locally free and $G_x \to E_x$ is injective, we deduce that $Tor^1(k, \mathcal{E}/\mathcal{G}) = 0$ hence $\mathcal{E}/\mathcal{G}$ is locally free. It follows that $\mathcal{E}/\mathcal{G}$ splits as a direct summand of $\mathcal{E}$ so $\mathcal{G}$ is also (locally) free. We then can replace $\mathcal{E}$ by $\mathcal{G}$ i.e. we may assume the map $\mathcal{F} \to \mathcal{E}$ is surjective.

Take now $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{E} \to 0$ a short exact sequence, and because $\mathcal{E}$ is (locally) free, this sequence splits, so $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{F}'$. Now by tensoring with $k$, by get $\mathcal{F}'_x = 0$ hence by Nakayama $\mathcal{F}' = 0$. This finishes the proof of the lemma.

**Corollary.** If $\mathcal{F} \to \mathcal{E}$ is a map of coherent sheaves over a Noetherian sheaf $X$ such $\mathcal{E}$ is locally free and all fibers $\mathcal{F}_x \to \mathcal{E}_x$ (it suffices to consider only the closed points $x$ if $X$ is affine or of finite type over an integrally closed field) are injective, then $\mathcal{F} \to \mathcal{E}$ is injective, $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are also locally free.

Now we return to the theorem. We apply it to the map $I/I^2 \to \Omega_{X'/S}$. It suffices to show that $(I/I^2)_x \to (\Omega_{X'/S})_x$ is injective.

Because $X$ is flat over $S$, the short exact sequence $0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ can be tensored with $k_s$ to produce $0 \to I \otimes k_s \to \mathcal{O}_{X'} \otimes \mathcal{O}_S \to \mathcal{O}_{X} \otimes \mathcal{O}_S \to 0$. This reduces to the case $S = Spec(k)$, and we can assume that $k$ is algebraically closed because $\overline{k}$ is free over $k$.

106
Then in the short exact sequence $0 \to I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$ the last two rings are regular. It follows that the dimension of $I \otimes k$ (over $k$) equals the difference of the (Krull) dimensions i.e. $m - n$. But then by computing dimensions, we deduce that $(I/I^2)\otimes k \to m'/m^2 \to m/m^2 \to 0$ is actually short exact. In particular this means that $I/I^2 \to \Omega_{X'/k}$ is injective on the fiber, and this finishes the proof. □

**Corollary.** Let $X \hookrightarrow X'$ be a closed embedding of schemes over a base $S$, given by an ideal sheaf $\mathcal{I}$, and assume $X$ is flat over $S$. If $X'$ is smooth of relative dimension $m$ (over $S$), then the following are equivalent:

i) $X$ is smooth of relative dimension $n$

ii) $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $m - n$

iii) $\mathcal{I}/\mathcal{I}^2 \to \Omega_{X'/S}$ is an injective bundle map.

(Recall that an injective bundle map is a map of vector bundles that is injective on fibers - or equivalently, locally it is a split injection).

Proof: we have $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X'/S} \to \Omega_{X/S} \to 0$ a short exact sequence. If i) holds, then the last two elements are locally free - it follows that locally this short exact sequence splits and this proves iii) and the weaker ii). Conversely, assuming ii) it will follow that the short exact sequence above will again split locally, hence $\Omega_{X/S}$ will also be locally free of rank $n$. Finally because $X$ is flat, the dimension of fibers cannot decrease by more that the rank of $\mathcal{I}/\mathcal{I}^2$. □

Here are some properties of relative smoothness:

- stable under base-change: if $X' \to X$ is a cartesian diagram and $X$ is smooth over $S$ then $X'$ is smooth over $S'$ (provided that $S' \to S$ is of finite type).

Indeed, flatness is stable under base change, and the geometric fibers are the same: if we have $s'$ a point of $S'$ that maps to $s \in S$ then $k_{s'} \sim k_s$ and $X' \otimes_k k'_s \cong X \otimes_k k_s$.

- stable under products: if $X_1, X_2$ are smooth over $S$, then $X_1 \times_S X_2$ is smooth, too.

It follows from the fact that $\Omega_{X_1 \times_S X_2/S} \cong p_1^*(\Omega_{X_1/S}) \oplus p_2^*(\Omega_{X_2/S})$ hence it will also be free of appropriate rank - and the dimension clause follows from the fact that $A \otimes B$ has dimension at least $\dim(A) + \dim(B)$ using $m_{A\otimes B} = m_A \otimes B + A \otimes m_B$.

For proving the identity $\Omega_{X_1 \times_S X_2/S} \cong p_1^*(\Omega_{X_1/S}) \oplus p_2^*(\Omega_{X_2/S})$ it is enough to work over affines. Then it is easy to see that the natural map $p_1^*(\Omega_{A/C}) \oplus p_2^*(\Omega_{B/C}) \to \Omega_{A\otimes B/C}$ is surjective, as $d(a \otimes b) = (1 \otimes b)d(a) + (a \otimes 1)d(1 \otimes b)$.

- transitive If $X$ is smooth over $Y$, $Y$ is smooth over $S$, then $X$ is smooth over $S$.

Proof: flatness is transitive, for the rest it suffices to base change so we can assume $S$ is Spec$(k)$ where $k$ is algebraically closed.

For any closed point $y \in Y$, $\dim(X) = \dim(Y) + \dim(X_y)$ where $X_y$ is the fiber. This follows from a previous proposition as $X$ is flat over $Y$. This implies the dimension clause, as we can work with every irreducible component of $X$.

It remains to show that $\Omega_{X/k}$ is locally free of correct rank (sum of relative dimensions).

Recall the exact sequence $f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$.

Now $\Omega_{X/Y}$ is locally free so it splits locally as a direct summand of $\Omega_{X/k}$

On fibers, we have the same exact sequence $(f^*\Omega_{Y/k})_x \to (\Omega_{X/k})_x \to (\Omega_{X/Y})_x \to 0$.

We obtain $\dim((\Omega_{X/k})_x) \leq \dim((\Omega_{X/Y})_x) + \dim((\Omega_{Y/k})_{f(x)})$ - and the LHS equals $\dim m_x/m_x^2$ whereas the RHS equals $\dim(X)$. Since in general, the reverse inequality takes place (why?) this si actually an equality hence $X$ is regular thus smooth. □
Proposition. Let $X \xrightarrow{f} Y$ be a morphism of smooth schemes over $S$. Then the following are equivalent:

i) $f$ is smooth of relative dimension $n = \text{rel.dim}(X) - \text{rel.dim}(Y)$

ii) $\Omega_{X/Y}$ is locally free of rank $n$

iii) $f^*\Omega_{Y/S} \to \Omega_{X/S}$ is an injective bundle map.

Proof: i) $\implies$ ii). On fibers, we can assume $S$ is the spectrum of an algebraically closed field. The exact sequence $f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$ means that the free vector bundle $f^*\Omega_{Y/S}$ surjects onto $\text{Ker}(\Omega_{X/S} \to \Omega_{X/Y})$ which is also a vector bundle because it is the kernel of a surjection of vector bundles (we’ve seen this before). Moreover these two vector bundles have the same dimension, therefore they must be equal (the kernel is 0 by Nakayama). This implies iii)

iii) $\implies$ i) We need to show that $f$ is flat, the dimension clause following then automatically. First we do it for $S = \text{Spec}(k)$, $k$ algebraically closed.

Recall the definition of tangent space - at a point $x$ of a scheme $X$, the tangent space $T_xX$ is $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. It’s easy to see that a map of schemes $f: X \to Y$ induces a differential map on tangent spaces $T_xX \xrightarrow{df} T_{f(x)}Y$. If $X$ is finite over an algebraically closed field smooth $\text{Spec}(k)$, to $k$ is the residue field at every point, then $T_xX$ is actually the $k$-vector space dual of $(\Omega_{X/k})_x$.

Recall that to show that a map is flat, as $X$ and $Y$ are both regular, it suffices to show that for any $y \in Y$, $\text{dim}(X_y) = \text{dim}(X) - \text{dim}(Y)$. A priori we know $\text{dim}(X_y) \geq \text{dim}(X) - \text{dim}(Y)$

But $\text{dim}(X_y) \leq \text{dim}_k((\Omega_{X/Y})_y) = \text{dim}(X) - \text{dim}(Y)$ where the last equality follows from the short exact sequence $0 \to f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0$

The case of general $S$ will be dealt with later. □

Theorem. Over a field of characteristic zero (in fact, over any perfect field), every integral scheme $X$ (of finite type) is generically smooth, i.e. has an open subscheme $\tilde{X}$ which is smooth.

Proof: flatness is automatic over a field. Let $n = \text{dim}(X)$, which is also the transcendence degree of the fraction field of $X$. We want $\Omega_{X/k}$ to be locally free of rank $n$ over $X$. Any coherent sheaf is locally free over some open, because it is free at the generic point and we need to choose a small enough open where a chose basis at the generic point lifts. We only need the right rank, i.e. we want $\Omega_{X/k} \otimes O_X^* = \Omega_{K(X)/k}$ to have rank $n$. This will follow immediately from the following lemma:

Lemma. Let $K/k$ be a separably generated extension of transcendency degree $n$ - i.e. a separable algebraic extension of $k(x_1, \ldots, x_n)$. Then $\Omega_{K/k}$ has rank $n$ over $K$.

Proof of lemma: Let $K' = k(x_1, \ldots, x_n)$ be the intermediate field extension, then $\Omega_{K'/k}$ has rank $n$ over $K'$. We have the exact sequence $\Omega_{K'/k} \otimes K' \to \Omega_{K/k} \to \Omega_{K'/K} \to 0$. Now $\Omega_{K'/K}$ is zero because the extension is separable. We have done it indirectly before, as $\text{Spec}(K')$ was shown to be smooth over $\text{Spec}(K)$ in this case, and the relative dimension must be because the geometric fiber clearly has dimension 0. But this can be done directly in fact, and will be shown later.

We deduce that $\Omega_{K'/k} \otimes K \to \Omega_{K/k}$ hence the rank is at most $n$. But this map must also be injective. Again, this was proven indirectly as $\text{Spec}(K)$ is clearly smooth over $\text{Spec}(K')$ and we apply part iii) of the previous proposition. □

4/01/2010

Definition. A scheme morphism $f: X \to Y$ is faithfully flat if and only it is flat and is surjective on the level of points.

Theorem. A morphism $f: X \to Y$ is faithfully flat if and only if $f^*$ is exact and conservative - where conservative means $\mathcal{F} = 0 \implies f^*\mathcal{F} = 0$
This was proved in the first semester.

**Lemma.** Assume \( f : X \to Y \) is a faithfully flat morphism of Noetherian schemes. If \( F \) is a coherent sheaf of \( Y \) such that \( f^*(F) \) is locally free then \( F \) is locally free.

**Proof:** Note that if \( f \) is faithfully flat, then a sequence \( F_1 \to F_2 \to F_3 \) on \( Sh(Y) \) is exact if and only if \( f^*(F_1) \to f^*(F_2) \to f^*(F_3) \) is exact. Indeed, one direction follows by \( f^* \) being exact. The other follows by it being conservative - if \( F_1 \to F_2 \) has non-zero kernel then so does \( f^*(F_1) \to f^*(F_2) \), so \( F_1 \to F_2 \) is injective. Similarly \( F_2 \to F_3 \) is surjective, and the composite map \( F_1 \to F_3 \) is zero as it having non-zero image would mean \( f^*(F_1) \to f^*(F_3) \) would have non-zero image. So \( \text{Im}(F_1 \to F_2) \subseteq \text{Ker}(F_2 \to F_3) \) but this map is surjective as its cokernel dies under \( f^* \).

Because being locally free for coherent sheaves is equivalent to being flat, we want the functor \( \mathcal{G} \to \mathcal{G} \otimes F \) to be exact. Because of what was just said, this is equivalent to the functor \( \mathcal{G} \to f^*(\mathcal{G} \otimes O_Y) \to F \) being exact, but that rewrites as \( \mathcal{G} \to f^*(\mathcal{G} \otimes O_X) \) being exact, which follows by \( f^* \) being exact and \( f^*F \) being locally free. \( \square \)

**Corollary.** If \( f \) is faithfully flat, then a map of coherent sheaves \( F_1 \to F_2 \) is an injective bundle map if and only if so is \( f^*F_1 \to f^*F_2 \).

**Proof:** the equivalence of injectivity was proven in the course of the proof of the previous lemma. Assuming injectivity, let \( F_3 \) be the cokernel, then the map is an injective bundle map if and only if \( F_3 \) is locally free, which is equivalent to \( f^*F_3 \) being locally free by the previous lemma. \( \square \)

**Theorem.** Let \( A, B \) be local Noetherian rings and consider a faithfully local map between them (i.e. the map \( \text{Spec}(B) \to \text{Spec}(A) \) is faithfully flat). Then \( B \) regular implies \( A \) regular.

**Proof:** as shown before, in the case of local rings, regularity (which is the same as having finite cohomological dimension by Serre’s theorem) is equivalent to \( \text{Tor}_i(k_A, M) \) living in finitely many cohomological degrees, for all \( M \).

The map \( k_A \to k_B \) is injective since the map is local, and therefore \( k_B \) is free over \( k_A \), which implies \( k_B \) equals a direct sum of copies of \( k_A \) as an \( A \)-module too. Thus it’s enough to show \( \text{Tor}_i^A(k_B, M) \) lives in finitely many cohomological degrees.

The ”baby version” of projection formula yields \( \text{Tor}_i^A(N, M) \cong \text{Tor}_i^B(N, B \otimes M) \) for \( N \in B - \text{mod}, A \in A - \text{mod} \).

Indeed, both sides compute the cohomologies of \( R\Phi_*(N) \otimes_A B \) and \( R\Phi_*(N \otimes_A B \otimes Lf^*(M)) \) - note that in this case \( \Phi_* \) is automatically exact and \( R\Phi_* \) is just the functor that views \( B \)-modules as \( A \)-module, while \( Lf^* = f^* = - \otimes_A B \) because \( B \) is flat over \( A \) by assumption.

Alternatively, this identity can be proven by simpler means - to compute the RHS we take a free resolution of \( M \) which becomes a free resolution of \( B \otimes M \) by flatness, so it can be used to compute the RHS too and gives the same result.

In particular, taking \( N = k_B \) we deduce that \( \text{Tor}_i^A(k_B, M) \cong \text{Tor}_i^B(k_B, B \otimes M) \) and the conclusion follows from \( B \) being of finite cohomological dimension. \( \square \)

**Remark:** in local rings, faithfully flat implies flat and local (surjectivity of map on points implies the map being local). The converse is not true, for example in the case of the map \( k \to k[\epsilon]/\epsilon^2 \).

**Lemma.** A faithfully flat map of rings sends zero-divisors to zero-divisors non-zero divisors to non-zero divisors. In particular, if \( A \to B \) is faithfully flat and \( B \) is a domain, then so is \( A \).

**Proof:** \( a \in A \) is a zero-divisor if and only if \( A \twoheadrightarrow A \) is injective, and under \( f^* \) this maps transforms to \( B \to B \) which immediately implies \( a \) is a zero-divisor if and only if \( f(a) \) is. In particular, if \( B \) is a domain then any element in \( A \) that does not get mapped to 0 will get sent to a non-zero divisor. It remains to verify that no \( a \neq 0 \) gets sent
to 0. Indeed, assume it does, then there is a short exact sequence $0 \to I \to A \xrightarrow{a} aA \to 0$ with $aA \neq 0$ which after applying $f^*$ transforms to $0 \to I \otimes B \to B \xrightarrow{f(a)} aA \otimes B \to 0$ and the map $\frac{f(a)}{A}$ is zero hence $aA \otimes B$ must be zero but it is not zero since $f^*$ is conservative. □

**Proposition.** Assume $A \to B$ is faithfully flat, and that $A$ is regular in codimension 1. If $B$ is normal, then so is $A$.

**Proof:** $A$ is a domain by the previous lemma. R1 also follows by regularity. For $S2$ we consider $M$ of support of codimension $\geq 2$. Note that a flat map sends a point to a point of $\leq$ codimension, because for flat maps the going down theorem holds. It follows that the codimension of the support of $f^*(M)$ is $\geq 2$. Then we have $\text{Ext}^2(M, A) \otimes B \cong \text{Ext}^2_B(B \otimes M, B)$ - this is easily proved by choosing projective resolutions, and since the morphism is faithfully flat, the right-hand side is zero so the left-hand side is zero. □

Let's return to properties of smoothness.

**Theorem.** If $X$ is of finite type over $S$. The following are equivalent:

i) $X$ is smooth of relative dimension $n$

ii) $X$ is flat, for any $s \in S$, $X \times S \xrightarrow{k_s} k_s$ has dimension $n$ (meaning all irreducible components have dimension $n$), and $\Omega_{X/S}$ is locally free of rank $n$.

iii) Locally, for every/some closed embedding $X \hookrightarrow A^n_X = X'$ of schemes over $S$ we have

$$0 \to I/I^2 \to \Omega_{X'/S} \otimes \mathcal{O}_X \to \Omega_{X/S} \to 0$$

is a short exact sequence of locally free sheaves.

**Corollary.** If $S$ is regular, and $X$ is smooth over $S$ (of relative dimension $m$), then $X$ is regular.

**Proof:** this is local, we can assume $X \subseteq A^n_X = X'$ s.t. that the ideal $I$ of $X$ in $X'$ can be generated by $f_1, \ldots, f_{m-n}$ such that for any $x \in X$, $\overline{f_i} \in m'_x/m'^2$ are linearly independent - this follows from part iii) of the previous theorem.

So now we have a general fact, if we have $A = A'/I$ where $A'$ is local regular of dimension $n$, $I$ is generated by $m-n$ elements that are linearly independent in $m'/m'$ and $\text{dim}(A) = m$, then $A$ is regular. This fact is known in commutative algebra (ref. Sherry's notes) □

Now say that $X$ is a scheme of finite type over $k$.

**Corollary.** If $X$ is smooth, then it is regular.

Now suppose $X$ of finite type over $k$ and regular of dimension $n$. Locally, $X \subseteq A^n_k = X'$. Locally the ideal $I$ of $X$ can be generated by $m-n$ functions $f_1, \ldots, f_{m-n}$. Indeed, that’s enough to show on stalks so we have to show that of $0 \to I \to A \to A' \to 0$ where $A, A'$ are regular local rings of dimension $m, n$ then $I$ can be generated by $m-n$ sections.

Indeed, let’s build the sequence $0 \to (I + m^2)/m^2 \to m/m^2 \to m'/m'^2 \to 0$ thus the dimension over $k$ (the quotient field of $A$ and $A'$) of $(I + m^2)/m^2$ is $m-n$, but $(I + m^2)/m^2$ is $I/(I \cap m^2)$. Clearly $I \cap m^2$ hence a set of $m-n$ of generators of $I/(I \cap m^2)$ lifts to a set of generators of $I$ by Nakayama.

Moreover, this set of generators projects in $m_A/m_A^2$ to an independent set, as easily seen from the above short exact sequence. Note: smoothness is equivalent to it projecting to an independent set in $\Omega_{A/k}$ (using the line bundle injection definition). This is stronger than regularity because the map $m_A/m_A^2 \to \Omega_{A/k}$ need not be an isomorphism.

**Lemma.** (PSET) Let $A$ be a local ring with a nilpotent maximal ideal, which is also an algebra over a field $k$. Assume that $k_A/k$ is finite and separable. Then the structure map $k \to A$ extends uniquely to a map $k_A \to A$, which is the right inverse to the canonical projection.
Proof: Let \( k_A = k(t) \) where \( t \) satisfies a minimal equation \( f \) over \( k \) - such a \( t \) always exists. Regarding \( k \) as a subring of \( A \), the statement asks us to show that there is a unique \( u \in A \) such that \( f(u) = 0 \) and \( u \mod m_A = t \). This is shown by a Hensel's lemma argument.

Let's show this assertion for \( A \) replaced by \( A/m_A^p \) by induction on \( p \) - and \( m_A^p = 0 \) so we will be done eventually. Base \( p = 1 \) is clear. Now assume that we have found \( v \in A/m_A^p \) with this property and let's choose \( u \in A/m_A^{p+1} \). Let's choose a lift \( u_1 \) of \( v \) to \( A/m_A^{p+1} \) so that \( f(u_1) \in m_A^p \). Observe that \( f(u_1 + x) = f(u_1) + xf'(u_1) + x^2E \) where \( E \) is an expression in \( A \). Let's choose \( x = -\frac{f(u_1)}{f'(u_1)} \in m_A^p \) - this can be done because \( f'(u_1) \) is invertible as it projects to the non-zero element \( f'(t) \in k_A \). We conclude that \( f(u_1 + x) \cong 0 \mod x^2A \) but \( x^2 \in m_A^{2p} \subset m_A^{p+1} \) hence \( u = u_1 + t \) satisfies the condition. For the uniqueness, observe that if \( f(u') \in m_A^{p+1} \) then \( u' \cong u \mod m_A^{p+1} \) by the induction hypothesis and \( f(u') = f(u) + (u - u')f'(u) + (u - u')^2E \) hence again we conclude \( u - u' \in m_A^{p+1} \) which means the two elements are the same in \( A/m_A^{p+1} \). □

**Lemma.** Let \( A \) be a \( k \)-algebra, \( m \in \text{Spec}(A) \) a maximal ideal, such that \( k_m \) is a separable extension of \( k \). In this case the natural map

\[
\frac{m/m^2}{A} \to \frac{\Omega_{A/k}}{\Lambda k}
\]

is an isomorphism.

Proof: The map \( \phi \) is given by \( x \in m \) mapping to \( dx \otimes 1 \). Note that \( da(b) \otimes 1 = da \otimes b + db \otimes a = 0 \) as \( a, b \) project to 0 in \( k_A \) for \( a, b \in m \) hence the map factors through \( m^2 \).

Let's prove the map is surjective. Recall the embedding \( \psi: k_A \hookrightarrow A \) constructed in the previous lemma. Let's prove \( \phi \circ \psi = 0 \). Indeed, choose \( t \in k_A \) and let it satisfy a minimal polynomial \( p \) over \( k \) with \( p'(t) \neq 0 \) as the extension is separable. Then \( p'(t) \) is invertible in \( A \), and then \( dp(t) \otimes 1 \) can be computed to be \( p'(t)dt \otimes 1 \) and since \( p'(t) \) is invertible in \( k \) so in \( k_A \) we conclude \( \psi(t) = 0 \).

The module \( \Omega_{A/k} \otimes_A k_A \) is generated as a \( k_A \)-vector space by the elements \( dx \otimes 1, x \in A \). Now if we let \( y = \psi(x (mod m)) \) we get \( x - y \in m \) and \( dy = 0 \) hence \( dx \otimes 1 = d(x - y) \otimes 1 = \phi(x - y) \) so the map is surjective.

For injectivity, we construct a right inverse map. We want a map in \( \text{Hom}_k(\Omega_{A/k} \otimes_A k_A, m/m^2) = \text{Hom}_A(\Omega_{A/k}^{1}, m/m^2) = \text{Der}_A(m/m^2) \). This derivation is given by \( x \to (x - \psi(x (mod m))) \) \( (mod m^2) \). It is immediate that it is a derivation: if \( \psi(x (mod m)) = x , \psi(y (mod m)) = y \) then \( xy \) gets sent to \( xy - xy = 0 \) which modulo \( m^2 \) is the same as \( x(y - y) + y(x - x) \) and the difference between them is \( 0 + 0 \) \( \in m^2 \). [Remark: the map \( \psi \) is defined only if \( m \) is nilpotent, but as we know from the previous lemma it can be defined into \( A/m^2 \) and that’s enough for our purposes]

It remain to see that the composite of these maps is the identity. Using the way the universal property was defined, we see that the map sends \( dx \otimes 1 \) to \( x - \psi \) \( (mod m^2) \). In particular if \( x \in m \) then \( \psi = 0 \) so \( x \) gets sent by \( \phi \) to \( dx \otimes 1 \) and then back to \( x \), which finishes the proof. □

In particular, applying this lemma to the local ring of a scheme at a point, we obtain the following:

**Proposition.** Let \( X \) be a scheme of finite type over any field \( k \). Recall that for any \( x \in X \) we have the map

\[
\frac{m_x/m_x^2}{\text{O}_{X,x}} \to \frac{\Omega_{X/k}}{\text{O}_{X,x}}
\]

- obtained by sending \( u \) to \( du \otimes 1 \)

If \( k_x/k \) is separable then this map is an isomorphism.

**Corollary.** If \( k \) is perfect, then regularity is equivalent to smoothness.

**Definition.** An irreducible scheme \( X \) (over a field) is called generically smooth if it’s smooth at the generic point (which is equivalent to its stalk at the generic point being a field which is a separable extension of the base field).

**Lemma.** \( X \) is generically smooth if it’s generically reduced and \( \text{dim}_K(\Omega_{X/k}) = \text{dim}(X) \)
Proof: follows from the theorem that smoothness can be tested by $\Omega_{k/k}$ - which is always locally free, note that the dimension clause follows. Flatness is equivalent to $K$ being a field i.e. $X$ being generically reduced. Note that, in general, smooth implies regular implies reduced. □

Now let $K$ be a finitely generated field extension of $k$. Then $K$ is an algebraic extension of some $K_0 = k(x_1,\ldots,x_m)$ with $m = tr.deg(K/k)$.

**Definition.** $K$ is called separably generated over $k$ if there exists such a $K_0$ such that $K/K_0$ is a separable extension.

**Lemma.** If $k$ is perfect, then any finitely generated extension is separably generated.

Proof: Choose any $x_1,\ldots,x_m$ algebraically independent, and let $K_0' = k(x_1,\ldots,x_m)$. It is enough to show that for any $y \in K$, we can replace it by $K''_0 = k(x'_1,\ldots,x'_r)$ of the same form, such that $K''_0(y)/K_0'$ is separable. Then by repeating the argument for a finite set of $y$ that generate $K_0$, we can conclude.

Look at the minimal polynomial of $y$ over $K_0'$.

If it’s separable, we take $K''_0 = K_0'$. Otherwise all exponents of $y$ are multiples of $p$. If all exponents of every $x_i$ are multiples of $p$, we can extract a $p$-th root as $k$ is perfect, which would contradict minimality. If $x_i$ appears with exponent non-divisible by $p$, its minimal polynomial over $k(x_1,\ldots,x_{i-1},x_{i+1},\ldots,y)$ is separable so we can swap $x_i$ and $y$.

Now by letting $y$ run over a finite set of generations of $K_0$ over $k$, we will obtain a field extension $k(x_1,\ldots,x_m)$ such that every element from that set is either contained in $k(x_1,\ldots,x_m)$ or is separable algebraic over it, which finishes the proof. □

**Lemma.** If $K/k$ is separably generated, then

$$dim_k(\Omega_{K/k}) = tr.deg.(K/k)$$

Proof: choose $K_0 \subset K$ of the form $k(x_1,\ldots,x_n)$ such that $K$ is algebraic over it. We have the exact sequence

$$K \otimes K_0/\Omega_{K_0} \to \Omega_{K/k} \to \Omega_K/K_0 \to 0$$

The latter is zero because $K/K_0$ is separable hence the map $K \otimes K_0/\Omega_{K_0} \to \Omega_K/K_0$ is surjective and so $dim_k(\Omega_{K/k}) \leq tr.deg.(K/k)$. However, we have the reverse inequality:

**Lemma.** $dim_k\Omega_{K/k} \geq tr.deg.(K/k)$

Proof: Choose an affine integral scheme $X = Spec(A)$ such that $K$ is the fraction field of $A$. We need to show that the rank of $\Omega_X/Spec(k)$ at the generic point is $\geq dim(X)$. Base change it to $\overline{k}$. But $\Omega_{X/Spec(\overline{k})}$ is $m/m^2$ that has dimension $\geq dim(X)$ □

**Corollary.** Over a perfect field a scheme of finite type is generically smooth if and only if it’s generically reduced.

Proof: $dim_k(\Omega_{K/k}) = tr.deg. = dim(X)$ and we use a previous lemma. □

**Corollary.** $X$ is generically smooth if $X \otimes \overline{k}$ is generically reduced.

Proof: an algebraically closed field is perfect, so generically smooth is equivalent to generically reduced. It remains to show that $X$ is generically smooth if and only if $X \otimes \overline{k}$ is generically smooth - but both are equivalent to $K(X) \otimes \overline{k}$ being regular.

**Corollary.** $dim_k\Omega_{K/k} = tr.deg.(K/k)$ is equivalent to $K \otimes \overline{k}$ being a sum of fields for all algebraic extensions $k'$. 

112
Proof: \( k' \otimes \Omega_{K/k} = \Omega_{K' \otimes k'/k} \) and according to the previous corollary \( \dim_K \Omega_{K/k} = \text{tr.deg.}(K/k) \) is equivalent to \( X \otimes k' \) being generically reduced, i.e. \( K \otimes k \) being reduced - in particular, \( K \otimes k' \) being reduced. But this latter is equivalent to it being a direct sum of fields, because a finite algebra over a field is a direct sum of Artinian rings which are reduced if and only if they are fields. \( \square \)

**Lemma.** \( K/k \) is separably generated if and only if

\[
\dim_K \Omega_{K/k} = \text{tr.deg.} K/k
\]

One direction was already done.

Let’s complete the proof of a proposition from the previous lecture.

**Theorem.** Let \( f: X \to Y \) be a morphism of schemes smooth over \( S \) of relative dimension \( d_X, d_Y \). The following are equivalent:

i) \( f \) is smooth of relative dimension \( d_X - d_Y \)

ii) \( \Omega_{X/Y} \) is locally free of rank \( d_X - d_Y \)

iii) \( f^* \Omega_{Y/S} \to \Omega_{X/S} \) is an injective bundle map.

Proof: the implication i) \( \Rightarrow \) iii) was done before. The implication iii) \( \Rightarrow \) ii) follows from the sequence \( f^* \Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0 \)

It remains to prove ii) \( \Rightarrow \) i). This was proven before for \( S = \text{Spec}(k) \), as follows:

Smoothness is equivalent to flatness and local freeness of the differential. The second condition is automatically satisfied.

We know that flatness is equivalent to dimension not jumping up: \( \forall y \in Y, \dim(f^{-1}(y)) \geq \dim(X) - \dim(Y) \) and we want equality.

But \( \text{rk} \Omega_{X/Y} \geq \dim(f^{-1}(y)) \geq \dim(X) - \dim(Y) \) and we actually have equality.

It follows that the map is flat.

Now we consider arbitrary \( S \). Again we need to show flatness. Let \( y \in Y, s \) the image of \( y \) in \( S \). Consider

\[
\mathcal{O}_X \otimes_{\mathcal{O}_Y} k_y = \left( \mathcal{O}_X \otimes_{\mathcal{O}_Y} k_s \right) \otimes_{\mathcal{O}_{Y_S}} k_y
\]

- because \( X \) is flat over \( S \) it follows that \( \mathcal{O}_X \otimes_{\mathcal{O}_Y} k_s = \mathcal{O}_X \otimes_{\mathcal{O}_Y} k_s = \mathcal{O}_{X_s} \) thus we obtain \( \mathcal{O}_{X_s} \otimes_{\mathcal{O}_{Y_S}} k_y \) which by the field case equals \( \mathcal{O}_{X_s} \otimes_{\mathcal{O}_{Y_S}} k_y = \mathcal{O}_{X_s} \otimes_{\mathcal{O}_Y} k_y \) which shows that the map is flat as all the higher Tors from \( \mathcal{O}_X \) to \( k_y \) are zero (where \( X_s, Y_s \) are \( X \times_S k_s, Y \times_S k_s \)). \( \square \)

**Definition.** A map \( X \to Y \) is etale if it’s smooth of relative dimension zero.

**Theorem.** \( X \to Y \) is smooth (of relative dimension \( n \)) if and only if locally on \( X \), it factors as \( X \xrightarrow{\text{etale}} \mathbb{A}^n_Y \to Y \)

This theorem will be proven later. Now we sketch how this theorem applies to prove the other direction of the lemma stated before, that \( \dim_k(\Omega_{K/k}) = \text{tr.deg.}(K/k) \Rightarrow K \) is separably generated over \( k \):

Choose \( \text{Spec}(A) \) with \( \text{Frac}(A) = K \). \( X \) is generically smooth so we can localize and assume that \( X \) is smooth. Again by localizing if necessary, we factor \( X \xrightarrow{\text{etale}} \mathbb{A}^n_k \to k \). It remains to note that a map of fields is etale if and only if it is a separable finite extension.

**Flatness and smoothness handout.**

This is a summary of the material discussed before, with some completions. Many propositions are given without proof, most of them have been proven before.
Let $A$ be a commutative ring. Recall that an $A$-module $M$ is flat if and only if the functor $M \otimes_A -$ is exact. The following is known from commutative algebra:

**Proposition. (1.1.1., 1.1.2.)** i) A module if flat if and only if its localization at any prime (or maximal ideal) is flat.

ii) $M$ is flat if and only if $\text{Tor}_1(M, A/I) = 0$ for any ideal $I \subset A$.

iii) If $A$ is Noetherian, a module $M$ is flat if and only if $\text{Tor}_1(M, A/p) = 0$ for any prime $p$.

iv) If $A$ is Noetherian a module $M$ is flat if and only if $\text{Tor}_1(M, k_p) = 0$ for any prime $p$ and $i \in \mathbb{N}$.

v) If $M$ is finitely generated over $A$, it is flat if and only if $\text{Tor}_1(M, A/m) = 0$ for all maximal ideals $m$ (the finitely generated assumption is necessary).

**Definition.** A scheme morphism $f: X \to Y$ is flat if, locally on $X$, the structure sheaf $\mathcal{O}_X$ is flat as an $\mathcal{O}_Y$ module. If is flat at $x$ is the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$.

It is easy to see that a morphism is flat if and only if it is flat at every point.

More generally, we can define the notion of $Y$-flatness (resp. $Y$-flatness at a point $X$) aka flatness with respect to $f$ (resp. flatness with respect to $f$ at a point $x$) for $\mathcal{F} \in \text{QCoh}(X)$:

Recall that a map is faithfully flat if it is flat and surjective. It is know that faithfully flat maps are open and surjective.

**Proposition. (1.3.1.)** Assume $Y$ is Noetherian, and $\mathcal{F} \in \text{QCoh}(X)$.

i) $\mathcal{F}$ is $Y$-flat if and only if $\text{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, k_y)$ vanish (as q.c. sheaves on $X$) for all $y \in Y$ and $i \in \mathbb{N}$.

ii) Assume $X, Y$ are both of finite type over a field, and $\mathcal{F}$ is coherent on $X$. Then $\mathcal{F}$ is flat if and only if $\text{Tor}_i^{\mathcal{O}_Y, \mathcal{O}_Y}(\mathcal{F}, k_y)$ vanishes for all closed points $y$ in $Y$.

**Theorem (Generic flatness, 1.3.2.).** Let $f: X \to Y$ be a morphism of finite type with $Y$ integral and Noetherian. Let $\mathcal{F}$ be a coherent sheaf of $X$. Then there exists a non-empty open subscheme $\tilde{Y} \subset Y$ such that $\mathcal{F} |_{f^{-1}(\tilde{Y})}$ is $Y$-flat (or $\tilde{Y}$-flat).

**Corollary. (1.3.3.)** Let $f: X \to Y$ be a morphism of finite type with $Y$ Noetherian. Let $\mathcal{F}$ be a coherent sheaf on $X$. Let $y \in Y$ be a point such that for every $x \in X$ with $f(x) = y$, the sheaf $\mathcal{F}$ is $Y$-flat at $x$. Then $y$ admits a Zariski neighborhood over which $\mathcal{F}$ is flat.

Here is a basic fact about the interaction between the notions of flatness and dimension.

If $X$ is a Noetherian scheme and $x \in X$ is a point, we denote by $\text{dim}(X)_x$ the ”dimension of $X$ at $x$”, i.e. the dimension of the Noetherian local ring $\mathcal{O}_{X,x}$.

**Proposition. (1.4.1.)** Let $f: X \to Y$ be a morphism between Noetherian schemes. Let $x \in X$ be a point and $y = f(x)$ its image in $Y$. Let $X_y$ be the fiber of $X$ over $y$, i.e. $X \times k_y$.

i) We have the inequality $\text{dim}(X_y)_x + \text{dim}(Y)_y \geq \text{dim}(X)_x$.

ii) If $f$ is flat at $x$, the above inequality is an equality.

iii) If $Y$ is regular at $y$ and $X$ is CM at $x$, then the assertion of ii) is ”if and only if”.

**Proposition. (1.5.1.)** Let $X \to Y$ be a faithfully flat map with both $X$ and $Y$ locally Noetherian. If $X$ is i) reduced ii) integral iii) regular iv) $R_n$ v) $S_n$ vi) locally factorial, then the same ii true for $Y$.

Proof: i), ii), iii) were done before. Let’s prove v). The assertion is local, so we can assume both $X$ and $Y$ are affine. Property $S_n$ means that for all $\mathcal{F} \in \text{Coh}(Y)$ with $\text{codim(supp}_Y(\mathcal{F})) \geq n$, we have $\text{Ext}^k(\mathcal{F}, \mathcal{O}_Y) = 0$ for $k \leq n - 1$. By faithful flatness, it’s enough to show that $f^*(\text{Ext}^k_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_Y)) = 0$ while the latter, by flatness and the fact that $\mathcal{F}$ is coherent, is isomorphic to $\text{Ext}^k_{\mathcal{O}_X}(f^*(\mathcal{F}), \mathcal{O}_X)$. By part ii) of the previous proposition, we deduce that $\text{codim(supp}_X(f^*(\mathcal{F}))) = \text{codim(supp}_Y(\mathcal{F})))$ so the latter group vanishes since $X$ was $S_n$. 

114
Let’s prove vi). Let \( j: \hat{Y} \hookrightarrow Y \) be an open subset whose complement is of codimension \( \geq 2 \). Let \( \mathcal{L} \) be a line bundle over \( \hat{Y} \). We need to show that it admits an extension to a line bundle on the entire \( Y \). We claim that \( j_* (\mathcal{L}) \) does the job. Indeed, since \( f \) is faithfully flat, it’s enough to show that \( f^*(j_*(\mathcal{L})) \) is a line bundle on \( X \). Since \( f \) is flat, by the trivial case of the projection formula, we have \( f^*(j_*(\mathcal{L})) = j_*(f^*(\mathcal{L})) \) where \( j: \hat{X} \hookrightarrow X, \hat{X} = f^{-1}(Y) \). However, \( \text{codim}(X - \hat{X}) = \text{codim}(Y - \hat{Y}) \) by the previous proposition, so we know that \( f^*(\mathcal{L}) \) admits an extension to a line bundle \( \mathcal{L}' \) on \( X \). Since \( X \) is normal, we also know that this extension must coincide with \( \overline{j_* (f^*(\mathcal{L}))} \). □

We next discuss smoothness over a field. First, we give a refresher on the behavior of dimension for schemes of finite type over a field.

**Proposition. (2.1.1.)** Let \( X \) be a scheme of finite type over a field \( k \). Then

i) If \( X \) is irreducible, then \( \text{dim}(X)_{x_1} = \text{dim}(X)_{x_2} \) for any two closed points \( x_1, x_2 \in X \).

ii) If \( X \) is integral, then \( \text{dim}(X) = \text{tr.deg.}(K(X)/K) \) where \( K(X) \) is the field of fractions of \( X \).

Let first \( k \) be an algebraically closed field, and \( X \) a scheme of finite type over \( k \).

**Definition.** We say that \( X \) is smooth of dimension \( n \) if it is regular as a scheme and has dimension \( n \).

**Lemma. (2.2.2.)** Assume that every irreducible component of \( X \) is dimension \( \geq n \). Then the following conditions are equivalent:

i) \( X \) is smooth of dimension \( n \).

ii) \( \Omega^1_{X/k} \) is a locally free sheaf of dimension \( n \).

The proof follows the fact that for any closed point \( x \in X \), the natural map \( m_x/m_x^2 \to (\Omega^1_{X/k})_x \) is an isomorphism.

Let \( k \) be an arbitrary field, let \( X \) be a scheme of finite type over \( X \).

**Definition.** We say that \( X \) is smooth of dimension \( n \) over \( k \) if \( X \times_k \overline{k} \) is smooth of dimension \( n \) over \( \overline{k} \).

From the previous lemma we obtain:

**Corollary. (2.4.2.)** A scheme \( X \) is smooth of dimension \( n \) over \( k \) if and only if each of its irreducible components has dimension \( \geq n \), and \( \Omega^1_{X/k} \) is locally free of rank \( n \).

As seen before, an field extension \( k'/k \) which is not separable provides an example of a scheme over \( k \) which is regular but not smooth.

**Proposition. (2.4.3.)** Let \( X \) be a scheme of finite type over \( k \).

i) Any scheme smooth over a field is regular.

ii) Let \( k'/k \) be a separable field extension. If \( X \) is regular, then so is \( X' = X \times_k k' \)

iii) Over a perfect field, a scheme is smooth if and only if it’s regular.

Let’s recall the following assertion about regular local rings:

**Proposition. (2.5.1.)** Let \( A' \to A \) be a surjection of local rings with \( A' \) regular of dimension \( m \). Then the following conditions are equivalent:

i) \( A \) is regular of dimension \( n \).

ii) The ideal \( \text{ker}(A' \to A) \) can be generated by \( m - n \) elements \( f_1, \ldots, f_{m-n} \), whose images in \( m'/m'^2 \) are linearly independent.

Here is an analogue of this assertion when regularity is replaced by smoothness.

**Theorem. (2.5.2.)** Let \( X \) be a scheme of finite type over \( k \). Then the following conditions are equivalent:

i) \( X \) is smooth over \( k \).
ii) Locally on $X$, for any closed embedding $X \hookrightarrow X'$ with a sheaf of ideals $\mathcal{J}$ and $X'$ smooth, the sequence

$$\mathcal{J}/\mathcal{J}^2 \to \Omega_{X'/k} |_{X} \to \Omega_{X/k} \to 0$$

is a short exact sequence of vector bundles.

Proof: the implication $\text{ii}) \Rightarrow \text{i})$ follows from the criterion about dimension and local freeness of the Kahler differential. Local freeness is clear, and dimension follows from the fact that $\mathcal{J}$ is generated by the right number of generators, hence the dimension decreases by at most this number. For the implication in the other direction, we can base change to $k$. In the latter case we deduce it from the previous proposition (proven before):

**Lemma.** (2.5.3.) Let $\alpha: \mathcal{F} \to \mathcal{E}$ be a map of coherent sheaves on a locally Noetherian scheme $X$, where $\mathcal{E}$ is locally free. Assume that for every $x \in X$, the fiber $\mathcal{F}_x \to \mathcal{E}_x$ is injective. Then $\alpha$ is an injective bundle map, i.e.

i) The map $\alpha$ is injective
ii) $\mathcal{F}$ is locally free
iii) $\text{coker}(\alpha)$ is locally free

Moreover, the above condition is sufficient to check for closed points. □

**Definition.** We say that $X$ is generically smooth of dimension $n$ over $k$ if $X$ contains a dense Zariski open $\overset{\circ}{X}$, which is smooth of dimension $n$ over $k$.

**Lemma.** (2.6.2.) Let $X$ be an irreducible scheme of finite type over $k$.

i) The generic rank of $\Omega_{X/k}$ is $\geq \dim(X)$
ii) The equality in i) holds if and only if $X$ is generically smooth.

The proof follows by base change to $\bar{k}$, noting that being locally free is "generic" (for example by generic flatness, or by explicit choice of a base). □

**Definition.** A finitely generated field extension $K/k$ is said to be separably generated if it can be written in the form $K \subset K_0 = k(x_1, \ldots, x_n)$, where $K/K_0$ is a finite separable extension.

**Lemma.** (2.6.4.) Let $K/k$ be a finitely generated field extension.

i) We have $\dim_K(\Omega_{K/k}) \geq \text{tr.deg.}(K/k)$
ii) If $K/k$ is separably generated, then the inequality in i) is an equality (in fact, the converse is also true).

Proof: to prove i) choose an integral scheme of finite type over $X$ with field of fraction $K$, then the assertion follows from the previous lemma. To prove ii), we consider the exact sequence

$$K \otimes_{K_0} \Omega_{K_0/k} \to \Omega_{K/k} \to \Omega_{K/K_0}$$

and the assertion follows from counting dimensions. □

**Lemma.** (2.6.5.) If $k$ is perfect, then any finitely generated field extension is separably generated.

**Corollary.** (2.6.6.) Any integral scheme over a perfect field is generically smooth.

Now we discuss smoothness in full generality.

Let $f: X \to S$ be a morphism of schemes. We implicitly assume the base $S$ is locally Noetherian. Assume $f$ is of finite type (smoothness is only defined for morphisms of finite type).

**Definition.** We say that $f$ is smooth of relative dimension $n$ if the following conditions hold:

i) $X$ is flat over $S$.
ii) For every point $s \in S$, the base change $\overset{\circ}{X} \times_S k_s$ is a smooth scheme of dimension $n$ over the residues field $k_s$.

**Theorem.** (3.1.2.) Let $f: X \to S$ be a morphism of finite type. The following conditions are equivalent:
i) Condition ii) in the previous definition holds, and $\Omega_{X/S}$ is locally free of rank $n$.

ii) $f$ is smooth of rel. dim. $n$.

iii) $X$ is flat and locally on $X$, we can find a closed embedding $X \hookrightarrow X' := A^n_S$ (compatible with the projection to $S$), so that if we denote by $J_{X,X'}$, the corresponding sheaf of ideals on $X'$, the sequence

$$J_{X,X'}/J_{X,X'}^2 \to \Omega_{X'/S} |_{X} \to \Omega_{X,S} \to 0$$

is a short exact sequence of vector bundles.

Proof: the fact that i) implies ii) follows from the dimension criterion (why does it hold?). The implication $\text{ii) } \Rightarrow \text{ i) }$ is easy. So its enough to show that ii) implies iii). We use the following generalization of a previous lemma:

**Lemma. (3.1.3.)** Let $\alpha: F \to \mathcal{E}$ be a map of coherent sheaves on a locally Noetherian scheme $X$, where $\mathcal{E}$ is locally free. Assume that for every $x \in X$, there exists a closed subscheme $Z_x \subset x$, such that the resulting map $\alpha |_{Z_x}: F |_{Z_x} \to \mathcal{E} |_{Z_x}$ is injective and the quotient is locally free on $Z_x$. Then the map $\alpha$ is an injection of vector bundles. Hence, it is sufficient to check this condition for closed points only.

We apply this lemma to the map $J_{X,X'}/J_{X,X'}^2 \to \Omega_{X'/S} |_{X}$. For every $x \in X$, we take $Z_x := X_f(x)$. We need to verify that the conditions of the lemma holds. First, since $X$ is flat over $Y$, for every $y \in Y$, the natural map $J_{X'_y} \to J_{X_y} |_{X'_y}$ is an isomorphism, and hence

$$(J_{X,X'}/J_{X,X'}^2)_{x_y} \cong J_{X'_y} / J_{X'_y}^2$$

The assertion now follows from a previous proposition. □

Note: from what we proved it follows that this theorem can be rephrased as follows: $X \to S$ is smooth if and only if it is flat, and (locally) for any closed embedding $X \hookrightarrow X'$ with $X'$ smooth over $S$, the sequence in part iii) of the theorem is short exact.

Here is a differential criterion for smoothness of a map:

**Proposition. (3.2.1.)** Let $f: X \to Y$ be a map of schemes smooth over a base $S$. The following are equivalent:

i) $f$ is smooth of relative dimension $\dim_{rel.}(X,S) = \dim_{rel.}(Y,S)$

ii) $\Omega_{X/Y}$ is locally free of rank $\dim_{rel.}(X,S) = \dim_{rel.}(Y,S)$

iii) The map $f^*(\Omega_{Y/S}) \to \Omega_{X,S}$ is an injection of vector bundles.

Proof: That i) implies ii) is immediate from a previous theorem (also done before).

For ii) implies iii), first we have a little lemma:

If $\mathcal{E}_1 \to \mathcal{E}_2$ is a surjection of sheaves, and $\mathcal{E}_1, \mathcal{E}_2$ are vector bundles of the same rank, then the map is actually an isomorphism.

Indeed, being an isomorphism can be checked locally or even on stalks, so we can assume we have a surjection of free modules $A^n \to A^n$ over $A$. This surjection corresponds to a matrix $X \in M_n(A)$ and surjectivity implies $M$ has a left inverse. But then as we know, this would be a left inverse as well.

To apply this lemma to our case, consider the exact sequence $f^*(\Omega_{Y/S}) \to \Omega_{X,S} \to \Omega_{X/Y} \to 0$ and let $K$ be the kernel of the first map so that we get a short exact sequence $0 \to K \to \Omega_{X,S} \to \Omega_{X/Y} \to 0$. Locally, the second and third term of the sequence are free which means that the first term is also free - because the short exact sequence splits and hence it is locally projective but locally projective implies flat implies locally free (on Noetherian schemes). Moreover, we claim the dimensions add up: indeed, otherwise we would get $\Omega_{X/S}$ locally isomorphic to $A^n$ on one hand, and to $A^m$ on the other hand. We claim $A^n$ and $A^m$ are never isomorphic over $A$: the reason being, if $n > m$ then $A^n \to A^m$ it is never an injection, as seen by playing with minors in the standard way.

Therefore it follows that $K$ is a vector bundle of rank $r.d.\Omega_{X/S} - r.d.\Omega_{X/Y} = r.d.\Omega_{Y/S}$. Since $f^*(\Omega_{Y/S})$ surjects onto $K$, it is actually isomorphic to it by the lemma, and we are done.

Finally, we do iii) implies i).
Being smooth is equivalent to \( \Omega_{X/Y} \) being free of the appropriate rank and to \( X \) being flat over \( Y \). The first part follows immediately from the condition, as a cokernel of an injection of vector bundles is vector bundle (of the appropriate rank).

It remains to check flatness. As we know, flatness can be checked pointwise but if we are at \( x \in X \) with \( f(x) = y \) we can simply replace \( S \) by \( S \) the point which is the image of \( x \) (or \( y \)). The justification for this is as follows:

We have the following criterion for flatness: for every \( x \in X, f(x) = y \in Y \),

\[
\dim(X_y) + \dim(Y)_y = (\dim X)_x
\]

The proposition applies because \( X \) and \( Y \) being smooth over \( S \), in particular are regular at \( x \) and \( y \). In fact, it is enough to check it for \( y \) closed because flatness can be only checked at maximal ideals.

Indeed, we always know \( \dim(X_y) + \dim(Y)_y \geq (\dim X)_x \). But we also have the inequality \( \text{rank} \Omega_{X/Y} \geq \dim(X_y) \). Indeed, a previous lemma tells that \( \text{rank} \Omega_{X_y/k_y} \geq \dim(X_y) \) but we check that \( \text{rel} \Omega_{X/Y} = \Omega_{X_y/k_y} \) where \( i \) is the closed embedding of \( X_y \) into \( X \), for the following reason: locally we have \( B \rightarrow A \) and we pick a prime ideal \( y \), so that we consider \( A \otimes_B k_y \). Now let \( M = \Omega_{A/B} \). Then \( \text{Hom}_{A\otimes_{k_y}}(M \otimes_A (A \otimes_{k_y} N) \cong \text{Hom}_A(M, N) = \text{Der}_B(A, N) \cong \text{Der}_{k_y}(A \otimes_{k_y} N) \) which shows that \( M \otimes_A A \otimes_{k_y} k_y \) is \( \Omega_{A\otimes_{k_y}/k_y} \). We need to show the last equality, which is obtained as follows: if we start with a \( B \)-derivation \( d \) of \( A \) into \( N \) we make a \( k_y \) derivation of \( A \otimes_{k_y} N \) into \( N \) by sending \( a \otimes x \) to \( x da \) - this is well-defined as only the residue class of \( x \) matters since \( y \) acts trivially on \( N \). Also the map is easily seen to be bilinear and a derivation. Conversely, starting from a derivation \( d \) of \( A \otimes_B k_y \) into \( N \) we produce a derivation that sends \( a \) to \( d(a \otimes 1) \).

Therefore \( \text{rank} \Omega_{X/Y} \geq \dim(X_y) \) but \( \text{rank} \Omega_{X/Y} = \text{rel}\dim(X,Y) – \text{rel}\dim(Y/S) = (\dim X)_x – (\dim Y)_y \) however the reverse inequality holds \( \dim(X_y) \geq (\dim X)_x – (\dim Y)_y \) so it is an equality. \( \square \)

**Final Project: Generic Smoothness and Bertini’s Theorem.**

The aim of this project is to present two theorems: generic smoothness and as a consequence of it, Bertini’s theorem. The latter has a simpler particular case which can be proven without involving generic smoothness, however the generalization is probably a more correct way to look at it. For this general version, we will introduce the notion of linear systems of divisors. These theorems are only true for integral schemes of finite type over an algebraically closed field of characteristic 0. The references for this project are Hartshorne sections II.8. and III.10., Shafarevich section I.6.3. and the notes on smoothness and flatness (the latter will be the default source for any proposition mentioned).

**Generic smoothness**

**Lemma.** Let \( f : X \rightarrow Y \) be a dominant morphism of integral schemes of finite type over an algebraically closed field \( k \) of characteristic 0. Then there is a non-empty open set \( U \) in \( X \) such that \( f|_U \) is smooth.

**Proof:** Because over a perfect field, every scheme of finite type is generically smooth (corollary 2.6.6. in the notes), we may assume \( X \) and \( Y \) are smooth. Since the field is perfect, the extension of the function fields is separable (lemma 2.6.5. in the notes) which implies (lemma 2.6.4.) that the \( \Omega_{X/Y} \) has the correct rank at the generic point. It follows that \( \Omega_{X/Y} \) is free of the correct rank at some open set around the generic point, which by theorem 3.2.1. implies that the map is smooth on some open set.

**Corollary.** Let \( f : X \rightarrow Y \) be a morphism of integral schemes of finite type over an algebraically closed field of characteristic 0. For any integer \( r \in \mathbb{N}_0 \), let \( X_r \) denote the set of closed points \( x \in X \) such that the image of the tangent space \( T_x \) at \( x \) under \( f \) has rank at most \( r \). Then \( f \) \( X \) has dimension at most \( r \).

**Proof:** Take \( Y' \) any irreducible component of \( f(X_r) \). Note that \( X_r \) maps into \( f(X_r) \), because the preimage of \( f(X_r) \) is closed and contains \( X_r \). There must be an irreducible component \( X' \) of \( X_r \) which maps into \( Y' \) and dominates it. Indeed, all irreducible components of \( X_r \) must map inside one of the irreducible components of \( f(X_r) \) - in virtue of their irreducibility, and at least one of them must map into \( Y' \) because otherwise \( Y' \) could just be removed.
from the set of irreducible components. At least one such must be dominant: if the image of every such map is just a single (closed) point, then we could replace \( Y \) with the union of these points, which is of course impossible.

Having these irreducible closed subsets \( X', Y' \) mapping into each other, we give them their reduced induced structure and consider the induced dominant morphism \( f': X' \to Y' \). By the previous lemma \( f'|_{U'} \) is smooth for some dense \( U' \) in \( X' \) - moreover we may assume that \( X' \) and \( Y' \) are smooth because they are generically so and we are taking their intersections with some open set.

In the following diagram (for a closed point \( x \in U' \cap X, y = f(x) \in Y' \))

\[
\begin{array}{c}
T_{x,U'} \longrightarrow T_{x,X} \\
T_{f',x} \downarrow \quad \quad \downarrow T_{f,x} \\
T_{y,Y'} \longrightarrow T_{y,Y}
\end{array}
\]

the horizontal maps are injective by default (because \( U' \) and \( Y' \) are locally closed subschemes of \( X, Y \)) and the rank of \( T_{f,x} \) is at most \( r \) by definition. It follows that the rank of \( T_{f,x} \) is at most \( r \). We claim this latter map is surjective. Indeed, since we assumed \( X' \) and \( Y' \) are smooth we have an injection of vector bundles \( f^* \Omega_{Y/k} \hookrightarrow \Omega_{X/k} \) which then can be tensored with \( k \) the residue field of \( x \) to yield \( f^* \Omega_{Y/k} \otimes k \hookrightarrow \Omega_{X/k} \otimes k \). As we know, \( \Omega_{A/k} \otimes_A k \cong m/m^2 \) for \( A \) a local \( k \)-algebra of finite type, therefore the above injection reads as \( m_y/m^2_y \hookrightarrow m_x/m^2_x \) and now we take duals to conclude surjectivity. It thus follows that \( T_{y,Y} \) has dimension at most \( r \) - which by a combination of Nakayama’s lemma and Krull’s Hauptidealsatz implies that \( Y'_y \) has dimension at most \( r \) and as we know (2.1.1.) this equals the dimension of \( Y' \) itself. This finishes the proof.

Before moving to generic smoothness, we will give a criterion for a map of smooth varieties over a field to be smooth in terms of maps of tangent spaces. Recall that for a point \( x \in X \) with local ring \( O_{X,x} \) and maximal ideal \( m_x \), the tangent space is the dual to the \( k_x \) vector space \( m_x/m^2_x \). If \( f: X \to Y \) is a map with \( f(x) = y \), it induces a local map \( O_{Y,y} \to O_{X,x} \) and therefore an inclusion \( k_y \subset k_x \) and a map \( m_y/m^2_y \to m_x/m^2_x \) - which by taking duals induces a map \( T_{f,x}: T_{x,X} \to T_{y,Y} \).

The criterion goes as follows:

**Theorem.** Let \( X, Y \) be a morphism of smooth integral schemes of finite type over an algebraically closed field \( k \). Then \( f \) is smooth of relative dimension \( \dim X - \dim Y \) if and only if for every closed point \( x \in X \), the map \( T_{f,x} \) is surjective.

**Proof:** assume the map is smooth. Then by proposition 3.3.1 we have an injection of vector bundles \( f^* \Omega_{Y/k} \hookrightarrow \Omega_{X/k} \) which then can be tensored with \( k \) the residue field of \( x \) to yield \( f^* \Omega_{Y/k} \otimes k \hookrightarrow \Omega_{X/k} \otimes k \). As we know, \( \Omega_{A/k} \otimes_A k \cong m/m^2 \) for \( A \) a local \( k \)-algebra of finite type, therefore the above injection reads as \( m_y/m^2_y \hookrightarrow m_x/m^2_x \) and now we take duals to conclude surjectivity.

Conversely, assume \( T_{f,x} \) is surjective for every point \( x \). First we prove that the map is flat - and it suffices to show that \( O_x \) is flat over \( O_y \) for every closed point \( x \in X \), because flatness, like injectivity, can be checked at the maximal ideals.

Both \( O_x \) and \( O_y \) must be regular as the schemes are smooth, and they must be of appropriate dimension by 2.1.1. - note that \( y \) is also closed because the map is proper. As \( T_f \) is surjective, the map \( m_y/m^2_y \to m_x/m^2_x \) is injective.

Let \( t_1, \ldots, t_r \) be independent elements of \( m_y/m^2_y, r = \dim Y \). Their images in \( m_x/m^2_x \) will also be independent, hence they will form a regular sequence (Matsumura, p.121. theorem 36).

Now \( O_x/(t_1, \ldots, t_r) \) is flat over \( O_y/(t_1, \ldots, t_r) \cong k \). We will prove by descending induction on \( i \) that \( O_x/(t_1, \ldots, t_i) \) is flat over \( O_y/(t_1, \ldots, t_i) \) and the case \( i = 0 \) will show that \( O_x \) is flat over \( O_y \).

Indeed, we have a more general lemma: if \( A \) is local Noetherian and \( t \in A \) is a non-unit and non-zero divisor, then a finitely generated module \( M \) over \( B \) (where \( A \to B \) is a local map of Noetherian rings) is flat if and only if \( t \) is not a zero divisor and \( M \) and \( M/tM \) is flat over \( A/tA \).

119
The proof follows from the local criterion for flatness. If $M$ is flat then the short exact sequence $0 \rightarrow A \xrightarrow{t} A \rightarrow A/tA \rightarrow 0$ can be tensored with $A$ to produce $0 \rightarrow M \xrightarrow{t} M \rightarrow M/tA \rightarrow 0$. In particular $t$ is a non-zero divisor and $M/tM$ is flat because $\text{Tor}^1(k, M/tM) = 0$ from the short exact sequence of Tor, and $A, A/fA$ have the same residue field. Conversely, if $t$ is a non-zero divisor and $M/tM$ is flat then we have the short exact sequence $0 \rightarrow M \xrightarrow{t} M \rightarrow M/tM \rightarrow 0$. We write out the Tor long exact sequence that in particular reads $\text{Tor}^1(k, M) \xrightarrow{t} \text{Tor}^1(k, M) \rightarrow 0$. However multiplication by $t$ is 0 because multiplication by $t$ is 0 on $k$, and exactness implies $\text{Tor}^1(k, M)$.

Next, we have the exact sequence $f^*\Omega_{Y/k} \otimes k(x) \rightarrow \Omega_{X/k} \otimes k(x) \rightarrow \Omega_{X/Y} \otimes k(x) \rightarrow 0$ and we can conclude that it’s also exact at the left from the conclusion, thus $\Omega_{X/Y} \otimes k(x)$ has dimension $\dim X - \dim Y$. Particularly since $f$ is flat it is non-constant thus dominant and surjective.

Also, $\dim_K(\Omega_{X/Y} \otimes K_X) \geq \dim X - \dim Y$ according to the exact sequence above.

This is enough to show that $\Omega_{X/Y}$ is locally free. Indeed, choose a closed point $x$ and an affine around it. The localization of $\Omega_{X/Y}$ at $x$ produces a finitely generated module $M$ over $O_x$ such that $M \otimes k(x)$ has dimension $\dim X - \dim Y$ but $M \otimes O_x K_X$ has dimension $\geq \dim X - \dim Y$. The first condition together with Nakayama’s lemma produces a set of $\dim X - \dim Y$ generators of $M$ over $O_x$ thus $O_x^{\dim X - \dim Y} \rightarrow M$ is surjective so we get a short exact sequence $0 \rightarrow N \rightarrow O_x^{\dim X - \dim Y} \rightarrow M \rightarrow 0$, and by tensoring with $K_X$ we deduce that the second map must be an isomorphism this $N \otimes K_X = 0$ however $N$ is a submodule of $O_x^{\dim X - \dim Y}$ is torsion-free which implies $N = 0$ thus $M$ is free.

Combined with proposition 3.3.1. this finishes the proof.

**Theorem (Generic Smoothness)** Let $f : X \rightarrow Y$ be a map of schemes of finite type over an algebraically closed field $k$ of characteristic 0, and assume that $X$ is smooth. Then there is a nonempty open subset $V$ of $Y$ such that the map $f^{-1}V \rightarrow V$ is smooth.

**Proof:** Since $Y$ is generically smooth, we might assume $Y$ is smooth. Let $r$ be its dimension, and define $X_{r-1}$ as in the previous corollary. Since $f(X_{r-1})$ has dimension at most $r - 1$, we can remove it and shrink $Y$ so that we assume that the rank of $T_f$ is at least $r$ for every closed point of $X$. But since $Y$ is smooth, the dimension of the tangent space at every closed point of $Y$ equals $r$ so that $T_f$ is surjective for every closed point of $X$. This is enough to prove that the map is smooth, according to the criterion established before.

Generic smoothness is particularly useful for group schemes, because finding an open subset where a morphism is smooth allows us to translate it using the action of $G$

**Theorem.** Let $X$ and $G$ be schemes of finite type over an irreducible field $k$ of characteristic 0, and suppose $G$ is a group scheme acting on $X$ such that the action of $G(k)$ on $X(k)$ is transitive ($X$ is called a homogeneous space of $G$). Consider two smooth varieties $Y, Z$ and let $f : Y \rightarrow X, g : Z \rightarrow X$ we two morphisms. Every element $\sigma \in G(k)$ (i.e. a closed point of $G$) induces a morphism $\text{Spec}(k) \rightarrow G$ and therefore by base change and action of $G$ a composite morphism $\sigma : X \rightarrow G \times X \rightarrow X$. Define now $Y^\sigma$ be $Y$ with the morphism $\sigma \circ f \rightarrow X$. Then there is a non-empty open subset $V \subseteq G$ such that for every $\sigma \in V(k)$, $Y^\sigma \times_X Z$ is either empty or smooth of dimension $\dim Y + \dim Z - \dim X$.

**Proof:** consider the composite morphism $h : G \times Y \rightarrow X$ obtained by composing $f$ with the group action $\theta : G \times X \rightarrow X$. Next, note that $X$ and $G$ are both smooth - more generally every homogeneous space is smooth - it has a smooth open subgroup by generic smoothness, and the transitive action of $G$ will cover the variety by translations of this subset (using the fact that closed points are dense in the variety because a point is closed if and only if its residue field is $k$ so being closed is local, at least in varieties over an algebraically closed field). We now apply generic smoothness to $h$ producing an open subset $U \subseteq X$ such that $h : h^{-1}(U) \rightarrow U$ is smooth. Take the action of $G$ on $G \times Y$ be left multiplication, and the action of $G$ on $X$ by $\theta$ and these actions are compatible with $h$: $h \circ (\sigma(k) \times 1) = \sigma(k) \circ h$. Indeed, this is just the translation on the level of $k$-points of the identity $h \circ m \times 1 = \theta \circ h$ as morphisms from $G \times G \times Y$ to $X$, where $m$ is the multiplication map in $G$. This follows immediately from the definition of $h$ as $\theta \circ (1 \times f)$ and the axioms of an action of a group scheme. It follows that for any $\sigma \in G(k)$, $h : h^{-1}(U^\sigma) \rightarrow U^\sigma$ is smooth too. Because the action is transitive, $U^\sigma$ cover $X$ which implies that $h$ is smooth.
Now let \( W = (G \times Y) \times_X Z \) with projection maps \( g', h' \) to \( G \times Y, Z \). Because \( h: G \times Y \to X \) is smooth, so is its base extension \( h' \). Because \( Z \) is smooth, this implies that \( W \) is also smooth.

Next consider the morphism \( q: W \to G \) which is the composition of \( g' \) with the projection \( G \times Y \to G \). Applying generic smoothness, there is an open subset \( V \subseteq G \) such that \( q: q^{-1}(V) \to V \) is smooth. In particular for every closed point \( \sigma \in V(k) \), its fiber \( W_\sigma \subset W \) must be non-singular (it may not be irreducible, but each component is smooth). But \( W_\sigma \) is just \( Y^\sigma \otimes_X Z \) which implies what we wanted.

For the dimension of \( W_\sigma \), we know that the relative dimension of \( h \) is \( \dim G + \dim Y - \dim X \) (by generic smoothness which transports to the entire scheme by the group action) hence so is the relative dimension of its base change \( h' \). It follows that \( \dim (W) - \dim (Z) = \dim G + \dim Y - \dim X \). On the other hand \( q \) (where smooth) must have relative dimension \( \dim W - \dim G \) so from here we compute \( \dim (W_\sigma) = \dim W - \dim G = \dim Y + \dim Z - \dim X \).

Bertini’s theorem

Bertini’s theorem is one of the classic results of algebraic geometry of the form ”if a variety in projective space satisfies some property \( P \), then its intersection with a generic hyperplane will satisfy \( P \) too”. As it is often the case, this is better understood in a more general setting, with hyperplanes giving way to linear systems. But first, the original Bertini’s theorem.

A geometric preliminary.

**Theorem.** Let \( f: X \to Y \) be a closed and surjective map map between integral irreducible schemes of finite type over an algebraically closed field. Then \( \dim X \geq \dim Y \) and the dimension of the fiber of every closed point of \( Y \) is at least \( \dim X - \dim Y \), and equality holds for closed points in a non-empty (thus dense) open set \( U \subset Y \).

**Proof:** Let \( x \) be a closed point of \( X \) and \( f(x) = y \). Because the map is closed, so is \( y \). Then \( \dim X_x = \dim X, \dim Y_y = \dim Y \), according to proposition 2.1.1. from the Smoothness and Flatness handout. The first part follows now from proposition 1.4.1. together with the Generic Flatness theorem (first semester, problem set 9 problem 6) applied to the sheaf \( f_* \mathcal{O}_X \) (which is coherent according to Jonathan’s project from last semester).

For an alternative proof, see Shafarevich theorem 7 from section I.6.3. (p. 76).

**Corollary.** Let \( f: X \to Y \) be a map of projective schemes over an algebraically closed field. If \( Y \) is irreducible, and every fiber is irreducible of the same dimension \( d \), then \( X \) is irreducible of dimension \( \dim(Y) + d \).

**Proof:** Let \( X_i \) be the irreducible components of \( X \), so that \( X_i \) are also projective and thus the composite maps \( f_i: X_i \to Y \) are proper: any map between projective schemes over the same base is proper, as it becomes proper when composed with the map to the base, and problem 5 from PS 11 first semester finishes it. It follows that they are closed and hence \( f(X_i) \) are closed subsets that cover \( Y \), in particular at least one of them equals \( Y \) by irreducibility. For such \( i \), we can find \( U_i \) as in the previous theorem (where the dimension of the fibers does not jump up, say it equals \( d_i \)), for all other \( i \) we can take \( U_i \) the complement of \( f(X_i) \). All these \( U_i \) are non-empty hence their intersection is non-empty. Choose \( y \) in the intersection. The irreducibility of \( f^{-1}(y) \) implies that \( f^{-1}(y) = f_i^{-1}(y) \) for some \( i \). By the choice of the \( U_i \), this implies that \( f_i \) is surjective, and now we can apply the previous theorem to the map \( f_i \). It follows that every fiber (of a closed point) of \( f_i \) has dimension at least \( d \) (note that \( d = d_i \)) but since all fibers of \( f \) have dimension exactly \( d \), the fibers of \( f \) and \( f_i \) must coincide which implies \( X_i = X \) so \( X \) is irreducible.

**Linear systems**

The main observation underlying hyperplanes is that they are effective divisors equivalent to each other - and they form a complete class of such effective divisors. These classes can be characterized in terms of varieties.

Assume \( X \) is smooth over an algebraically closed field \( k \). This means it is regular and the notions of Weil and Cartier divisor coincide. Now let \( D_0 \) be a divisor on \( X \) and \( \mathcal{L} \cong \mathcal{O}(D_0) \) be the line bundle associated to it.
Lemma. The set of all effective divisors on $X$ that are linearly equivalent to $X$ is identified with $(\Gamma(X,\mathcal{L}) - \{0\})/k^*$, via assigning each global section a divisor on $X$.

Proof: If $s$ is a global section of $\mathcal{L}$ then embedding $\mathcal{L}$ into $\mathcal{H}$ will make $s$ into a rational function $f \in K$. We immediately see that the divisor associated to $f$ is generated by $\{p\}$ finite-dimensional projective space equivalent to some divisor $D$. $k$ is a global invertible section of $L$ a global section of $s$ corresponds to a global section $v$ of $V$. The properties:

- $V$ in some $P$ is the same as injecting a vector bundle into $V$. Such a linear system defines a morphism to projective space. Indeed, recall that mapping into projective space $L$ projects inside $\ast = L$. We have such a map since $P$ is regular at every point.

Conversely if $(D) = (f) + D_0$ then we reverse the above process and see that $f$ as a global section of $\mathcal{H}$ is actually a global section of $\mathcal{L} \hookrightarrow \mathcal{H}$ as well.

Finally, two divisors will be the same if and only if the corresponding ratio $f/f'$ has associated zero divisor i.e. is a global invertible section of $X$. However projective varieties over algebraically closed fields have global sections equal to $k$, which finishes the proof.

We can now define what linear systems are.

Definition. A complete linear system of a nonsingular projective variety is the set of all effective divisors linearly equivalent to some divisor $D_0$. It is denoted by $|D_0|$ and corresponds via the above lemma to the closed points of finite-dimensional projective space $\mathbb{P}\Gamma(X,\mathcal{L})$ for $\mathcal{L} = \mathcal{O}(D_0)$. A linear system is a subset of a complete linear system, corresponding to a linear subspace of $\mathbb{P}\Gamma(X,\mathcal{L})$. In other words it consists of all divisors associated to the vectors in some $V$, where $V \subseteq \Gamma(X,\mathcal{L})$. The dimension of the linear system $d$ is its dimension as a projective space i.e. $dim_k V - 1$.

Definition. A point $P \in X$ is called a base point of a linear system $d$ if it satisfies any of the equivalent properties:

1) It is in $Supp D$ for all $D \in d$.
2) $V$ projects inside $m_P\mathcal{L}_P$ in the local ring of $P$.

In particular, a linear system is base point free if and only if $\mathcal{L}$ is generated by the global sections in $V$.

Such a linear system defines a morphism to projective space. Indeed, recall that mapping into projective space is the same as injecting a vector bundle into $V^* \otimes \mathcal{O}_X$ - or equivalently constructing a surjection of vector bundles $V \otimes \mathcal{O}_X \rightarrow \mathcal{L}$. We have such a map since $V \subseteq \Gamma(X,\mathcal{L})$ and the condition that $\mathcal{L}$ is generated by the global sections exactly implies that this map is a surjection. Moreover, hyperplanes of $\mathbb{P}V^*$ correspond to elements of $d \cong (V - \{0\})/k^*$ - more precisely if $D \in d$ corresponds to a vector $v \in V$ (defined up to multiplication by scalars), then there is a unique hyperplane $H$ in $\mathbb{P}V^*$ corresponding to $v$ and the ideal sheaf $D$ is precisely $X \otimes_{\mathbb{P}V^*} H$ - this can be easily shown locally, but apparently it was also done in class last semester.

Now, for Bertini’s theorems themselves.

Theorem. Let $X$ be a smooth closed subscheme of $\mathbb{P}^n k$, where $k$ is an algebraically closed field (we do not need characteristic 0 for this version). Then there is a hyperplane $H \subseteq \mathbb{P}^n k$ not containing $X$, such that the scheme $H \cap X$ is regular at every point.

Proof: Hyperplanes can be characterized as closed points of $Gr^{n-1}V \cong \mathbb{P}^n V^*$ - so in the following we will work with $\mathbb{P}^n V^*$ as the source of hyperplanes.

For a closed point $x \in X$ consider the set $B_x$ consisting of hyperplanes $H$ that fail the conclusion of the theorem at $x$ i.e. $H \supseteq X$ or $X \not\supseteq X$ but $x \in H$ and $H \cap X$ is not smooth at $x$. $B_x$ will actually form a linear system. Indeed, a hyperplane $H$ is defined by a vector $f \in V$. Let’s fix $f_0$ such that $f$ is not in the hyperplane determined by $f_0$ i.e. $x(f_0) \neq 0$. We can then define a map $\phi_x$ from $V$ to $\mathcal{O}_{x,X}/m_x^2$ by sending $f$ to the image of the regular function $f/f_0$ modulo $m_x^2$. Then the scheme $H \cap X$ is defined at $x$ by the ideal generated by $f/f_0$ and so we see that the local ring will be non-regular if and only if $\phi_x(f) \in m_x^2$, and if $\phi_x(f) \not\in m_x$ implies $x \not\in H \cap X$. Note that if $\phi_x(f) = 0$ then $X \subset H$. It follows that $B_x$ corresponds to $f \in Ker \phi_x$ which shows that $B_x$ is indeed a linear system. Moreover, $\psi_x$
is surjective because \( m_x \) is generated by sections in \( V \) as \( x \) is a closed point. And since \( \dim \mathcal{O}_x/m_x^2 = \dim X + 1 \) we conclude that \( \dim \ker(\psi_x) = \dim V - \dim X = n - r \) so that \( B_x \) has dimension \( n - r - 1 \).

Consider now the complete linear system \( |H| \) as the projective space \( \mathbb{P}^n \), and form the subset \( B \subset X \times |H| \) consisting of all pairs \( x, H \in B_x \) where \( x \) is a closed point. This defines a classical variety, and therefore via the correspondence between varieties and projective schemes \( B \) is the set of closed points of a closed subset of \( X \times |H| \).

We denote this subset by \( B \) too, and give it the reduced induced scheme structure. As \( r < n \), \( B_x \) is non-empty for every \( x \) so that the projection \( B \to X \) is surjective on all closed points so it is surjective (once again a subvariety is completely determined by its closed points). The fiber is a projective space of dimension \( n - r - 1 \). It follows that \( B \) is irreducible of dimension \( n - 1 \) and hence the dimension of its projection to \( |H| \) is at most \( n - 1 \), in particular it cannot be everything which finishes the problem.

Even more, as \( X \) is projective, the base change map \( X \times |H| \to |H| \) is proper so the image is closed in \( H \), thus the set of hyperplanes satisfying the condition of the problem is an open dense subset of \( |H| \).

In characteristic zero, the problem can be generalized to arbitrary linear systems without base points - every element \( D \in \mathfrak{d} \) corresponds to a closed subscheme of \( X \) determined by the line bundle \( \mathcal{L}(-D) \). This is consistent with our previous notation.

**Theorem.** Let \( X \) be a smooth projective scheme of finite type over an algebraically closed field \( k \) of characteristic 0. Let \( \mathfrak{d} \) be a linear system without base points. Then for almost every element of \( \mathfrak{d} \) (i.e. on open dense subset of \( \mathfrak{d} \) considered as projective space), the closed subscheme associated to it is nonsingular.

**Proof:** The linear system \( \mathfrak{d} \) defines a morphism \( f : X \to \mathbb{P}^n \). Now consider the action of \( PGL(n) \) on \( \mathbb{P}^n \) - it makes \( \mathbb{P}^n \) into a homogeneous space. Take \( H \) a hyperplane of \( \mathbb{P}^n \) and let \( h \) be the inclusion map. According to a previous theorem, for almost all \( \sigma \in PGL_n(k) \), \( X \times_{\mathbb{P}^n} H^\sigma \) is non-singular. It remains to observe that the subspaces \( X \times_{\mathbb{P}^n} H^\sigma \) correspond precisely to elements of the linear system \( \mathfrak{d} \).

04/0/2010

**Theorem.** Let \( k \) be a perfect field. A scheme \( X \) of finite type over \( k \) is smooth if and only if it’s regular.

**Proof:** Let’s prove first that smooth implies regular.

We claim that the map \( X \otimes_k \overline{k} \to X \) is faithfully flat, which would imply the conclusion as regularity is transferred by faithfully flat maps as shown before.

Indeed, it is flat because \( \overline{k} \) is free over \( k \). Also for every prime ideal \( p \) of \( X \) its preimage is \( Spec(k_p \otimes k\overline{k}) \) - now the latter is a direct sum of fields isomorphic to \( \overline{k} \) because \( k_p \) is separable. In particular, it has prime ideals so that the map is surjective.

Now let’s prove the other direction. First, as \( Spec(k_p \otimes_k \overline{k}) \) is a direct sum of fields, we readily conclude what the prime ideals are - in fact they must be equal to the maximal ideals as the residue ring is a quotient of a direct sum of fields i.e. a direct sum of fields which can be integral if and only if it is a field. So then we deduce that no two points above \( p \) contain each other, thus the dimension of every point in \( X \otimes_k \overline{k} \) equals the dimension of its image in \( X \) (in fact we have stated this fact in class for faithfully flat maps).

We now have the correct dimension, so it is enough to show that \( \Omega_{X \otimes_k \overline{k}/\overline{k}} \) is locally free of the same rank as \( \Omega_{X/k} \). In fact, we claim \( \Omega_{X \otimes_k \overline{k}/\overline{k}} \cong \Omega_{X/k} \otimes_k \overline{k} \), at least locally - this will do the job because by smoothness the latter is isomorphic to \( m/m^2 \) which is free over \( k_m \) of the correct rank.

Indeed, using the universal property it is enough to prove that \( A \otimes_k \overline{k} \) derivations into \( M \) are the same as \( A \) derivations into \( M \), where \( M \) is an \( A \)-module. But this is true: starting with an \( A \otimes_k \overline{k} \) derivation into \( M \) we produce an \( A \)-derivation into \( M \) by restriction, and conversely from an \( A \) derivation into \( M \) we produce an \( A \otimes_k \overline{k} \) derivation into \( M \) by tensoring up. □

**Proposition.** (PSET) Let \( X \) be an integral scheme of finite type over a field \( k \), let \( k \subset k' \) be a field extension, denote \( X' = X \times_k k' \)
a) If $X$ is generically smooth then $X'$ is locally integral.

b) If $k'/k$ is a finite separable extension, then $X'$ is always locally integral.

Proof: a) First, a lemma: Let $p$ be a polynomial in $k[t_1, \ldots, t_n]$. Assume that $q$ is a factor of $p$ in $k'[t_1, \ldots, t_n]$. Then $q$ has all coefficients algebraic over $k$ - i.e. $q \in k[t_1, \ldots, t_n] -$ up to changing it by a constant.

Proof: Let's perform the substitution $t_i = x^{n_i}$ such that all monomials appearing in $p, q, r = x^q$ will be mapped to monomials of different degrees. This immediately reduces the problem to the case $n = 1$. In that case the problem is trivial as $p$ factors into linear factors in $k[t_1, \ldots, t_n].$

A refinement of the lemma is as follows: let $p$ be a polynomial in $A[t]$ where $A$ is integral of finite type over $k$. It $q$ is a factor of $p$ in $A \otimes_k k'$ then $q$ (up) to constants is in $A \otimes_k k[t].$

Proof: Let $pr = q$. By surjecting $k[x_1, \ldots, x_n]$ we lift $p, q, r$ to $P, Q, R$ and we clearly can change $R$ modulo the kernel such that we still have $P, Q, R$. The refinement now follows immediately from the previous lemma.

By Noether Normalization, we may assume that $A = k[x_1, \ldots, x_m]$ adjoined $y_1, \ldots, y_k$ where $y_i$ are algebraic over $k[x_1, \ldots, x_m].$

Choose an affine $Spec(A)$ with $A$ integral, so that $A_f$ is smooth. Note that $f$ will be a non-zero divisor on $A \otimes_k k'$ because as an $A$-module the latter is free (since $k'$ is a free $k$-module).

By Noether Normalization, $A = k[t_1, \ldots, t_n]$ adjoined $y_1, \ldots, y_k$ which are algebraic over $k[t_1, \ldots, t_n]$. Let's prove $y_i$ are separable over $k'[t_1, \ldots, t_n]$. Indeed, if they were not then for example the minimal polynomial $f$ of $y_1$ (which is the same as the minimal polynomial over the fraction field by Gauss lemma) would have a double factor which would then be over $k[t_1, \ldots, t_n]$ by the lemma. Let's write $f = \prod p_i$ where $p_i$ are powers of distinct irreducibles - in particular they are pairwise coprime and by localizing at an appropriate element of $k[t_1, \ldots, t_n] \subset A$ we may assume any two generate the unit ideal together - we may now also localize at $f$. Then the Chinese remainder Theorem applies which says that $A' = k[t_1, \ldots, t_n]$ adjoined $y_1$ (and localized) is isomorphic to $\oplus k[t_1, \ldots, t_n, t_{n+1}]$. But if $p_i$ is not irreducible but a higher power, by tensoring up with $\bar{k}$ we produce a nilpotent element (namely the root of $p_i$ modulo $p_i$). This is impossible as regular implies reduced.

Let's now prove by induction on $k$ that $k'[t_1, \ldots, t_n]$ adjoined $x_1, \ldots, x_k$ is a disjoint union of integral domains. The basis is true as $f_1$ may decompose into pairwise coprime polynomials irreducible in $k'[t_1, \ldots, t_{n+1}]$ - and then we can apply the Chinese Remainder theorem (see the last paragraph in part b) for details). For the induction step, take $f_{k+1}$ the minimal polynomial of $x_{k+1}$ over one of the direct summands of $k[t_1, \ldots, t_n]$ adjoined $y_1, \ldots, y_k$. We claim it is separable in the new ring as well - by the same reasoning as above, otherwise we would produce something which is nilpotent in $A \otimes_k \bar{k}$. Then it would also be a nilpotent in $A_f \otimes_k \bar{k} = (A \otimes_k \bar{k})_f$ and it would be non-zero since $f$ is be a non-zero divisor on $A \otimes_k \bar{k}$ because as an $A$-module the latter is free (since $\bar{k}$ is a free $k$-module). Contradiction.

b) Again let's choose an affine $Spec(A)$ with $A$ an integral domain. If $k'/k$ is finite separable, $k'$ is generated by adjoining a single element i.e. $k' = k[t]/(p)$ where $p$ is irreducible and monic in $k[t]$. Moreover since the extension is separable the discriminant of $p$ is non-zero, in particular even regarded as a polynomial in $A[t]$ it has no multiple factors. Let's factor $p = f_1 \ldots f_k$ in $A[t]$. We claim that $f_i$ generate the unit ideal as $A[t]$ is a UFD and PID (at least when it comes to monic polynomials) hence the Chinese Remainder theorem applies and $A \otimes_k k' = A[t]/(p) \cong A[t]/f_i$ so it is a direct sum of integral domains - which shows that $Spec(A) \otimes_k k'$ is a disjoint union of integral schemes. Hence $X \otimes_k k'$ is locally integral. We may now write $X \otimes_k k'$ as a disjoint union of integral schemes as follows. Around every point, consider an integral open affine. Now observe that if two integral affines meet they meet in their common generic point which means that the relation of intersecting is transitive and so we can group all affines intersecting each other into an integral scheme, which provides the desired decomposition.

It remains to verify that $f_i$ indeed generate the unit ideal in $A[t]$. A priori, by division algorithm, we know the ideal they generate contains an ideal $aA[t]$ where $a$ is some linear combination of the polynomials obtained from the division algorithm. But observe that all the coefficients of these polynomials are in $\bar{k}$ - according to the lemma from part a). It follows that performing the same division algorithm over $\bar{k}$ we will end up with something in $\bar{k}$ so $a$ is integral over $k$. But everything integral over $k$ is invertible - every element divides the free coefficient of a polynomial it satisfies and we have a polynomial whose coefficient over $k$ is non-zero so invertible. □
Proposition. Let \( X' \) be a smooth scheme over a base \( S \), and let \( X \hookrightarrow X' \) be a closed embedding with the sheaf of ideals \( J \). Assume that
\[
\overline{\mathcal{J}}_{X,X'/S}^2 \to \Omega_{X'/S} |_{X} \to \Omega_{X/S} \to 0
\]
is a short exact sequence of vector bundles. Then \( X \) is smooth over \( S \).

Consider a scheme \( X \) over a field \( k \), which happens to be a subscheme of \( \mathbb{A}^m_k \), cut out by the ideal \((f_1, \ldots, f_m)\) such that \((df_1)_x, \ldots, (df_n)_x \in (\mathcal{O}_{\mathbb{A}^m_k})_x\) are linearly independent for each \( x \in X \). Then \( x \) is smooth over \( k \) of dimension \( n \), because the exact sequence of differentials of a closed embedding means that \( \Omega_{X/k} \) is locally free of \( n \), and also every irreducible component of \( X \) has dimension at least \( n \) since it’s cut by \( m - n \) elements.

We now introduce the concept of etaleness which is a loose analog of local isomorphisms. 

Namely if \( f: X \to Y \) is a morphism of smooth schemes over \( S \), there are analogues to the following theorems in differential geometry:

- (Implicit function theorem) If \( f \) induces an isomorphism \( f^*\Omega_{Y/S} \to \Omega_{X/S} \) then \( f \) is etale.
- (Inverse function theorem.) If \( f \) induces an injective bundle map \( f^*\Omega_{Y/S} \to \Omega_{X/S} \) then locally the map factors through an etale map: \( X \xrightarrow{\text{etale}} \mathbb{A}^n_S \to Y \)
- (Theorem about immersions) If \( f \) is a closed embedding then locally we can construct a diagram of the type:

\[
\begin{array}{ccc}
X & \xrightarrow{\text{etale}} & Y \\
\downarrow \text{etale} & & \downarrow \text{etale} \\
\mathbb{A}^n_S & \to & \mathbb{A}^m_S
\end{array}
\]

Definition. A map \( X \to Y \) is etale if it is smooth of dimension 0, i.e. for any \( x \in X \) with image \( y \in Y \), \( \mathcal{O}_{X,x} \) is flat over \( \mathcal{O}_{Y,y} \) and \( \Omega_{X,x/Y,y} = 0 \).

Definition. A morphism \( X \to Y \) is called unramified if \( \Omega_{X/Y} = 0 \).

Note that etale morphism are unramified but the converse is not true, for example any closed embedding is an unramified morphism.

Lemma. Let \( A \to B \) be a local map of local rings. It is unramified (i.e. the corresponding map \( \text{Spec}(B) \to \text{Spec}(A) \)) is if and only if:
- i) \( k_B/k_A \) is separable
- ii) \( m_A \cdot B = m_B \)

Proof: By Nakayama it is enough to inquire when
\[
\Omega_{X/Y} \otimes_B k_B = 0
\]
i.e.
\[
(\Omega_{X/Y} \otimes_A k_A) \otimes_{k_A} k_B = 0
\]

Consider the following Cartesian diagram
\[
\begin{array}{ccc}
\text{Spec}(B/m_A B) & \to & \text{Spec}(B) \\
\downarrow & & \downarrow \\
\text{Spec}(k_A) & \to & \text{Spec}(A)
\end{array}
\]
so \( \Omega_{B/A} \otimes_A k_A \cong \Omega_{C/k} \) where \( C = B/m_A B \).
So we are reduced to the case when $A$ is a field, i.e. a map $k \to C$.

We want to know when $\Omega_{C/k} = 0$ i.e. $C$ is smooth of dimension 0 over $k$. This is equivalent to being regular, with no nilpotents. Since an algebra of finite type over $k$ with no nilpotents is a direct sum of fields, we get $C$ being a direct sum $\oplus k'$ where $k'/k$ is separable - which is equivalent to what we need.

A more direct proof is obtained by looking at $k \to C \to k_C$ which yields $m_C/m_C^2 \to \Omega_{C/k} \otimes_{C} k_C \to \Omega_{C/k} \to 0$

By Nakayama, $\Omega_{C/k} = 0 \iff \Omega_{C/k} \otimes_{C} k_C = 0$, which is equivalent to $m/m^2 = 0$ (i.e. $m = 0$ which is condition ii) ) and $\Omega_{C/k} = 0$ (i.e. $k_C/k$ separable which is condition i) ). Indeed, if $\Omega_{C/k} = 0$ then $k_C/k$ is separable hence from a previous proposition $m_C/m_C^2 \xrightarrow{\sim} \Omega_{C/k} \otimes_{C} k_C = 0$. Conversely, if both outer objects are zero so is the middle one. □

Examples:

1) Consider the multiplicative group $G_m = \text{Spec}(k[t, t^{-1}])$ and the map $G_m \xrightarrow{x \mapsto x^n} G_m$ given by the map $k[t, t^{-1}] \to k[t, t]^{-1}$ defined by $t \to t^n$.

We claim that the map is etale (or unramified, in this case they are equivalent since the map is always flat) if and only if $p \not| n, p = \text{char}(k)$.

Indeed, $\Omega_{k[t, t^{-1}]/k[t^n, t^{-n}]}$ is generated over $k[t, t^{-1}]$ by a single element $dt$ subject to $d(t^n) = nt^n dt = 0$ and $nt^n - 1$ is invertible for $p \not| n$ and zero otherwise.

2) Consider the additive group $G_a = \text{Spec}(k[t])$, $\text{char}(k) = p \neq 0$ and we take the map $G_a \xrightarrow{x \mapsto x^p - x} G_a$ given by $k \xrightarrow{id} k, t \to t^p - t$.

This map is always etale (and a group homomorphism). Indeed, $(t^p - t) = 0$ yields $pt^p - 1 dt - dt = 0$ i.e. $dt = 0$. The map is also flat as $k[t]$ free as a module over $k[t^p - t]$ - a basis is $1, t, \ldots, t^{p-1}$.

Lemma. Consider a map $f: X \to Y$ of schemes smooth over a base $S$. Then it is etale if and only if $f^*\Omega_{Y/S} \to \Omega_{X/S}$ is an isomorphism.

This follows immediately from the criterion of $f$ being smooth of dimension 0 as $f^*\Omega_{Y/S} \to \Omega_{X/S}$ being an injection of line bundles of dimension 0.

For example, in the above examples 1) and 2), $\Omega_{X/k}$ is $\mathcal{O}_X \cdot dt$ and the map $f^*\Omega_{X/k} \to \Omega_{X/k}$ is just the usual differentiation, given by $dt \to nt^n - 1 \cdot dt$ and $dt \to (pt^p - 1)dt = -dt$.

A third example would be that of an elliptic curve: the multiplication by $n$ in $E$ turns out to be etale if and only if $n$ is not divisible by the characteristic of the field.

Proposition. Let $f: X \to Y$ be etale at $x$, $y = f(x)$. Assume that $k_x = k_y$. Then $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces an isomorphism of their completions

$$\hat{\mathcal{O}}_{Y,y} \cong \hat{\mathcal{O}}_{X,x}$$

Proof: let’s rephrase, with a map $A \to B$ of local rings. We know that $m_B = B \cdot m_A$ by etaleness. It’s enough to show that the map $A/m^n_A \to B/m^n_B$ is an isomorphism i.e. $A/m_A^n \to B/m_B^n \cdot B$ is an isomorphism, for any $n > 0$ (note that it holds for $n = 1$ by $k_A = k_B$).

This map is surjective: indeed, as $B/m_B^n$ is an extension of $m_B^k/k_B^{k+1}$ which are finite $A$-modules (in fact finite $k_A = k_B$-modules), so we can apply Nakayama: $A/m_A^n \otimes_A k_A \to B/m_B^n \otimes_A k_A$ becomes $k_A \to B/m_B^n \cdot B$ i.e. $k_A \to k_B$ which is a surjection.

Now we show injectivity. By replacing $A, B$ with $A/m_A^n, B/m_B^n$ (note that is is still flat as we just tensor both rings with $A/m_A^n$) we have to prove that if $A \to B$ is an etale map of local Artinian rings with $k_A \cong k_B$, then $A \cong B$.

Indeed, $B$ is finite as a module over $A$ as it’s an extension of $m_B^k/m_B^{k+1}$, but then $B$ being flat over $A$ is equivalent to $B$ being free over $A$ - in particular the map will be injective (if $a \in A$ is in the kernel then the basis $b_1, \ldots, b_m$ of $B$ over $A$ is killed by $a$ which is impossible). This finishes the proof (surjectivity was shown before). □
Proposition. If

then $g$ is etale as well.

Proof: the map $g^*\Omega_{X_2/S} \to \Omega_{X_1/S}$ is the map $0 \to 0$ so is an isomorphism. (Note: in the same way, if $X_2$ is etale over $Y$ and $X_1$ is smooth over $Y$ then $X_1$ is smooth over $X_2$ of the same relative dimension). □

Recall than if $X \to Y$ is smooth, certain properties of $X$ transfer to $Y$. When the map is etale, properties of $Y$ transfer to $X$ as well:

Theorem. Let $X \to Y$ be etale. Then if $Y$ is

i) reduced ii) regular iii) $R_n$ iv) $S_n$ v) locally factorial

the same is true for $X$.

Proof: all properties are local so we can assume that we have a map $A \to B$ of regular local rings. Properties i)-iv) are cohomological (why is being reduced cohomological?). Ten it suffices to prove the following identities:

$$Tor_i(k_A, k_A) \otimes k_B \xrightarrow{\sim} Tor_i(k_B, k_B)$$

$$Ext^i(k_A, A) \otimes k_B \xrightarrow{\sim} Ext^i(k_B, B)$$

$$Ext^i(k_A, k_A) \otimes k_B \xrightarrow{\sim} Ext^i(k_B, k_B)$$

Choose a (finite) projective resolution $P^\bullet$ of $k_A$, then because $B$ is flat over $A$ $P^\bullet \otimes_A B$ is a projective resolution of $k_B$ as a $B$ module (note that $k_A \otimes B = k_B$ as the map is unramified). This shows the first identity.

The third identity will follow from the second using a resolution of $k_B$ and the long exact sequence of $Ext$ (noting that the $\to$ map exists naturally). So it suffices to show the second identity.

Indeed, the LHS is the cohomology of $\text{Hom}_A(P^\bullet, A) \otimes k_B$ which as $P^\bullet$ are finitely generated is the same as $\text{Hom}_A(P^\bullet, B) \cong \text{Hom}_B(P^\bullet \otimes B, B)$ whose cohomologies compute the RHS since $P^\bullet \otimes B$ is a projective resolution of $k_B$. □

Theorem. $X \to S$ is smooth if and only if locally on $X$ it factors as $X \xrightarrow{\text{etale}} S^n \to S$

Proof: one direction is easy as composition of smooth morphisms is smooth. For the other, locally on $X$ embed it into some $k_S^n$ - then as we know $X$ (locally) can be generated by $f_1, \ldots, f_m-n \in A[t_1, \ldots, t_m]$ such that $(df_1)_x, \ldots, (df_{m-n})_x$ are linearly independent at all points $x \in X$. On a Zariski open, we can some selection of $t_i$ - say $t_1, \ldots, t_n$ such that $(df_1)_x, \ldots, (df_{m-n})_x, (dt_1)_x, \ldots, (dt_n)_x$ form a basis of the differential ring at every point.

We then use $t_1, \ldots, t_n$ to define (locally) the map $X \to k^n_S$. We claim the map is etale - it is easy to see that $f^*$ induces an isomorphism $f^*\Omega_{Y/S} \to \Omega_{X/S}$. □

As shown before, this theorem implies that a finitely generated extension $K.K$ is separably generated if and only if $\text{dim}_K(\Omega_{K/k}) = \text{tr.deg.}(K/k)$.

Proposition. (PSET) Let $X, Y$ be schemes smooth over $X$. Let $f : X \to Y$ be a map.

i) $f$ is etale if and only if the map $f^*(\Omega_{Y/S}) \to \Omega_{X/S}$ is an isomorphism.

ii) If $X$ and $Y$ are etale over $S$, then $f$ is automatically etale.

iii) If $X$ is etale and separated over $S$, the diagonal map $X \to X \times_S X$ is an embedding of a union of connected components.
Proof: i) Follows immediately from theorem 3.2.1.c) from the notes: we have the exact sequence \( f^*\Omega_{Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0 \). If the map is etale then the last term is 0 and the first map is an injection whose cokernel is 0 so \( f^*\Omega_{Y/S} \to \Omega_{X/S} \) is an isomorphism. Conversely, if \( f^*\Omega_{Y/S} \to \Omega_{X/S} \) is an isomorphism then its cokernel \( \Omega_{X/Y} \) is 0 and \( f^*\Omega_{Y/S} \to \Omega_{X/S} \) is an injective bundle map, so the map is smooth of relative dimension \( \text{rank}(\Omega_{X/Y}) = 0 \) meaning the map is etale.

ii) Follows from i) as then the map is 0 \( \to 0 \) which is an isomorphism.

iii) The map \( X \times_S X \to X \) is the base change of the etale map \( X \to S \) hence it is etale. It follows that the map \( X \times_S X \to S \) is also etale being the composition of two etale maps, and then using part b) we deduce that the intermediate map \( X \to X \times_S X \) is also etale.

But this map is also a closed embedding. It remains to prove that an etale closed embedding is the inclusion of a union of connected components. Indeed, such a map should be a closed map but also an open map because it is a flat map of finite type. In particular it must send every closed and open set in \( X \) to a closed and open set in \( X \otimes_S X \) - so it sends connected components to connected components, and we are done.

It remains to show the claim that a flat morphism of finite type is open.

The image of an open set is constructible from Chevalley’s theorem, and we need to show it’s stable under generization - i.e. if \( q \) is a point such that \( p \) is in the closure of \( q \) and \( p \) is in the image, the so is \( q \). Localize at \( p \) and its preimage \( p' \) then we get a flat local map of local rings - and therefore it must be faithfully flat, which implies that it is surjective. In particular \( q \) which corresponds to an ideal in the localization must be in the image which concludes the proof. □

**Proposition. (PSET)** If \( X \to Y \) is smooth, and \( Y \) is i) regular ii) \( R_n \) iii) \( S_n \) iv) reduced v) normal, then so does \( X \).

Proof: Normality follows from ii) and iii) as it is equivalent to \( R_0 \) and \( S_1 \). So does being reduced, because it is equivalent to being \( R_0 \) and \( S_1 \). Let’s prove this.

Assume that the ring is reduced. Localizing at a minimal prime produces a reduced ring with a single prime ideal, which then must be 0 since anything in that ideal is nilpotent as it belongs to all the prime ideals - thus the ring is a field, which implies \( R_0 \). For \( S_1 \), localize at a prime of height at least 1, thus we have a reduced local ring of non-zero dimension. It having dimension at least 1 is equivalent to it having no elements \( a \) annihilated by the maximal ideal. Indeed, such an \( a \) must be non-invertible so in the maximal ideal, implying \( a^2 = 0 \) but the ring is reduced contradiction.

Conversely, assume that the ring is both \( R_0 \) and \( S_1 \). We claim that it is reduced. Say \( a \) is nilpotent and let \( J \) be its annihilator. By a standard argument done before, we can assume \( J \) is maximal and prime. Then we can localize at that prime and obtain \( a \) which is annihilated by the maximal ideal. In particular, this implies that the depth of the localized ring is 0 - which by \( S_1 \) implies that the ring has dimension 0, but then \( a \) is a non-zero nilpotent which is impossible as a regular local ring of dimension 0 is a field and has no nilpotents.

It suffices to show i), ii) and iii) then.

We can factor a smooth morphism \( X \to Y \) into \( X \to \mathbb{A}^nY \to Y \) where \( X \to \mathbb{A}^nY \) is etale. So it suffices to prove the proposition for etale maps and for maps \( \mathbb{A}^nY \to Y \).

The etale case was done in class, and here’s how we did it.

Since all properties are local, let’s choose a point \( x \in X \) and let \( y \in Y \) be its image. By looking at stalks, we get a flat local map of local rings \( A \to B \).

Consider the residue fields \( k_A, k_B \) and let’s choose a projective resolution of \( k_A: \ldots \to P_1 \to P_0 \to k_A \)

Because the map is flat, \( \ldots \to P_1 \otimes_A B \to P_0 \otimes_A B \to k_A \otimes_A B \) is a projective resolution of \( k_A \otimes_A B = k_B \) (because the map is etale so unramified).

Now assume \( A \) is regular - then it is of finite cohomological dimension. It follows that the resolution above is finite and thus \( Tor_i(k_B, k_B) \) is eventually zero, which implies that the ring has finite cohomological dimension so by Serre’s theorem is regular.

128
Assume $A$ is $R_n$. This either means that the Krull dimension of $A$ is less than $n$, or that $A$ is regular. The first possibility implies that the Krull dimension of $B$ is also less than $n$ - because in fact they are equal (according to proposition 1 for example, as $B \otimes_A k_A$ is a direct sum of fields since $B$ is flat over $A$ and finite thus free, so has dimension 0). The second possibility was taken care of before.

For the $S_n$ case, like above we assume that $A$ has Krull dimension at least $n$, and then $\text{depth}(A,m_A) \geq n$ meaning $\text{Ext}^i(k_A,A) = 0$ for $i \leq n$. But using the resolution above, we see that $\text{Ext}^i(k_B,B) = \text{Ext}^i(k_A,A) \otimes_A B = \text{Ext}^i(k_A,A) \otimes_{k_A} k_B$ (because $\text{Hom}_A(P^*,A) \otimes_A B = \text{Hom}_A(P^*,B) = \text{Hom}_B(P^* \otimes_A B, B)$ as the modules in the resolution are finitely generated). This implies $B$ has depth at least $n$, as desired.

Now let’s prove the case $k^n Y \to Y$. It suffices to consider $n = 1$ and like above we look at the map $A \to A[t]$ where $A$ is a local Noetherian ring.

If $A$ is regular we show $A[t]$ is regular by copy-pasting from Matsumura. Let $q$ be a prime ideal of $A[t]$ lying above $p$ a prime ideal of $A$, then we can localize at $A - p$ and assume that $p$ is actually the maximal ideal of $A$. Then $A[t]/pA[t] = k_A[t]$ and $q$ is a prime ideal of that which can be either 0 or generated by an irreducible polynomial, and this shows that $q = p \cdot A[t]$ or $q$ is generated by $p$ and some (irreducible) monic polynomial $f(t)$. Let $d$ be the height of $p$ so that $p$ is generated by $d$ elements by regularity, meaning that $p \cdot A[t]$ is generated by $d$ elements and $(p \cdot A[t], f \cdot A[t])$ is generated by $d + 1$ elements. On the other hand, it’s clear that these ideals have height at least $d$ and $d + 1$ (the chains can be “imported” from $A$). Since in general, the least number of generators is greater than or equal to height, we have equality which implies that $A[t]_q$ is regular.

The $R_n$ case is similar: if $A$ is $R_n$ then every prime $q$ of $A[t]$ lies above a prime $p$ of heigh at most $n$, in particular by localizing we are basically in the previous case.

I don’t know how to do the $S_n$ case. □

**Proposition. (PSET)** If $X \to X'$ is a closed embedding given by a nilpotent ideal, then the categories of schemes etale over $X$ and etale over $X'$ are canonically equivalent.

Proof: Assume that $\overline{Y} \to Y$ is a closed embedding given by a nilpotent sheaf of ideals. The map of categories in one direction is given by base change: if we have $X \to Y$ etale then $\overline{X} \to \overline{Y}$ is etale where $\overline{X} = X \otimes_Y \overline{Y}$. Note that $\overline{X} \to X$ is a closed embedding given by a nilpotent sheaf of ideals as well: on affines we pass from $A$ to $A \otimes_B B/I \cong A/I$ where $I$ is nilpotent and if $I^k = 0$ then $(IA)^k = 0$.

Let’s prove that it is fully faithful.

Assume that $\overline{X}_1 \to \overline{X}_2$ is a map of etale schemes over $\overline{Y}$.

We consider the following diagram

$$
\begin{array}{ccc}
\overline{X}_1 & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & Y
\end{array}
$$

where the map $\overline{X}_1 \to X_2$ is the composite map $\overline{X}_1 \to \overline{X}_2 \to X_1$.

Because $X_2 \to Y$ is etale thus formally etale, there is a unique extension $X_1 \to X_2$ that makes the diagram commute which implies our claim.

Now it remains to prove essential surjectivity. First, we claim it is enough to do it locally. Indeed, say we have an etale map $\overline{X} \to \overline{Y}$ and we want to find $X$. Assume we have found open subschemes $Y_i$ of $\overline{Y}$ - which then come from (non-unique) open subschemes $Y_i$ of $Y$ such $\overline{X}_i$, the restriction of $\overline{X}$ to the preimage of $Y_i$ extends to a commutative diagram

$$
\begin{array}{ccc}
\overline{X}_i & \longrightarrow & X_i \\
\downarrow & & \downarrow \\
Y_i & \longrightarrow & Y_i
\end{array}
$$

129
This gives maps $X_i \rightarrow Y$. Also if we restrict to $Y_i \cap Y_j$ then the preimages of $Y_i \cap Y_j$ inside $X_i$ and $X_j$ are canonically isomorphic - because their pullbacks to $\overline{Y}_i \cap \overline{Y}_j$ are $\overline{X}_i \cap \overline{X}_j$ and we use the result which we just prove above. Hence They glue to a scheme $X$ that maps to $Y$ and receives a map from $Y$, the scheme being also etale over $Y$ as it is locally so.

It remains to prove the assertion locally. We can choose a small enough open affine subscheme $\overline{Y}_i$, coming from an open affine subscheme $Y_i$ of $Y$, such that $\overline{X}_i$ embeds into $\mathbb{A}^n \overline{Y}_i$, and moreover the ideal defining it is generated by $n$ elements $\overline{f}_1, \ldots, \overline{f}_n$ whose images in $\Omega_{\mathbb{A}^n \overline{Y}_i/\overline{Y}_i}$ are independent - which as $\Omega_{\mathbb{A}^n \overline{Y}_i/\overline{Y}_i}$ is free on $dt_1, \ldots, dt_n$ means that $\text{det}(\partial f_i/\partial x_j)$ is an invertible element of $O_Y$. We now lift $\overline{f}_i$ to global sections $f_i$ $\mathbb{A}^n Y_i$ and take $X_i$ to be the subscheme of $\mathbb{A}^n Y_i$ determined by $f_i$. We immediately see that $X_i$ is smooth of relative dimension 0 over $Y_i$ hence it is etale over $Y_i$ and moreover it clearly satisfies the required conditions. □

**Proposition. (PSET)** Let $f: X \rightarrow Y$ be a finite morphism between Noetherian schemes. We say that $f$ is radial if i) it induces a bijection at the level of points and ii) for any $x \in X$ the field extension $k_x/k_{f(x)}$. Then a base change of a radial morphism is radial.

Proof: Assume that $X \rightarrow S$ is radial, and let $Y \rightarrow S$ be another map with $Z = X \times_SY$ the cartesian product. We need to prove that $Z \rightarrow Y$ is radial.

Pointwise surjectivity is done as follows. Let $y \in Y$ be a point that maps to a point $s \in S$ and choose $x \in X$ that maps to $s$ by surjectivity. We can now choose small affines $\text{Spec}(B), \text{Spec}(C), \text{Spec}(A)$ around $y, s, x$ such that $\text{Spec}(B), \text{Spec}(A)$ map into $\text{Spec}(B)$. Then $\text{Spec}(B \otimes_C A)$ will be an open subscheme of $Z$, and we want to prove it contains a prime ideal whose contraction to $B$ is $p$ where $p$ corresponds to $y$. Let $q$ be the ideal corresponding to $x$, so that the contractions of $p$ and $q$ to $C$ coincide (let’s call this contraction $a$).

An ideal contracting to $p$ corresponds to an ideal of $k_p \otimes_B B \otimes_C A = k_p \otimes_C A$ - so it suffices to prove this ring is not zero. We can go further and tensor it up with $k_p$ producing the ring $k_p \otimes_C k_q$. It is clear that the tensor product over $C$ is equivalent to the tensor product over $k_p$ as the map from $C$ to $k_p, k_q$ factors through $C/\mathfrak{a}$ and also anything not in $\mathfrak{a}$ is invertible in $k_p, k_q$. In this case, the tensor product is non-zero as $k_p, k_q$ are free over $k_s$.

For pointwise injectivity we claim it’s enough to consider the affine case. Indeed, let’s choose $x, y, s$ as above and let $z$ be a preimage of $x$. We need to show that it is unique. Let $z’$ be another preimage. Because the original map is injective, the projection of $z$ to $X$ must be the point $x$. Now choosing $\text{Spec}(B), \text{Spec}(C), \text{Spec}(A)$ as above we conclude that both $z$ and $z’$ must be in the preimage of $\text{Spec}(B)$ and $\text{Spec}(C)$ and we know that this preimage is $\text{Spec}(B \otimes_C A)$ so we are fine with the affine case. Like above, the assertion reduces to investigating the fiber product $k_p \otimes_{k_s} k_q$ knowing that the extension $k_q/k_\mathfrak{a}$ is purely inseparable.

More concretely, we have the following lemma:

If $k’, k”$ are extensions of $k$ with $k’$ finite and purely inseparable then $k’ \otimes_k k’’$ is a local Artinian ring of dimension 0 (so it has only one prime ideal which implies injectivity).

Let’s embed $k’, k”$ into a larger field $K$ (note that all such embedding are conjugate as every $k$-automorphism fixes $k’$). We have the natural multiplication map $k’ \otimes_k k” \rightarrow K$. Let $R$ be the image and $I$ be the kernel. We claim that $I$ is the only prime ideal of the ring. First, it is prime because it is the kernel of a map into $K$ which is integral. Now let’s prove that it is nilpotent. Every element $a$ in $I$ can be written as a sum of $c_i \otimes x_i$ where $c_i$ is a chosen fixed basis of $k’$ over $k$. In particular $(c_i)^m \in k$ for some sufficiently big $m$, and this implies that $a^{k”}$ lands inside $k \otimes_k k” \cong k”$. But $k” \rightarrow K$ is injective, so the only way this can happen is if $a^{k”}$ is 0, which implies the claim.

It remains to prove that the extension is radical. With the work we’ve done, it is easy: the residue field of $z$ is the field of fractions $k’ \otimes_k k”/I \cong R$. But $R$ inside $K$ is generated with respect to $k”$ by elements of $k’$, which are purely inseparable over $k$ hence purely inseparable over $k’$ as well.

Finally, the map is finite because being finite is stable under base change.

Remark: it is not clear from above why $R$ is a field. It actually is, because we’ve proven a while ago that finite morphisms yield finite field extensions, and it was done by showing that every prime ideal in the corresponding
Artinian algebra is maximal (since an integral domain which is finite over a field is a field), and this shows that $R$ the residue ring is a field. □

**Proposition. (PSET)** Let $X$ be an irreducible scheme of finite type over a field $k$, let $k'/k$ be a field extension, and let $X' = X \times_k k'$. Then every connected component of $X'$ is irreducible.

Proof: First we claim it’s enough to do this for affine schemes. Indeed, let $Y$ be a connected component of $X'$ and say $Y_1, Y_2$ are irreducible components of $Y$. Then at any point of $Y_1 \cap Y_2$ one has a small enough affine for which we can prove the problem, meaning that both $Y_1$ and $Y_2$ contain that affine, so that $Y_1 \cap Y_2$ is actually open. On the other hand is closed contradicting the fact that $Y$ is a connected component.

So it is enough to prove it for $X$ affine.

Next, observe that if we prove it for $k/k'$ and $k'/k''$ then we can prove it for $k/k''$. This follows from working with each connected component of $X'$ which will be irreducible so the condition of the theorem holds.

Since any extension is an algebraic extension of a purely transcendental extension, it is enough to prove it for the case when the extension is purely transcendental and for the case when it’s algebraic.

In the case it’s algebraic, we claim it’s enough to show it when it’s finite. For this, we give an alternative definition of what it means to have irreducible connected components.

Namely, if $x$ and $y$ are elements in $A$ such that $xy = 0$, then there are ideals $I, J$ such that $A = I \oplus J$ and $x^n \in I, y^n \in J$ for some $n \in \mathbb{N}$. Let’s show why this is equivalent to the original definition. Observe that having irreducible connected components is the same as being a direct sum of rings with prime nilradical - every connected component must be affine because the embedding is a closed embedding so an affine map, and the spectrum of a ring is irreducible if and only if the nilradical is prime. If $A = \oplus A_i$ with this property and $x = (x_i), y = (y_i)$ satisfy $xy = 0$ i.e. $x_iy_i = 0$, then for every $i$ it means that some power of $x_i$ or $y_i$ is 0. Since there are finitely many $i$, we choose a high enough power $n$ such that $x^n$ and $y^n$ will map to distinct components $A_i$, and now it’s clear how to select $I, J$.

Conversely, let’s divide the ring into connected components yielding $A = \oplus A_i$. If $A_i$ does not have prime nilradical we find $x, y$ such that $xy = 0$ but $x, y$ are not in the nilradical. Such $x, y$ contradict the assumption because there are no $I, J$ with that property: every $I, J$ will come from $I_i, J_i$ on $A_i$, but since $A_i$ is connected, one of $I_i, J_i$ must be zero, impossible.

In this setting, say we want to show $A \otimes_k k'$ has irreducible connected components where $k'$ is an algebraic extension but not necessarily finite. Choose $x, y$ such that $xy = 0$ then they come from $A \otimes_k k''$ where $k''$ is a finite intermediate extension. We now choose $I, J$ for $x, y$ in $A \otimes_k k''$ and $I \otimes_k^e k', J \otimes_k^e k'$ will be the required ideals that will show the proof.

Since every finite algebraic extension can be obtained by a finite separable extension followed by a finite purely inseparable extension, and every finite purely inseparable extension can be obtained by successive extensions of the form $k[t]/(t^p - a)$ we have the following cases:

i) the extension is purely transcendental, $k' = k(x_i)$. In that case $B = A \otimes_k k' = A[x_i]$ localized at $k[x_i]$. If $\text{Spec}(A)$ is irreducible, the nilradical $p$ is prime so $A/p$ is an integral domain. Clearly $pB$ is nilpotent, but $B/pB$ is integral as it equals $A/p[x_i]$ localized at $k[x_i]$ and $A/p[x_i]$ is integral. In particular, $pB$ is the nilradical and it is prime.

ii) The extension is finite separable, so $k' = k[t]/(f)$ where $f$ is monic. It follows that $B = A \otimes_k k' = A[t]/(f)$ and $B/pB = (A/p)[t]/(f)$.

Over $A$, $f$ factors as a product of irreducibles $p_i$ and as it was shown in the previous problem set, we have $A \otimes_k k' = A[t]/(f) \cong \oplus A[t]/(f_i)$ and the latter are irreducible as $B_i = A[t]/(f_i)$ satisfies $B_i/pB_i = (A/p)[t]/(f_i)$ and these are integral domains. (Again, this has been done in a previous problem set)

iii) The extension is $k' = k[t]/(t^p - a)$. Then $B = A \otimes_k k' = A[t]/(t^p - a)$. If $a$ has a $p$-th root in $A$ equal to $b$ then this is just $A[t]/(t - b)^p \cong A[t]/t^p$ which is irreducible as its nilpotent ideal is generated by $p, t$ and the quotient is $A/p$ which is integral. The same argument works is $a$ has a $p$-th root in $A/p$. Otherwise, $t^p - a$ is irreducible in $A/p$: every factor of it must be a power of $(t - b)$ where $b$ is the root of $a$ in some algebraic extension of $A/p$ and then
it is \((t - b)^k\) where \(k\) is coprime to \(p\) so by algebraic manipulations we can obtain \(t - b\) so \(b\) is in the fraction field of \(A/p\) which implies that \(b\) is actually in \(A/p\).

So we obtain that \(B/pB = (A/p)[[t]]/(t^p - a)\) which is an integral domain, and this finishes the proof.

Note: we have used the fact that over an integral domain of characteristic \(p\) which is finitely generated over \(k\) \(t^p - a\) either has a root or is irreducible, if \(a \in k\). Let’s prove it. Indeed, if the polynomial is not irreducible then as above there is a root \(b\) in the fraction field, so that \(b^p = a\). But then as the polynomial is not irreducible it has a factor, whose free coefficient is \(b^k\) which is in the integral domain, and then \(b = b^k + pr = b^k a^r\) is in the integral domain as desired (note that \(a\) is invertible).

We will now discuss the functorial properties of smoothness and etaleness.

Let \(F \leftarrow G\) be a map (i.e. a natural transformation) between functors from \(Aff.Schemes^{op} = Rings\) to \(Sets\).

**Definition.** \(\phi\) is formally smooth if the following happens:

Given a closed embedding \(Spec(A) \hookrightarrow Spec(A')\) given by a nilpotent ideal \(I\), and a diagram

\[
\begin{array}{ccc}
Spec(A) & \longrightarrow & F \\
\downarrow & & \downarrow \\
Spec(A') & \longrightarrow & G
\end{array}
\]

there is a lift \(Spec(A') \rightarrow F\) that extends it:

\[
\begin{array}{ccc}
Spec(A) & \longrightarrow & F \\
\downarrow & & \downarrow \\
Spec(A') & \longrightarrow & G
\end{array}
\]

(here, by a map \(Spec(A) \rightarrow F\) we mean a natural transformation from the Yoneda embedded functor \(Hom(\_ , Spec(A))\) to \(F\), which is simply \(F(Spec(A))\) by Yoneda’s lemma. If \(F\) is representable - which often happens, it corresponds to an honest morphism between objects in the category)

The justification for this definition is the following ”valuative criterion” for smoothness, due to Grothendieck:

**Theorem.** If \(F, G\) are represented by Noetherian smooth schemes (in the larger category of schemes), and \(\phi\) is of finite type, then \(\phi\) is smooth if and only if it is formally smooth.

This is similar to the functorial definition of separated morphisms. The analog to the functorial definition of proper morphisms is for etale morphisms - map is etale if and only if in the above diagram the lifting is unique (formal etaleness).

For example, \(\mathbb{A}^n_S \rightarrow S\) is formally smooth. Indeed, since the functors are representable it is enough to work with morphisms of schemes instead of morphisms of functors.

First assume \(S = Spec(B)\) is affine.

If we have a diagram

\[
\begin{array}{ccc}
Spec(A) & \longrightarrow & \mathbb{A}^n_S \\
\downarrow & & \downarrow \\
Spec(A') & \longrightarrow & S
\end{array}
\]

we want to find a lifting \(Spec(A') \rightarrow \mathbb{A}^n_S\).
On affines, we have a diagram

\[
\begin{array}{ccc}
A & \xleftarrow{\delta} & \mathcal{B}[t_1, \ldots, t_n] \\
\downarrow & & \\
A' & \xleftarrow{\delta} & \mathcal{B}
\end{array}
\]

we want to map \(\mathcal{B}[t_1, \ldots, t_n]\) to \(A'\). This is equivalent to lifting the images of \(t_i\) in \(A\) to \(A'\), hence the lifting exists.

For the non-affine case, we glue - noting that the \(t_i\) above lift to global sections of \(\mathbb{A}^n_Y\).

04/08/2010

In the infinitesimal lifting properties considered previously, it is enough to consider \(I^2 = 0\). Let’s now prove that smoothness/etaleness implies formal smoothness/formal etaleness.

It is enough to consider etaleness, because any smooth map factors locally as \(X \xrightarrow{\text{etale}} \mathbb{A}^n_Y \rightarrow Y\) with \(\mathbb{A}^n_Y \rightarrow Y\) being formally smooth as proven before (why is formal smoothness local?)

If \(X \rightarrow Y\) is etale, then locally \(X\) can be embedded into \(\mathbb{A}^n_Y\) with ideal \(J\) such that \(J\) locally can be generated by \(f_1, \ldots, f_n\) with \(\delta f_i\) being linearly independent in \(\Omega_{\mathcal{O}_Y, [t_1, \ldots, t_n]}\). If \(t_i\) get mapped to \(a_i\) in \(A\) we want to lift \(a_i\) to \(a'_i\) in \(A'\) such that \(f_i(a'_1, \ldots, a'_n) = 0\). Assuming \(I^2 = 0\) we choose any lifting \(a'_i\) then we want to correct them by \(r_i \in I\) such that \(f_i(a'_1, \ldots, a'_n) + \sum_j \frac{\partial f_i}{\partial t_j} r_j = 0\) - this choice exists and is unique because the matrix \((\frac{\partial f_i}{\partial t_j})\) is invertible.

Hence liftings exists locally, and the glue because of the uniqueness clause.

Remark: this uniqueness clause gluing argument implies that formal etaleness work for not just embeddings \(\text{Spec}(A^v) \hookrightarrow \text{Spec}(A')\) but for general closed embeddings of schemes \(X \hookrightarrow X\) given by nilpotent ideals.

Now say \(\overline{Y} \rightarrow Y\) is a nilpotent embedding of (Noetherian) schemes, i.e. a closed embedding given by a nilpotent ideal sheaf.

**Theorem.** There is an equivalence of categories between etale schemes over \(Y\) and etale schemes over \(\overline{Y}\), map into one direction being the cartesian product: \(X \rightarrow \overline{X} = X 	imes_Y \overline{Y}\)

Proof: First we show that the functor is fully faithful. Assume that \(X_1 \rightarrow Y, X_2 \rightarrow Y\) are two etale morphisms with a map \(\hat{X}_1 \rightarrow \hat{X}_2\). The following diagram together with formal etaleness immediately imply there is a unique map \(X_1 \rightarrow X_2\) corresponding to it:

\[
\begin{array}{ccc}
\overline{X}_1 & \xrightarrow{\delta} & \overline{X}_2 \\
\downarrow & & \\
X_1 & \xrightarrow{\delta} & Y
\end{array}
\]

where the map \(\overline{X}_1 \rightarrow X_2\) is the composite \(\overline{X}_1 \rightarrow \overline{X}_2 \rightarrow X_2\).

Note that then \(\overline{X}_1 \rightarrow \overline{X}_2\) will be the base change of \(X_1 \rightarrow X_2\): indeed, if \(X_1', X_2' \xrightarrow{\delta} \overline{X}_1, \overline{X}_2, \overline{Y}\) then we can fit the base changed map into the following diagram:

\[
\begin{array}{ccc}
X'_1 & \xrightarrow{\delta} & X'_2 \\
\downarrow & & \\
\overline{X}_1 & \xrightarrow{\delta} & \overline{X}_2
\end{array}
\]

and as before, formal etaleness implies that the bottom map can be uniquely recovered, hence it coincides with the original map \(\overline{X}_1 \rightarrow \overline{X}_2\) (whose base change is the upper map).

For essential surjectivity, assuming that \(\overline{X} \rightarrow \overline{Y}\) is etale, locally \(\overline{X}\) is given by a closed subscheme of \(\mathbb{A}^n_{\overline{Y}}\) given by \(\overline{f}_i\) such that \(\text{det}(\frac{\partial f_i}{\partial \overline{t}_j}) \in \mathcal{O}_X^\times\)

Locally we lift them to \(f_1, \ldots, f_n \in \mathcal{O}_{\mathbb{A}^n_Y}\) such that the determinant is still invertible, they define \(X \in \mathbb{A}^n_Y\). By full faithfulness these local constructions glue to a global scheme \(X\). \(\square\)
Recall that we have proved that if $A \to B$ is a local etale map with $k_A \xrightarrow{\sim} k_B$ then $A/m_A^n \xrightarrow{\sim} B/m_B^n$.

We can rephrase it as follows: if $k \hookrightarrow k'$ a separable extension, and a nilpotent algebra $A \to k$, then there is a unique algebra $A'$ smooth over $A$ that makes the following diagram commute:

\[
\begin{array}{ccc}
k' & \leftarrow & A' \\
\uparrow & & \uparrow \\
k & \leftarrow & A
\end{array}
\]

Indeed, one such algebra is $A \otimes_k k'$ and any other such algebra clearly maps to it and using the previous proposition it is isomorphic to it.

Now let $B$ be an $A$-algebra. A relative square zero extension of $B$ is a short exact sequence $0 \to I \to B' \to B \to 0$ where $B'$ is also an $A$-algebra and $B' \to B$ is a map of $A$-algebras, with $I^2 = 0$. A splitting is a map $B \to B'$ which is left inverse to $B' \to B$.

**Lemma.** The set of splittings (if non-empty) is acted on simply transitively by the abelian group $Hom_B(\Omega_{B/A}, I) \cong Der_A(B, I)$

Proof: we can modify any given map $B \to B'$ by a map (of $A$-modules) $B \to I$, and the difference between any two such maps is a map $B \to I$. For it to be an algebra map we need the Leibniz rule to hold.

**Proposition.** $B/A$ is formally smooth/etale if and only if for any $\text{Spec} \tilde{A}, A \to \tilde{A}$,

\[
\begin{array}{ccc}
\text{Spec} \tilde{B} & \longrightarrow & \text{Spec} B \\
\downarrow & & \downarrow \\
\text{Spec} \tilde{A} & \longrightarrow & \text{Spec} A
\end{array}
\]

any

\[
0 \to I \to \tilde{B}' \to \tilde{B} \to 0
\]

of $\tilde{A}$-algebras splits/uniquely splits.

Proof: Assume the map is formally smooth. Note that the base change of formally smooth/etale is also formally smooth - as easily shown by diagram chasing and the universal property. Thus $\tilde{B}$ is formally smooth over $\tilde{A}$, and the definition of formal smoothness yields a lifting

\[
\begin{array}{ccc}
\text{Spec} \tilde{B} & \longrightarrow & \text{Spec} \tilde{B} \\
\downarrow & & \downarrow \\
\text{Spec} \tilde{B}' & \longrightarrow & \text{Spec} \tilde{A}
\end{array}
\]

If the map is formally etale, the above splitting is unique.

Now we do the other direction. Assume we have a diagram

\[
\begin{array}{ccc}
\text{Spec} \tilde{A} & \longrightarrow & \text{Spec} B \\
\downarrow & & \downarrow \\
\text{Spec} \tilde{A} & \longrightarrow & \text{Spec} A
\end{array}
\]

with $0 \to I \to \tilde{A} \to \tilde{A} \to 0$ and $I^2 = 0$.

Take $\tilde{B} = B \times \tilde{A}$ and build the short exact sequence $0 \to J \to \tilde{B} \to B \to 0$ then $J^2 = 0$
As $A \to \tilde B$ (since $A \to B, A \to \hat A$) we get a relative square zero extension $B \to \tilde B$ which composed with $\tilde B \to \hat A$ yields the required lifting $B \to \hat A$ which shows smoothness. For etaleness, this map is unique as the splitting $B \to \tilde B$ is equivalent by the universal property to $B \to B$ (which must be the identity) and the map $B \to \hat A$.

Now we go back to smoothness implying local smoothness. Assume $X \to Y$ locally factors as $X \to \mathbb{A}_Y^n \to Y$. We then have a short exact sequence $0 \to J \to \mathcal{O}_{\mathbb{A}_Y^n} \to \mathcal{O}_X \to 0$

We recall that $X$ is smooth over $Y$ if and only if $J/J^2 \to \Omega_{\mathbb{A}_Y^n/Y}|_X$ admits a left inverse locally (i.e. splits).

This together with part ii) the following proposition for $X' = \mathbb{A}_Y^n$ will imply that smoothness yields formal smoothness:

**Proposition.** Assume $X$

$injX'$ is a closed embedding of schemes over a base $Y$.

i) If $X$ is formally smooth then $J/J^2 \to \Omega_{X'/Y}|_X$ admits a left inverse (locally)

ii) If $X'$ is formally smooth, then the point i) is if and only if.

Note that part i) shows that formal smoothness implies smoothness.

Proof: Locally, assume that $X' = Spec(B), X = spec(C), Y = Spec(A)$ so $A \to B \to C$ and $0 \to J \to B \to C \to 0$

What does it mean for $J/J^2 \to \Omega_{B/A} \otimes_B C$ to admit a left inverse?

In terms of the Hom functor, it is easy to see that this is equivalent to $Hom(\Omega_{B/A} \otimes C, N) \to Hom(J/J^2, N)$

Indeed, say we have a map from $J$ to $N$ killing $J^2$ - because $J^2$ is killed we can replace $N$ by $N/N^2$ i.e. we can assume $N^2 = 0$. Consider the pushout diagram $N \longrightarrow \hat{C}$, i.e. $\hat{C}$ being $N \oplus B$ modulo the antidiagonal image of $J$

( regarded as a submodule of bot $N$ and $B$). It is easy to see that we have a short exact sequence $0 \to N \to \hat{C} \to C \to 0$ and that $C \overset{id}{\longrightarrow} C$ completes to a map between short exact sequences. Then, $\hat{C}$ is a relative square zero extension and hence by formal smoothness there is a splitting $C \to \hat{C}$ or equivalently a splitting $\hat{C} \to N$ i.e. a map $B \to N$ of $A$-modules. It is easy to see that it satisfies the Leibniz rule hence gives an element in $Hom(\Omega_{B/A} \otimes C, N)$, as desired.

For part ii), assume that $A \to B$ is formally smooth and the above surjectivity $Hom(\Omega_{B/A} \otimes C, N) \to Hom(J/J^2, N)$ holds.

To prove that $A \to C$ is formally smooth, we want to lift a square zero extension $0 \to N \to \hat{C} \to C \to 0$. We again consider the pushout sequence $0 \to N \to \tilde{B} \to B \to 0$ which splits so we get a map $B \to \tilde{B}$ in particular a map $B \to \hat{C}$. This map may not factor through a map $C \to \hat{C}$ i.e. we need to to have the restriction map $\phi: J \to \hat{C}$ to be zero. We want to correct it to a map which will satisfy this property. Note that $\phi$ must land inside $N$ since its composition with $\hat{C} \to C$ must be zero as it is just the restriction to $J$ of the map $B \to C$. Moreover it factors through $J/J^2$ as $N^2 = 0$. By the surjectivity clause, it follows that we can lift this map to a map in $Hom(\Omega_{B/A} \otimes C, N)$ i.e. a derivation from $B$ to $N$. Subtracting this derivation from the splitting $B \to \tilde{B}$ (recall that any splittings differ by such a derivation), we get a splitting $B \to \tilde{B}$ that will now factor as needed. □

Now let $A$ be a finitely generated algebra over an algebraically closed field $k$, and $x \in X = Spec(A)$ a closed point.

**Theorem.** $X$ is smooth at $x$ if and only if for all local Artinian algebras $R_2 \to R_1$ there is a lifting in the following diagram:

$$
\begin{array}{ccc}
Spec(R_1) & \longrightarrow & Spec(A) \\
\downarrow & & \downarrow \\
Spec(R_2) & \longrightarrow & Spec(k)
\end{array}
$$
(here we consider maps \(Spec(R_1) \to Spec(A)\) such that the image of the closed point is \(x\))

Proof: Suppose \(A\) is smooth at \(x\). We know \(A/m^2 \cong k \oplus m/m^2\), and \(m\) is generated by \(x_1, \ldots, x_n\). Then the completion \(\hat{A}\) of \(A\) must be \(k[[x_1, \ldots, x_n]]\) (Koszul complex?). For Artinian algebras, mapping into \(A\) is the same as mapping into \(\hat{A}\), which can be lifted because it suffices to lift the images of \(x_i\).

For the other direction, choose \(R_1 = k \oplus m/m^2\) and more generally \(R_n = Sym(k/m^2)/Sym^{\geq n+1}(m/m^2)\).

The natural map \(A \to k \oplus m/m^2\) then lifts to maps \(A \to R_i\) - this clearly factors through \(A/m^i\). By passing to the limit we get \(\hat{A} \to Sym(k/m^2)\).

We then get a map \(gr(A) \to Sym_k(m/m^2)\) but there is also a canonical map \(Sym(m/m^2) \to gr(A)\), then \(gr(A)\) contains \(Sym(m/m^2)\) as a direct summand. Then this has to be an isomorphism, so \(A\) is regular. \(\square\)

\(04/13/2010\)

There is a Verdier duality (generalization of Poincare duality) for manifolds with corners, in geometry. There is an analogous theory for schemes, which will be developed here.

**Theorem.** Let \(X \xrightarrow{f} Y\) be a projective map of schemes. Then
i) \(Rf_*\) admits a right adjoint \(f!\)
ii) If \(f\) is smooth,
\[f^!(\mathcal{F}) \cong f^*(\mathcal{F}) \otimes \Omega^n_{X/Y}[n]\]
\((\Omega^n_{X/Y} = \Lambda^{top}\Omega_{X/Y})\)

Remark: it turns out that \(f!\) always exists, it is the left adjoint \(Rf^!\) that may not exist.

**Corollary.** Suppose \(Y = Spec(k)\). Then \(f^!(k) = \omega_X\) - what is known as the dualizing sheaf, satisfying by definition
\[Hom_{D(k-Vect)}(R\Gamma(X, \mathcal{F}),k) \cong Hom_{D(X)}(\mathcal{F}, \omega_X)\]

The right-hand side is clearly \(H^0(X, \mathcal{F})^\vee\) and in general if we replace \(\mathcal{F}\) by \(\mathcal{F}[i]\) we deduce \(H^i(X, \mathcal{F})^\vee \cong Ext^{-i}(\mathcal{F}, \omega_X)\)

Suppose now that \(\mathcal{F} \in D^b_{coh}(X)\). If \(\mathcal{F}_1 \in D^b_{coh}(X)\) we also have \(Hom(\mathcal{F}_1, \mathcal{F}_2) \in D_{QCoh}(X)\) determined by \(Hom(\mathcal{F}, Hom(\mathcal{F}_1, \mathcal{F}_2)) = Hom(\mathcal{F} \otimes \mathcal{F}_1, \mathcal{F}_2)\).

Define the functor \(D\) by \(D_X(\mathcal{F}) = Hom(\mathcal{F}, \omega_X)\).

**Theorem.** i) \(D_X\) maps \(D^b_{coh}(X)\) to \(D^b_{coh}(X)\)
ii) \(Id \cong D_X\)
iii) If \(f: X \to Y\) is proper, then \(D_Y \circ Rf_* \cong Rf_* \circ D_X\)

In particular, if in iii) we let \(f \) be \(X \to Spec(k)\) so \(f_* = \mathcal{F}\) and \(D_Y(\mathcal{F}) = Hom(\mathcal{F}, k)\), the above formula iii) yields \(Hom(R\Gamma(X, \mathcal{F}),k) = R\Gamma(X, D_X(\mathcal{F}))\) in particular for \(\mathcal{F} \to \mathcal{F}[i]\) we deduce that \((H^i(X, \mathcal{F}))^\vee = H^i(X, D_X(\mathcal{F}))\).

Assume that \(X\) is smooth so \(\omega_X = \Omega^n_{X/k}[n]\) so we deduce from a previous formula \((H^i(X, \mathcal{F}))^\vee \cong Ext^{n-i}(\mathcal{F}, \Omega^n_{X/k})\)

In particular assume \(\mathcal{F} = \mathcal{M}\) is a locally free sheaf then \(RHom(\mathcal{M}, \Omega^n_{X/k}) = RHom(\mathcal{M}, \Omega_X) \otimes \Omega^n_{X/k} = \mathcal{M}^\vee \otimes \Omega_{X/k}\) so we obtain Serre Duality
\[H^i(X, \mathcal{M})^\vee \cong H^{n-i}(X, \mathcal{M}^\vee \otimes \Omega)\]

If \(X\) is a complete curve and \(\mathcal{M}\) is a line bundle \(\mathcal{L}\), then we have
\[H^0(X, \mathcal{L}) \cong H^1(X, \mathcal{L}^\vee \otimes \Omega_X)\]

Particularly, using the Riemann-Roch Theorem
\[\dim(H^0(X, \mathcal{L})) - \dim(H^1(X, \mathcal{L})) = 1 - g + \deg(\mathcal{L})\]

136
yields
\[
dim(H^0(X, \mathcal{L})) - \dim(H^0(X, \mathcal{L}^{-1} \otimes \Omega_X)) = 1 - g + \text{deg}(\mathcal{L})
\]

Now let’s prove the first theorem, about the existence of the right adjoint functor \( f^! \).

First, assume that \( X = \mathbb{P}^n Y \). In general, if we have to check that a functor \( G \) is adjoint to a functor \( F \), we need to give the following data:

- a natural transformation \( F \circ G \rightarrow \text{Id} \) such that the composite map \( \text{Hom}(X, GY) \rightarrow \text{Hom}(FX, FGY) \) is an isomorphism.

We try \( f^!(\mathcal{F}) = f^*(\mathcal{F}) \otimes \Omega^n_{\mathbb{P}^n Y/Y}[n] \) (note that \( \Omega^n_{\mathbb{P}^n Y/Y} = \mathcal{O}(-n - 1) \)).

First, by the projection formula
\[
\text{Hom}(\mathcal{F}, \mathcal{G}) = \text{Hom}(\mathcal{F} \otimes \mathcal{G} \otimes \Omega^n_{\mathbb{P}^n Y/Y}[n], \mathcal{O})
\]

which gives the required natural transformation. We now need to check the isomorphism, i.e. that \( \text{Hom}(\mathcal{F}_X, \mathcal{F}_Y \otimes \mathcal{O}(-n - 1)) \rightarrow \text{Hom}(Rf_*(\mathcal{F}_X), \mathcal{F}_Y) \).

First we claim that \( \Omega^n_{\mathbb{P}^n Y/Y} \) is an isomorphism. It is the pullback of the sheaf of relative differentials on the left-hand side, hence the LHS becomes
\[
\text{Hom}(\mathcal{F}_Y \otimes \mathcal{R}_g^*(\mathcal{F}_X), \mathcal{O}_Y)
\]

but the latter pairing is perfect as shown last semester. This finishes the proof of the theorem for the projective space case.

For the general case, it is easy to see that if we have maps \( X \rightarrow Y \rightarrow Z \) and the theorem is true for \( f, g \) then it is true for \( f \circ g \). Note that this will imply our theorem as every projective map factors as the composition of a closed embedding and a map \( \mathbb{P}^n \rightarrow \mathbb{P}^n \), and for closed embeddings the existence of right adjoint to \( \mathbb{R}^n_* \) was shown long ago - the right derived functor of what was also called \( i^! \). \( \square \)

We now give a partial proof of the second theorem.

a) First assume that both schemes are projective of finite type. Then we show that \( Rf_* \circ \mathbb{D}_X \cong \mathbb{D}_Y \circ Rf_* \).

We want to show that \( \text{Hom}(\mathcal{F}_Y, Rf_* \circ \mathbb{D}_X (\mathcal{F}_X)) \cong \text{Hom}(\mathcal{F}_Y, \mathbb{D}_Y \circ Rf_* (\mathcal{F}_X)) \).

The LHS is \( \text{Hom}(Lf^*(\mathcal{F}_Y), \mathbb{D}_X (\mathcal{F}_X)) \) which by the definition of \( \mathbb{D}_X \) can be rewritten as \( \text{Hom}(Lg^*(\mathcal{F}_Y) \otimes \mathcal{F}_X, \omega_X) \).

Now because the composition of left adjoints is left adjoint to the composition, the sequence \( X \rightarrow Y \rightarrow pt \) implies \( f^!(\omega_Y) = \omega_X \) hence the LHS becomes \( \text{Hom}(Lf^*(\mathcal{F}_Y) \otimes \mathcal{F}_X, f^!(\omega_Y)) = \text{Hom}(Rf_*(Lg^*(\mathcal{F}_Y) \otimes \mathcal{F}_X), \omega_Y) \) which by projective formula equals \( \text{Hom}(\mathcal{F}_Y \otimes Rf_*(\mathcal{F}_X), \omega_Y) = \text{Hom}(\mathcal{F}_Y, H \circ Rf_*(\mathcal{F}_X), \omega_Y) = \text{Hom}(\mathcal{F}_Y, \mathbb{D}_Y \circ Rf_*(\mathcal{F}_X)) \).

b) Consider \( X \rightarrow \mathbb{P}^n \). Then \( i_* \) is conservative (on cohomologies) so to show \( \mathbb{D}_X (\mathcal{F}) \) is bounded it is enough to show that \( i_* \mathbb{D}_X (\mathcal{F}) = \mathbb{D}_X \circ i_*(\mathcal{F}) \) has bounded cohomologies. But since \( \omega_{\mathbb{P}^n} \) is locally free, we conclude.

The proof of c) is similar.

Let’s return to the claim that for \( X \rightarrow Y \) smooth of relative dimension \( n \) we have \( f^!(\mathcal{F}) = f^*(\mathcal{F}) \otimes \Omega^n_{X/Y}[n] \).

First we claim that \( \Omega^n_{X/Y} \cong \mathcal{O}(-n - 1) \).

Indeed, say \( \mathbb{P}^n = \mathbb{P}(V) \), so that \( \text{Hom}(\mathcal{O}(-1), \mathcal{O}) = V^* \)
We have the short exact sequence $0 \to \Omega^2_{\pi(V)} \to V^* \mathcal{O}(-1) \to \mathcal{O} \to 0$ and we use the fact that if $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ is a short exact sequence of vector bundles, then $\Lambda^{top}(\mathcal{E}_2) = \Lambda^{top}(\mathcal{E}_1) \otimes \Lambda^{top}(\mathcal{E}_3)$ thus $\Omega^n_{\pi(V)} \cong \Lambda^{n+1}(V^* \otimes \mathcal{O}) \otimes \mathcal{O}(-1)$.

Now using $\Lambda^i(\mathcal{M} \otimes \mathcal{L}) \cong \Lambda^i(\mathcal{M}) \otimes \mathcal{L}^\otimes i$, we get $\Omega^n_{\pi(V)} \cong \Lambda^{n+1}(V^*) \otimes \mathcal{O}(-n-1)$ whence the result.

We need to analyze the general case (i.e. add in closed embeddings).

**Definition.** Let $X \hookrightarrow Y$ be a closed embedding. We say that it's a regular embedding or a complete intersection if its ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_Y$ is locally given by a regular sequence of length $n$.

**Proposition.** Let $X \hookrightarrow Y$ be a regular embedding. Then:

i) $\mathcal{I}_X / \mathcal{I}_X^2$ is locally free of rank $n$ over $\mathcal{O}_X$. It is denoted $N^\vee_{X/Y}$

ii) $\Lambda^i N_{X/Y} \rightarrow Tor_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$

iii) $\Lambda^i N_{X/Y} \rightarrow Ext_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$

iv) $Ext^i(\mathcal{O}_X, \mathcal{O}_Y) = \begin{cases} 0, & i \neq n \\ \Lambda^n N_{X/Y}, & i = n \end{cases}$

The theorem follows from the fact that locally $\mathcal{O}_X$ as an $\mathcal{O}_Y$-module admits a resolution given by

$$\ldots \to \mathcal{O}_Y \oplus f_1 \oplus \ldots \oplus f_n \to \mathcal{O}_Y \to \mathcal{O}_X$$

- and particularly the monomials $f_{k_1} \cdot \ldots \cdot f_{k_i}$ with $k_1 < \ldots < k_i$ form a basis for $Ext^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ over $\mathcal{O}_X$. More precisely, we have the following:

**Proposition. (PSET)** Assume $Y$ is an affine scheme and $i: X \rightarrow Y$ is a closed embedding, whose ideal $I_X$ is generated by a regular sequence. Then:

- i) $N^\vee_{X/Y} = I_X / I_X^2$ is a free $\mathcal{O}_X$-module of rank $n$, with basis $f_1, \ldots, f_n$.

- ii) There is a canonical isomorphism $N_{X/Y} = Hom_{\mathcal{O}_X}(N^\vee_{X/Y}, \mathcal{O}_X) \cong Ext^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$

- iii) The cup product i.e. the composition

$$Ext^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \times \cdots \times Ext^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \rightarrow Ext^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$$

sends an element $f_{k_1} \times \ldots \times f_{k_i}$ to $f_{k_1} \cdot \ldots \cdot f_{k_i}$.

- iv) For any $i$, the image of $f_i \times f_i$ in $Ext^2_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ is zero

- v) There is an isomorphism $\Lambda^i_{\mathcal{O}_X}(N_{X/Y}) \rightarrow Ext^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$

Proof: a) It suffices to show $f_1, \ldots, f_n$ are linearly independent. We show this by induction on $n$ - i.e. that $f_1, \ldots, f_k$ are independent in $\mathcal{O}_Y/\langle f_1, \ldots, f_k \rangle \mathcal{O}_Y$. The base $k = 1$ is easy.

For the induction step. Assume $\sum f_{i+1} + \ldots + f_n = 0$. We lift it to $\mathcal{O}_Y$ and this induces $a_1 f_1 + \ldots + a_n f_n = \sum c_{i,j} f_i f_j$ - and by moving $c_{i,j} f_i f_j$ into $a_i$ or $a_j$ we may assume $a_1 f_1 + \ldots + a_n f_n = 0$. Because $f_n$ is injective on $\mathcal{O}_Y/\langle f_1, \ldots, f_{n-1} \rangle \mathcal{O}_Y$ we conclude $a_n = \sum_{i=1}^{n-1} b_i f_i$ and now by replacing $a_i$ by $a_i - a_n f_i$ we get $a_1 f_1 + \ldots + a_{n-1} f_{n-1} = 0$ and we are done by induction.

b) We have the short exact sequence $0 \to I_X \to \mathcal{O}_Y \to \mathcal{O}_X \to 0$ which by the long exact sequence of Ext gives the exact sequence $0 \to Hom_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \to Hom_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_X) \to Hom_{\mathcal{O}_Y}(I_X, \mathcal{O}_X) \to Ext^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) \to 0$.

In particular we have a surjective map $Hom_{\mathcal{O}_Y}(I_X, \mathcal{O}_X) \to Ext^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$. However there is also a canonical map $Hom_{\mathcal{O}_X}(I_X / I_X^2, \mathcal{O}_X) \to Hom_{\mathcal{O}_Y}(I_X, \mathcal{O}_X)$. In fact this map is an isomorphism: it is clearly injective and note that any $\mathcal{O}_Y$ map from $I_X$ to $\mathcal{O}_X$ must send $I_X^2$ to 0 as $I_X$ acts by 0 on $\mathcal{O}_X$, and then the $\mathcal{O}_Y$ action on such a map must factor through $\mathcal{O}_X$ since the $\mathcal{O}_Y$ action on $I_X / I_X^2$ and $\mathcal{O}_X$ do, which shows surjectivity.

Therefore we produce a canonical map $N_{X/Y} \to Ext^1_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$. From the long exact sequence it is surjective. It remains to show injectivity which is equivalent to the map $Hom_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_X) \to Hom_{\mathcal{O}_Y}(I_X, \mathcal{O}_X)$ from the short
exact sequence being 0 - but this is obvious since under any map of \( \mathcal{O}_Y \) to \( \mathcal{O}_X \), \( I_X \) must get sent to 0 as \( I_X \) acts trivially on \( X \).

c) We show it by induction on \( i \geq 1 \). The base is obviously true so let’s do the induction step. Let \( P^\bullet \) be the free resolution of \( \mathcal{O}_X \) that we are working with.

The morphism corresponding to \( f_{k_1} \times f_{k_2} \times \ldots \times f_{k_i} \) is, according to the induction step, represented by the product of the two lower huts in the diagram:

To construct the big hut we need to construct the two upper morphisms. One of them is obvious: it is the identity. The morphism labeled by "?" from \( P^\bullet \) to \( P^\bullet[i-1] \) is constructed as follows: the copy of \( \mathcal{O}_Y \) corresponding to the set \( \{k_1,k_2,\ldots,k_{i-1}\} \cap S \) will map isomorphically to the copy of \( \mathcal{O}_Y \) corresponding to the set \( S \). It is easy to check that this is indeed a morphism, that it makes the diagram commute, and that its composition with \( f_{k_i}[i-1] \) gives the morphism \( f_{k_i}f_{k_2}\ldots f_{k_i} \) which finishes the proof.

d) The proof is very similar to the proof above: indeed, note that we have not used the fact that \( k_i \) is distinct from all the other elements in the above construction. Therefore we can construct the same composite hut as above in our case, to conclude that the morphism is given by the map \( P \to P[2] \) that is the composition of "?" with \( f_i[1] \).

It remains to show that this composition is 0 which is obvious: "?" only hits the copies of \( \mathcal{O}_Y \) which correspond to \( S \) not containing \( i \), in particular \( S = \{i\} \) is not hit, but that is the only copy that maps non-trivially to \( \mathcal{O}_X[2] \) via \( f_i[1] \).

e) According to the previous two parts, the map in c) factors through \( \Lambda_{O_X}^i(Ext^1_{O_Y}(\mathcal{O}_X,\mathcal{O}_X)) \). Moreover, it is obvious that the map in c) is surjective hence the map \( \Lambda_{O_X}^i(Ext^1_{O_Y}(\mathcal{O}_X,\mathcal{O}_X)) \to Ext^i_{O_Y}(\mathcal{O}_X,\mathcal{O}_X) \). We have a surjective map between free modules over \( \mathcal{O}_Y \) of the same dimension - which implies that it must be an isomorphism (because it is given by a square matrix that has a right inverse but then that matrix is also the left inverse so the map is injective).

d) Follows immediately from b) and e). \( \square \)

**Proposition.** Let \( Y \) be an affine scheme and i: \( X \hookrightarrow Y \) a closed subscheme of \( Y \), whose ideal \( I_X \) is generated by a regular sequence \( f_1,\ldots,f_n \in \mathcal{O}_Y \). We have shown that for \( \mathcal{F} \in D_{QCoh}(Y) \), the natural map

\[
Ri^!(\mathcal{O}_Y) \otimes_{\mathcal{O}_X} L\iota^!(\mathcal{F}) \to Ri^!(\mathcal{F})
\]

is an isomorphism, and that \( Ri^!(\mathcal{O}_Y) \cong \mathcal{L}[-n] \) where \( \mathcal{L} \) is a line bundle on \( X \).

a) \( \mathcal{L} \cong \Lambda_{O_X}^n(N_{X/Y}) \)

b) There are canonical isomorphisms

\[
Tor^1_{O_X} (\mathcal{O}_X,\mathcal{O}_X) \cong \Lambda_{O_X}^n(N_{X/Y}^\vee)
\]

Proof: a) we will prove later that \( Ri^! \) is the right derived functor of \( Hom_A(A/I,-) \). Assuming that, \( H^i(Ri^!(\mathcal{O}_Y)) \cong Ext^i_A(A/I,A) \). Let \( P^\bullet \) be the Koszul complex of \( A/I \) as in the previous proposition, then \((P^\bullet)^\vee[n] \cong P^\bullet \) (cf. Eisenbud, Proposition 17.15), so we have \( Ext^i_A(A/I,A) \) for \( i < n \). We exhibit a canonical isomorphism \( Ext^i_A(A/I,A) \cong Ext^i_A(A/I,A/I) \) which proves the claim since the latter is isomorphic to \( \Lambda_{O_X}^n(N_{X/Y}^\vee) \) by f) of the previous proposition. The short exact sequence \( 0 \to I \to A \to A/I \) gives the long exact sequence

\[
Ext^0_A(A/I,I) \to Ext^0_A(A/I,A) \to Ext^0_A(A/I,A/I) \to 0
\]

139
We claim the first map is 0. To see this, make us of the Koszul complex again. Consider the diagram

\[
\begin{array}{ccc}
\text{Hom}(\Lambda^{n-1}(A^n), I) & \xrightarrow{-\circ d} & \text{Hom}(\Lambda^n(A^n), I) \\
\downarrow & & \downarrow \\
\text{Hom}(\Lambda^{n-1}(A^n), A) & \xrightarrow{-\circ d} & \text{Hom}(\Lambda^n(A^n), A)
\end{array}
\]

Since \(d(e_1 \wedge \ldots \wedge e_n) = \sum_{j=1}^{n} (-1)^{j-1} f_j \cdot e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n\) and \(I\) is generated by \(f_1, \ldots, f_n\), we see that

\[
\text{Im}(\text{Hom}(\Lambda^n(A^n), I) \to \text{Hom}(\Lambda^n(A^n), A) \subset \text{Im}(\text{Hom}(\Lambda^{n-1}(A^n), A) \to \text{Hom}(\Lambda^n(A^n), A))}
\]

which implies \(\text{Ext}^n(A/I, A) \to \text{Ext}^n(A/I, A)\) is the zero map.

b) Since the Koszul complex \(P^*\) is a free resolution of \(A/I\), the complex is determined up to unique homotopy by \(I\) (and hence does not depend of choice of generators). Now \(\text{Tor}_i^{O_Y}(O_X, O_X) \cong H^{-i}(P^* \otimes (A/I))\). The differential is zero on \(P^* \otimes (A/I)\). We also have \(\text{Ext}_i^{O_Y}(O_X, O_X) \cong H^i(\text{Hom}_A(P^*, A/I))\), where the differential of \(\text{Hom}_A(P^*, A/I)\) is again zero. We have a canonical isomorphism

\[
\text{Hom}_{A/I}(P^i \otimes A/I, A/I) \cong \text{Hom}_A(P^i, A/I)
\]

Letting \((-)^\vee\) denote \(\text{Hom}_{O_X}(-, O_X)\), we have shown

\[
\text{Tor}_i^{O_Y}(O_X, O_X)^\vee \cong (P^i \otimes O_Y)^\vee \cong \text{Hom}_A(P^i, A/I) \cong \text{Ext}_i^{O_Y}(O_X, O_X) \cong \Lambda^i_{O_X}(N_{X/Y})
\]

Since \(P^i \otimes O_X\) is a free \(O_X\)-module, and \(N_{X/Y} := (N_X^\vee)^\vee\) is also free, we apply \((-)^\vee\) again to get a canonical isomorphism

\[
\text{Tor}_i^{O_Y}(O_X, O_X) \cong (\Lambda^i_{O_X}(N_{X/Y}))^\vee \cong \Lambda^i_{O_X}(N_X^{\vee})
\]

\[\square\]

05/15/2010

A good reference for this material is Brian Conrad’s book “Grothendieck duality and base change”.

Recall that if \(X \rightarrow Y\) is proper (projective) then the functor

\[
Rf_* : \text{D}^+_{\text{QCoh}}(X) \to \text{D}^+_{\text{QCoh}}(Y)
\]

admits a right adjoint \(f^!\) which in \(X = \mathbb{P}_Y^N\) equals \(f^*(\mathcal{F}) \otimes \mathcal{O}(-n - 1)[n]\) and if \(i\) is a closed embedding is the right derived functor of the extension by zero.

We want to compare \(f^!(\mathcal{F})\) to \(f^*(\mathcal{F}) \otimes f^!(\mathcal{O}_Y)\) - sometimes it is true, but not in general.

For example, consider \(X = \text{Spec}(k) \to \text{Spec}(k[t]/(t^2))\), \(\mathcal{F} = k\).

Then

\[f^!(\mathcal{F}) = R\text{Hom}_{k[t]/(t^2)}(k, k) = k \oplus k[-1] \oplus k[-2] \oplus \ldots\]

\[f^*(\mathcal{F}) = k \oplus k \oplus k[1] \oplus \ldots\]

\[f^!(\mathcal{O}_Y) = R\text{Hom}_{k[t]/(t^2)}(k, k[t]/(t^2)) = k\]

since \(k[t]/(t^2)\) is injective.

**Proposition.** Let \(X, Y\) be Noetherian, \(f\) quasi-projective (actually, its enough for it to be of finite type).
Then \( f^! : D_{QCoh}^+(Y) \to D^+(X) \) is defined and if \( X \xrightarrow{f} Y \xrightarrow{g} Z \)
\[
f^! \circ g^! = (g \circ f)^!
\]

- If \( f \) is proper, then \( f^! \) is right adjoint to \( Rf_* \).
- If \( f \) is an open embedding, \( f^! = f^* \).

Proof: Let’s factor \( f \) as \( X \xrightarrow{j_1} \overline{X} \xrightarrow{j_2} Y \) where \( \overline{X} \xrightarrow{j_2} Y \) is projective.

We want to set \( f^! = j^* \circ \overline{f}^! \). We need to show that it’s independent of the choice of \( \overline{X} \) and that the composition holds.

It is enough to show that if we have open embeddings \( X \xrightarrow{j_1} \overline{X}_1, X \xrightarrow{j_2} \overline{X}_2 \) such that the following diagram commutes

\[
\begin{array}{ccc}
\overline{X}_1 & \xrightarrow{j_1} & X \\
\downarrow & & \downarrow \pi \\
\overline{X}_2 & \xrightarrow{j_2} & X
\end{array}
\]

and \( \pi \) is proper, then
\[
(j_2)^* = (j_1)^* \circ \pi^!
\]

To construct a map \( \to \) we use the adjunction: mapping \((j_2)^*(\mathcal{F}) \to (j_1)^* \circ \pi^!(\mathcal{F})\) is equivalent to mapping \( \mathcal{F} \) to \((j_2)_*j_1^! \pi^!(\mathcal{F}) = \pi_*(j_1)_*j_1^! \pi^!(\mathcal{F}) = \pi_! \pi^!(\mathcal{F})\) and we have a canonical map \( \mathcal{F} \to \pi_! \pi^!(\mathcal{F}) \).

It remains to prove that it is an isomorphism. Let \( Y_1 = \overline{X}_1 - X, Y_2 = \overline{X}_2 - X \). \( Y_1 \to Y_2 \).

Recall the admissible triple

\[
D_{QCoh, -Y_2}^+(X_2) \hookrightarrow D_{QCoh}^+(X_2) \xrightarrow{R(j_2)^*} D_{QCoh}(Y_2)
\]

We’ll show that both functors we compare vanish on \( D_{QCoh, -Y_2}^+(X_2) \) and that they are the same on \( D_{QCoh}(Y_2) \).

For the first part - an element in \( D_{QCoh, -Y_2}^+(X_2) \) is a direct image of some complex of sheaves in \( Y_2' \) (a different subscheme structure on \( Y_2 \)). Then \( j_2^*(i_2)_* = 0 \) and \( j_1^! \pi^!(i_2)_* \) is also true since \( \pi_!(i_2)_* \) is right adjoint to \( i_2^! \pi_* \) - which by base change is \( \pi_i i_1^! = i_1^! \pi^! \) and \( j_1^! (i_1)_* = 0 \).

Now for the second part. We compare \( j_2^*(j_2)_* = 1d \) to \( j_1^! \pi^!(j_2)_* \). Again \( \pi^!(j_2)_* \) is right adjoint to \( j_2^! \pi_* = j_2^! \) thus \( j_2^! \pi^!(j_2)_* = j_2^! (j_1)_* = 1d \). It follows (because on \( D_{QCoh, -Y_2}^+(X_2) \) the functors are zero) that \( j_1^! \pi^! = j_2^! \), as desired. \( \square \)

Now let \( X \) be a scheme of finite type over \( k, p : X \to Spec(k) \). We have the dualizing sheaf \( K_X = p^!(k) \). Recall the functor \( \mathbb{D}_X : D_{QCoh}^b(X) \to (D_{QCoh}^b(X))^{op} \) given by \( \mathcal{F} \to RHom(\mathcal{F}, K_X) \) i.e. we need to show there are finitely many \( Ext \).

We can extend this functor to \( D_{Coh}^+(X) \xrightarrow{\mathbb{D}_X} D_{Coh}^+(X) \) and \( D_{Coh}^-(X) \xrightarrow{\mathbb{D}_X} D_{Coh}^-(X) \).

**Proposition.** Let \( f : X \to Y \) be a quasi-projective map. Then \( f^! \circ \mathbb{D}_Y \cong \mathbb{D}_X \circ f^! \) as functors

\[
D_{Coh}^-(Y) \to D_{Coh}^+(X)
\]

and

\[
f^* \circ \mathbb{D}_Y \cong \mathbb{D}_X \circ f^!
\]

as functors

\[
D_{Coh}^+(Y) \to D_{Coh}^-(X)
\]

Proof: factor the map as \( X \xrightarrow{j} \overline{X} \xrightarrow{f} Y \) where \( j \) is an open embedding and \( f \) is open.
We have $\text{Hom}(\mathcal{F}', f^* \circ \mathbb{D}_Y(\mathcal{F})) \cong \text{Hom}(f_\ast \mathcal{F}', \mathbb{D}_Y(\mathcal{F})) \cong \text{Hom}(\mathcal{F}, \mathbb{D} \circ f_\ast(\mathcal{F}))$ and $\text{Hom}(\mathcal{F}', \mathbb{D}_X \circ f^*(\mathcal{F})) \cong \text{Hom}(f^* \mathcal{F}, \mathbb{D}_X(\mathcal{F}')).$

But $\text{Hom}(\mathcal{F}, \mathbb{D}_Y \circ f_\ast(\mathcal{F}')) \cong \text{Hom}(\mathcal{F}, f_\ast \circ \mathbb{D}_X(\mathcal{F}'))$ as shown before. □

**Theorem.** a) Let $X \to Y$ be a quasi-projective map between Noetherian schemes. Suppose $\tilde{X}$ is an open subscheme of $X$ such that $f \mid_{\tilde{X}}$ is smooth of relative dimension $n$. Then $f^!(\mathcal{F}) \mid_{\tilde{X}} \cong (f \circ j)^!(\mathcal{F})$ equals $f^!(\mathcal{F}) \mid_{\tilde{X}} \otimes \Omega^1_{\tilde{X}/Y}[n].$

b) Assume $X \overset{i}{\hookrightarrow} Y$ is a regular closed embedding. Then

$$Ri^!(\mathcal{F}) \cong Li^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \Lambda^n(N_{X/Y})[-n]$$

**Proposition.** Assume $X \overset{i}{\hookrightarrow} Y$ is a regular closed embedding of dimension $n$. Then:

$\text{Tor}^i_{\mathcal{O}_Y}(\mathcal{O}_X, -) = 0$ for $i > n$

$\text{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_X, -) = 0$ for $i > n$

Also, non-canonically:

$\text{Tor}^i_{\mathcal{O}_Y}(\mathcal{O}_X, i_!(\mathcal{F})) \cong \mathcal{F} \otimes_{\mathcal{O}_X} \Lambda^i(N_{X/Y})$

Also if $\mathcal{L}$ is a locally free sheaf, then

$$\text{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{L}) = \begin{cases} 0, & i \neq n \\ \text{(non-can.)} \mathcal{L} \mid_X, & i = n \end{cases}$$

Also, canonically $\mathcal{F} \otimes_{\mathcal{O}_X} \Lambda^i(N_{X/Y}) \cong \mathcal{F} \otimes_{\mathcal{O}_X} \Lambda^i(N_{X/Y})$

**Proposition.** Let $X \overset{i}{\hookrightarrow} Y$ be a regular embedding. Then there is a canonical isomorphism

$$i^* \mathcal{F} \otimes_{\mathcal{O}_Y} i^! \mathcal{O}_Y \cong i^! \mathcal{F}$$

Proof: to map $i^* \mathcal{F} \otimes_{\mathcal{O}_Y} i^! \mathcal{O}_Y$ to $i^! \mathcal{F}$ is equivalent by adjunction to mapping $i_!(i^* \mathcal{F} \otimes_{\mathcal{O}_Y} i^! \mathcal{O}_Y)$ to $\mathcal{F}$ which by the projection formula is equivalent to mapping $\mathcal{F} \otimes_{\mathcal{O}_Y} i_! i^! \mathcal{O}_Y \to \mathcal{F}$ and this map comes by adjunction. We have to show it’s an isomorphism. If $\mathcal{F}$ is a coherent sheaf we replace it by a projective resolution. Both functors have finite cohomological amplitude so we can truncate and we can work with finite complexes of rprojectives, then with free modules for which this is clear. □

**Corollary.** Let $X \overset{i}{\hookrightarrow} Y$ be a regular embedding. Then $f^!(\mathcal{F}) \cong f^*(\mathcal{F}) \otimes \Lambda^n N_{X/Y}[n]$

Now consider $X \overset{i}{\hookrightarrow} \mathbb{P}^n Y = Y' \to Y$
We then have \( f^! (\mathcal{F}) = i^! \circ \pi^! (\mathcal{F}) = i^! (\pi^* (\mathcal{F}) \otimes \Omega^m_{\mathcal{O}X/Y} [n]) \) which using the corollary equals \( i^* \pi^* (\mathcal{F}) \otimes \Omega^m_{\mathcal{O}X/Y} [n] \otimes \Lambda^m N_{Y/Y}[n - m] \) and we claim \( \Omega^m_{\mathcal{O}X/Y} [n] \otimes \Lambda^m N_{Y/Y}[n - m] \) is \( \Omega^{m-m}_{X/Y} \).

Indeed, this follows immediately from the sequence

\[
0 \rightarrow N^r_{X/Y} \rightarrow \Omega_{\mathcal{O}X/Y} |X \rightarrow \Omega_{X/Y} \rightarrow 0
\]

Thus we’ve shown that for \( X \xrightarrow{f} Y \) smooth

\[
f^! (\mathcal{F}) \cong f^* (\mathcal{F}) \otimes \Omega^m_{X/Y} [n]
\]

It depends on the factorization e.g. \( K_X = p^!(k) \cong \Omega^n_X [n] \). We want so show it’s canonical.

Say \( K_{X \times Y} = K_X \boxtimes K_Y \).

Consider the diagonal embedding \( X \xrightarrow{\Delta} X \times X \) then \( K_X \cong \Delta^! (K_X \boxtimes K_X) = \Delta^! (K_X \boxtimes K_X) \cong \Delta^* (K_X \boxtimes K_X) \otimes \Lambda^\text{top} N_{X/X \times X} \cong K_X \boxtimes K_X \otimes \Lambda^\text{top} N_{X/X \times X}
\]

Hence (as \( K_X \) is a line bundle) we deduce \( K_X = (\Lambda^\text{top} N_{X/X \times X})^{-1} \).

We claim \( \Lambda^\text{top} (N_{X/X \times X})^{-1} = \Omega^n_X \).

We use the sequence \( 0 \rightarrow N^r_{X/X \times X} \rightarrow \Omega_{X \times X} |X \rightarrow \Omega_X \rightarrow 0 \) but \( \Omega_{X \times X} |X = \Omega_X \oplus \Omega_X \) and hence \( N^r_{X/X \times X} \cong \Omega_X \).

Let \( X \) be a scheme of finite type.

**Proposition.** \( X \) is CM of dimension \( n \) if and only if \( \mathbb{D}_X (\mathcal{O}_X) \) is a single sheaf living in cohomological degree \(-n\).

This follows immediately from the following:

**Proposition. (PSET)** Let \( X \) be a scheme of finite type over a field \( k \), of dimension \( n \). Then \( K_X \) lives in cohomological degrees \([-n, 0]\). \( K_X \) lives purely in cohomological degree \(-n \) if and only if \( X \) is CM of dimension \( n \).

Proof (by Jonathan Wang): Consider an affine \( \text{Spec} (A) \xrightarrow{\Delta} X \) such that \( k[x_1, \ldots, x_m] \rightarrow A \). Let \( p: X \rightarrow \text{Spec} (k) \) so \( K_X = p^!(k) \). Since \( (p \circ j)^! \cong j^* \circ p^! \) we have \( K_X \mid \text{Spec} (A) \cong K_{\text{Spec} (A)} \).

For the first part, \( \text{dim}(A) \leq \text{dim}(X) \) so it’s enough to test cohomological degree of restriction to an open covering. For the second part, being CM is a local property. Thus we can assume \( X = \text{Spec} (A) \) is affine, and we have a closed embedding \( X \xrightarrow{\Delta} \mathbb{A}^m_k \). We know that as functors from \( D^-_{\text{coh}} (Y) \rightarrow D^-_{\text{coh}} (X) \), \( \Delta^! \circ D_{\text{coh}} \cong D_{\text{coh}} \circ \Delta^* \).

Applying this to \( \mathcal{O}_{\mathbb{A}^m_k} \), one one hand we have

\[
i^! (\mathcal{O}_{\mathbb{A}^m_k}) (\mathcal{O}_{\mathbb{A}^m_k}) = i^! \circ \mathcal{R} \text{Hom} (\mathcal{O}_{\mathbb{A}^m_k}, K_{\mathbb{A}^m_k}) \cong i^! (K_{\mathbb{A}^m_k}) = K_X
\]

On the other hand, this is \( \mathbb{D}_X (i^! (\mathcal{O}_{\mathbb{A}^m_k})) \cong \mathbb{D}_X (\mathcal{O}_X) = \mathcal{R} \text{Hom} (\mathcal{O}_X, K_X) \).

Since \( X \) is affine, this is

\[
\text{Hom}_{\mathbb{D}_X} (\mathcal{O}_X, K_X) \cong \text{Hom}_{\mathbb{D}_X} (\mathcal{O}_X, i^! K_{\mathbb{A}^m_k}) \cong \text{Hom}_{\mathbb{D}_{\mathbb{A}^m_k}} (i_* \mathcal{O}_X, \mathcal{O}_{\mathbb{A}^m_k} [m])
\]

and \( \mathbb{A}^m_k \) is smooth over \( k \) so \( K_{\mathbb{A}^m_k} \cong \Omega^n_{\mathbb{A}^m_k/\mathbb{A}^m_k} \cong \Omega^n_{\mathbb{A}^m_k} [m] \). Note that \( \mathcal{O}_* \) is exact because it is a closed embedding.

Thus the cohomologies of \( K_X \) are precisely \( \text{Ext}^i_{\mathbb{A}^m_k} (\mathbb{A}^m_k/\mathcal{O}_{\mathbb{A}^m_k}) \) where \( \text{dim}(\text{supp}(A)) = n \). Let \( A' = k[x_1, \ldots, x_m] \) and consider \( A \) as a f.g. \( A' \)-module. Recall that for \( i < \text{codim}(\text{Supp}(A)) \), \( \text{Ext}^i_{A'} (A, A') = 0 \). We have \( \text{codim}(\text{supp}(A)) \leq m - \text{dim}(\text{supp}(A)) = m - n \) and since \( \text{cd}(A') = m \), \( \text{Ext}^i_{A'} (A, A') = 0 \) for \( m > i \). Therefore \( \text{Ext}^i_{A'} (A, A') \) can only be nonzero for \( i \in [m - n, m] \). Equivalently, \( \text{Hom}_{\mathbb{D}_{\mathbb{A}^m_k}} (i_* \mathcal{O}_X, \mathcal{O}_{\mathbb{A}^m_k} [m]) \) lives in cohomological degrees \([-m, n]\).

For the second part, recall that \( A \) is CM of dimension \( n \) as a ring if and only if it is so as an \( A' \)-module. Take \( m \in \text{supp}(A) \). Then since \( A'_m \) is regular, \( A_m \) is CM of dimension \( n \) if and only if \( D(A_m) \) is concentrated in degree \( m - n \) (and non-zero there). Since \( H^i D(A_m) \cong \text{Ext}^i_{A_m} (\mathcal{O}_{\mathbb{A}^m_k}, A_m) \), we conclude that \( A \) is CM of dimension \( n \) if and only if \( \text{Ext}^i_{A'} (A, A') \) is non-zero exactly when \( i = m - n \), which finishes the proof. \( \square \)
**Proposition.** Let $X$ be a projective scheme over $k$, $\mathcal{L}$ an ample line bundle (i.e. the inverse image of $\mathcal{O}(1)$). Then the following are equivalent:

i) $X$ is CM of dimension $n$.

ii) $H^i(X, \mathcal{M} \otimes \mathcal{L}^{-k}) = 0$ for all $i \neq n, k \gg 0, \mathcal{M}$ locally free (equivalently, for any fixed $\mathcal{M}$).

Proof: $H^i(X, \mathcal{M} \otimes \mathcal{L}^{-k}) = \text{Ext}^{-i}((\mathcal{M} \otimes \mathcal{L}^{-k}, K_X) = H^{n-i}(X, K_X[n] \otimes \mathcal{M}^\vee \otimes \mathcal{L}^k)$ and we now use the previous result.

**Proposition. (PSET)** Let $f: X \to Y$ be a finite map between Noetherian schemes. In this case, the functor $f^*: D^+_{\text{QCoh}}(Y) \to D^+_{\text{QCoh}}(X)$ can be explicitly described as follows: locally, if $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$, the corresponding functor $D^+(A-\text{mod}) \to D^+(B-\text{mod})$ is the right derived functor of $M \to \text{Hom}_A(B, M)$

Proof: Consider the following diagram:

$$
\begin{array}{ccc}
\text{Spec}(A) & \xrightarrow{j_1} & X \\
\downarrow f' \quad & & \quad \downarrow f \\
\text{Spec}(B) & \xrightarrow{j_2} & Y
\end{array}
$$

We then deduce $j_2 f^* = j_1 f^!$. But $j_2, j_1$ are just $j_2, j_1$ i.e. restrictions as $j_1, j_2$ are open embedding. Therefore to prove the problem it suffices to work with $f'$ i.e. we can assume that we are in the case $\text{Spec}(B) \to \text{Spec}(A)$. In the following, let’s assume $X = \text{Spec}(B), Y = \text{Spec}(A)$.

Because the map is proper, we know that $f^!$ is the right adjoint to $Rf_*$ i.e. the right derived functor of the map that sends a $B$-module $M$ to $M$ regarded as an $A$-module. However in this case $Rf_* = f_*$ because the map is affine, or alternatively by direct computation as the functor does nothing to the complex except viewing it as consisting of modules over a different ring.

It suffices to show that the right derived functor $Rg$ of $M \to \text{Hom}_A(B, M)$ is the right adjoint to $f_*$. For this, we need a map $id \to Rg \circ f_*$ i.e. a map from $M$ to $\text{Hom}_A(B, M)$ where $M$ is a $B$-module. This natural map is simply the map $m \to (b \to bm)$.

Also, for $M^*$ a complex of $B$-modules and $N^*$ a complex of $A$-modules, we need the following composite map to be an isomorphism:

$$
\text{Hom}_{D(A-\text{mod})}(M^*, N^*) \to \text{Hom}_{D(B-\text{mod})}(Rg(M^*), Rg(N^*)) \to \text{Hom}_{D(B-\text{mod})}(M^*, Rg(N^*))
$$

Instead of doing this directly, we shall do something else: we will prove that $g$ id right adjoint to $f_*$ in the usual category, which means that there is an isomorphism as above in the original category, and the map and its inverse will therefore transcend to the derived category.

Indeed, let $M$ be a $B$-module and $N$ an $A$-module. Consider a map $h$ of $A$-modules from $M$ to $N$. We can create a map of $B$-modules from $M$ to $\text{Hom}_A(B, N)$ as follows: $m \in M$ goes to $(b \to b \cdot m)$. Conversely, assume we have a map $h'$ of $B$-modules from $M$ to $\text{Hom}_A(B, N)$. We then construct $h \in \text{Hom}_A(M, N)$ by sending $m$ to $h'(m)(1)$.

Remark: in fact this procedure can be done directly in the derived category, if we assume that $N^*$ consists of a complex of projectives. □

**Proposition. (PSET)** Let $f: X \to Y$ be an etale map.

a) In this case, $f^! \cong f^*$

b) If $f$ is also finite, then the functors $f_*$ and $f^*$ are both left and right adjoint to each other.

Proof: a) We prove before that for a smooth map, $f^! = f^* \otimes \Omega^\bullet_{X/k}[n]$ - particular since the map is etale, $n = 0$ so we get $f^! = f^*$.

b) One adjunction is old. The other adjunction holds in the derived category, because both maps are exact - $f_*$ is exact because the map is finite and $f^*$ is exact because the map is etale. It remains to see that being adjoint in the derived category implies adjoint in the original one. This is because $\text{Hom}_{\text{QCoh}(-)}(\mathcal{F}_1, \mathcal{F}_2) \cong \text{Hom}_{\text{DQCoh}(-)}(\mathcal{F}_1, \mathcal{F}_2)$ where $\mathcal{F}_1, \mathcal{F}_2$ are sheaves which are also regarded as one-object complexes. This was proven
indirectly using the functor $Ext$, since by definition $\text{Hom}_{DQCoh}(\mathcal{F}_1, \mathcal{F}_2) = Ext^0(\mathcal{F}_1, \mathcal{F}_2)$ and we know that $Ext^0(\mathcal{F}_1, \mathcal{F}_2) = Hom_{QCoh}(\mathcal{F}_1, \mathcal{F}_2)$.

Alternatively, we can construct a morphism in the derived category from a morphism in the usual one, and conversely we can construct a morphism in the usual category from the morphism in the usual one by looking at the cohomologies. We need to show that two huts in the derived category that induce the same morphism on cohomologies. Conversely, we can construct a morphism in the usual category from the morphism in the usual one by looking at the $Ext$-we know that degree is additive with respect to the tensor product because $2H^0(X, \mathcal{O}_X)$.

Proposition. (PSET) Let $X$ be a projective smooth connected curve over a field $k$.

a) $dim(H^0(X, \Omega_X)) = g$ - recall the notion of genus.

b) $deg(\Omega_X) = 2g - 2$

c) If $L$ is a line bundle with $deg(L) > 2g - 2$, then $H^1(X, \mathcal{L}) = 0$.

c') If $deg(L) = 2g - 2$, then $H^1(X, \mathcal{L}) = 0$ unless $\mathcal{L} \cong \Omega_X$

d) For any closed point $x \in X$, the line bundle $\mathcal{O}(x)$ is ample.

e) For any closed point $x \in X$, the complement $X - x$ is affine.

Proof: a) By Serre Duality, $H^1(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X^\vee \otimes \Omega_X) = H^0(X, \Omega_X)$ and therefore the conclusion follows since $H^1(X, \mathcal{O}_X)$ has dimension $g$ by definition.

b) By Riemann-Roch, $\dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = 1 - g + deg(\Omega_X)$. However, by Serre Duality, $\dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X) - dim H^0(X, \mathcal{O}_X) = g - 1$ and from here we deduce $deg(\Omega_X) = 2g - 2$.

c) By Serre duality, $H^1(X, \mathcal{L})$ is dual to $H^0(X, \mathcal{L}^\vee \otimes \Omega_X)$ and the latter is a line bundle of degree $2g - 2 - deg(\mathcal{L})$. We know that degree is additive with respect to the tensor product because $[\mathcal{L}_1 \otimes \mathcal{L}_2] = [\mathcal{L}_1] + [\mathcal{L}_2] - [\mathcal{O}_X]$ according to a previous problem set.

It remains to show that $H^0(X, \mathcal{L}) = 0$ if $\mathcal{L}$ has negative degree. Indeed, we know $\mathcal{L}$ is $\mathcal{O}(D)$ for some $D$ of positive degree $deg(D) = -deg(\mathcal{L})$. We can then map $\mathcal{O}(D) \to K_X \otimes \mathcal{O}_X$ and we see that a global section $s$ of $\mathcal{L}$ is naturally in $K_X$. For it to be in $\mathcal{O}(D)$ we must have $(s) - D$ effective where $(s)$ is the divisor associated to $s$. But we know that the degree of $(s)$ is $0$ hence $(s) - D$ has negative degree and is effective which concludes the proof.

c') As above, we now have $(s) - D$ of degree $0$ so it is effective if and only if it is $0$ i.e. $(s) = D$ so $D$ is principal meaning that $\mathcal{L} \cong \mathcal{O}_X$. Returning to the old notations, this means $\mathcal{L}^\vee \otimes \Omega_X \cong \mathcal{O}_X$ and tensoring with $\mathcal{L}$ we get $\mathcal{L} \cong \Omega_X$.

d) Recall (Hartshorne problem II.6.11) that every coherent sheaf $M$ can be put into a short exact sequence $0 \to \mathcal{O}(D) \to M \to T \to 0$ where $T$ is torsion, and so it suffices to show that $H^1(X, \mathcal{O}(D + nx)) = 0, H^1(X, T \otimes \mathcal{O}(nx)) = 0$ for $n \gg 0$. The first one follows from c). For the second, we use the fact that $T$ is a successive extension of skyscraper sheaves $\mathcal{O}_p$ and so it suffices to prove the assertion for these. Indeed, we have the short exact sequence $0 \to \mathcal{O}(-p) \to \mathcal{O} \to \mathcal{O}_p \to 0$ and so we have the short exact sequence $0 \to \mathcal{O}(nx - p) \to \mathcal{O}(nx) \to \mathcal{O}_p \otimes \mathcal{O}(-nx) \to 0$. By taking the long exact sequence of cohomology, $H^1(X, \mathcal{O}(nx)) = 0$ for large $n$ hence $H^1(X, \mathcal{O}_p \otimes \mathcal{O}(-nx)) = 0$ for large $n$ as well.

e) Consider the sheaf $\mathcal{O}(nx)$ which eventually will be very ample - by Sherry’s notes or Hartshorne II.7. We use the result in II.7.1. to construct a map into projective space: recall that there one took global sections $s_1, \ldots, s_n$ that generate the sheaf, and took $X_i$ be the locus of invertibility of these sections - and the construction identified $X_i$ with the preimage of the affine $U_i$ of the projective space. In particular, we can always add 1 to the set of sections (remember $\mathcal{O}_X \to \mathcal{O}(nx))$ so assume $s_1 = 1$. But the locus of invertibility of 1 in $\mathcal{O}(nx)$ is $X - \{x\}$ pretty much by definition. So $X - \{x\}$ is the preimage of the affine $U_1$ under the constructed closed embedding, so it is affine. (Note that using the construction in II.7.6., as long as some sections define a closed embedding, adding some more sections still keeps the corresponding morphism to projective space a closed embedding - or alternatively use II.7.3. to see that adding more sections still makes it a closed embedding.)

Proposition. (PSET) Let $f : X \to Y$ be a map between smooth schemes of finite type over a field $k$ with $\dim(X) = \dim(Y) = n$. Assume $X$ and $Y$ are connected and $f$ is dominant. Assume that $f$ is generically separable,
i.e. the fraction field extension $K(X)/K(Y)$ is separable. We define the ramification locus of $f$ as the complement of the maximal open subset $\overset{\circ}{X} \subset X$ such that $f|_{\overset{\circ}{X}}$ is etale.

a) The ramification locus coincides with the support of the coherent sheaf $\Omega_{X/Y}$.

b) The ramification locus of $f$ (if non-empty) is a union of irreducible components each of which has codimension 1 in $X$. The support of $\Omega_{X/Y}$ coincides with the locus of vanishing of the map of line bundles $\Omega_{X/k}^n \to \Omega_{Y/k}^n$.

c) If $f$ is etale off codimension 2, then it’s etale.

Proof: We can cover $X$ and $Y$ by affines that map into each other, i.e. affines $Spec(B) \to Spec(A)$. Moreover because the map is dominant the image is dense so for every $A$ there is at least one $B$ such that $Spec(B)$ maps to $Spec(A)$ - and this implies that $f^*(\Omega_{Y/k})$ is actually locally free of rank $n$. Also, as the extension $K(X)/K(Y)$ is separable, the map is etale and the generic point so $\Omega_{X/Y}$ is torsion.

a) As shown before, the map is etale (at $x$) if and only if $f^*(\Omega_{Y/k}) \to \Omega_{X/k}$ is an isomorphism i.e. $\Omega_{X/Y}$ is 0 (at $x$).

b) More generally, if we have an injection $\mathcal{F}_1 \to \mathcal{F}_2$ of locally free sheaves of rank $n$, then we claim that it is an isomorphism if and only if $\Lambda^n \mathcal{F}_1 \to \Lambda^n \mathcal{F}_2$ is an isomorphism (of vector bundles).

Indeed, locally on small enough affines $Spec(A)$ or on stalks we have a map $A^n \to A^n$ which corresponds to something in $M_{n,n}(A)$. The map is an isomorphism (locally) if and only if the determinant is invertible - which of course is equivalent to the map $\Lambda^n \mathcal{F}_1 \to \Lambda^n \mathcal{F}_2$ being an isomorphism as it is exactly equal to multiplication by the determinant.

Next, it suffices to show that if $0 \to \mathcal{L}_1 \to \mathcal{L}_2 \to T \to 0$ is a short exact sequence with $\mathcal{L}_1, \mathcal{L}_2$ vector bundles isomorphic at the generic point, then $Supp(T)$ is the union of irreducible components of codimension 1. Indeed, we need to show that the generic points of the irreducible components have codimension 1 - and this is enough to be done locally. Locally the map is $A \xrightarrow{a} A$ where $a$ is a non-zero divisor because the map is injective. The support of $T$ is $V(a)$ - i.e. all prime ideals such that $a$ is not invertible in the localization. Either $a$ is a unit in which case the support is non-empty or $a$ is a non-unit which is not zero (since the map is an isomorphism at the generic point). It follows that all minimal prime ideals containing $a$ have height 1 by Krull’s Hauptidealsatz and the conclusion follows since minimal primes correspond to the irreducible components.

c) The locus where $f$ is not etale, by the previous two parts, is either empty or the union of closed subsets of codimension 1. Since the locus where $f$ is not etale has codimension at least 2, it must be the former. $\Box$

**Proposition. (PSET)** Let $X, Y$ be two smooth connected projected curves over an algebraically closed field, and let $f : X \to Y$ be a dominant (i.e. non-constant) map. We define the degree of $f$ to be the degree of the field extension $K(X)/K(Y)$ or equivalently the rank of $f_* (\mathcal{O}_X)$ as a locally free sheaf on $Y$. (Note that $f$ is automatically flat by the dimension criterion!)

a) For a line bundle $\mathcal{L}$ on $Y$, we have $deg(f^*(\mathcal{L})) = deg(f) \cdot deg(\mathcal{L})$

Assume now that $f$ is generically separable. For a closed point $x \in X$ we consider the ramification index $r_x(f)$ as the length of the torsion coherent sheaf $\Omega_{X/Y}$.

b) (Hurwitz’s theorem)

\[(2g_X - 2) = (2g_Y - 2) \cdot deg(f) + \sum_x r_x(f)\]

c) For $x \in X$ and $y = f(x)$ let $t_y$ be a uniformizer at $y$ i.e. a local section of $\mathcal{O}_Y$ such that $v_y(t_y) = 1$, where $v_y$ is the valuation on the corresponding DVR. Then $r_x(f) \geq v_x(t_y \circ f) - 1$.

d) If $\text{char}(k) = 0$, the inequality in c) is an equality. (It is also an equality if the characteristic does not divide $v_x(t_y \circ f)$)

Proof: Apparently a map between projective schemes is proper, so the map is proper. Hence the map is proper so in particular its image is closed, but since it is dense, it is everything so the morphism is dominant. Also, it is flat by proposition 1.4.1. Indeed, let’s choose $x \in X$ that is not the generic point, so it maps to $y$ which is not the generic
point, and therefore the dimension of $X_x$ and $Y_y$ is 1. It remains to show that the dimension of $X \otimes k_y$ is 0, which is true: this set does not contain the generic point so its dimension is strictly smaller.

Next, we claim that the map is finite. It is proper so it suffices to show it’s quasi-finite, by problem 10a from PS 11 from the first semester. Indeed, the preimage of $y$ is $X \otimes k_y$ - which is a closed subscheme which must have finitely many irreducible components so is finite as every point is closed and irreducible (except the generic point, but the preimage of the generic point is the generic point).

Then $f_*O_X$ is locally free over $Y$, hence its rank is $d$ (which is the generic rank).

a) For $D = (y_0)$ where $y_0$ is the generic point of $x$ this is true because $f^*y_0 = x_0$ and $deg(x_0) = ddeg(y_0)$ since $d$ is the degree of the extensions of the local fields at $x_0$ and $y_0$.

For other points $y$, let’s choose $x$ in the preimage of $y$ and $U$ an affine around $x$ containing all preimages of $y$ - this can be done because removing only one point makes the scheme affine as shown in problem 3. Then locally we have a map $B \rightarrow A$ and we want to find all prime ideals of $B/pB$ where $p$ is a maximal ideal of $A$. We can localize at $p$ so we can assume that $A$ is a DVR.

In that case, because $B$ is finite over $A$, $B/pB$ is a a finite dimensional vector space over $k_p$. But as it is flat, $B$ is actually free over $A$ of dimension $d$.

We now know by the standard Dedekind domain theory that $pB = \prod q_i^{e_i}$ where $q_i$ are all primes above $p$ and $\sum e_if_i = \dim_{k_p} B \otimes k_p = d$ where $f_i$ is the residue field extension. But then $f^*(y) = \sum e_iq_i$ and as $deg(q_i) = f_*deg(p)$ we conclude that $deg(f^*(y)) = ddeg(y)$ and this finishes the proof as every line bundle is $O(D)$ for some $D$ and the degree of $O(D)$ equals the length of $O_D$ and the above calculation shows that it gets multiplied by $d$ via $f^*$.

b) By definition, the length of a torsion sheaf is the degree of the sheaf (where the length is the sum of the lengths of the stalks at the finitely many points in its support). We have the short exact sequence $0 \rightarrow f^*\Omega_{Y/k} \rightarrow \Omega_{X/k} \rightarrow \Omega_{X/Y} \rightarrow 0$. We take the degree of that, and use part a) together with the fact that the degree of $\Omega_{Z/k}$ is $2g_Z - 2$. This immediately gives us the result.

c) In the local case, we have the following situation $A \leftarrow B$ a map of DVR, with $v_B(t_y) = m$ where $t_y$ is the generator of the maximal ideal of $A$. We wish to estimate the length of $\Omega_{B/A}$. Consider the possibly smaller subring $A[t_x]$ of $B$ so we can assume $B = A[t_x]$. Let’s define a surjective map from $\Omega_{B/A}$ to $B/t_x^m$. This is the same as a derivation, and we simply send an element $c_0 + c_1t_x + \ldots + c_rt_x^r$ for $c_i \in A$ to $c_1 + 2c_2t_x + \ldots + mc_{m-1}t_x^{m-1}$. It is obviously a derivation, the subtler part is that it is well-defined i.e. independent of the choice of the $c_i$, representing the same element. Alternatively since the map is linear, if $c_0 + c_1t_x + \ldots + c_rt_x^r = 0$ we want $c_1 + 2c_2t_x + \ldots + mc_{m-1}t_x^{m-1}$ to be divisible by $t_x^m$. Indeed, in $B$ $c_0$ must be divisible by $t_x$ - but as $c_1$ is in $A$ it must be actually divisible by $t_y \sim t_x^m$. This means that $c_1t_x$ must be divisible by $t_x^2$ so now $c_1$ is divisible by $t_x^m$ and so on, meaning $c_0, \ldots, c_{m-1}$ are divisible by $t_x^{m-1}$ and then the conclusion follows as $B/t_x^m$ has length $m$ via the natural filtration by powers of $t_x$.

Note: we have said that we can assume $B = A[t_x]$. This is because in general, if $A \subset B \subset C$ then the length of $\Omega_{B/A}$ as a $B$-module is at most equal to the length of $\Omega_{C/A}$ as a $C$-module. The reason being simply any filtration $I_1 \subset \ldots \subset I_k$ lift to a filtration $CI_1 \subset \ldots \subset CI_k$. No two elements in the filtration can become equal because as easily seen in this case, $\Omega_{C/A} = \Omega_{B/A} \otimes_B C$.

d) Let’s keep the notations from the previous part. $\Omega_{B/A}$ is generated over $B$ by elements $dx$ for $x \in B$. Let’s prove that $du \in d(m_B)$ for $u$ a unit. Indeed, take $f$ the minimal polynomial of $u$ in $B/t_yB$ over $k_A$ then $df(u) \in d(m_B)$ as $f(u)$ is divisible by $t_y$ but $df(u) = f'(u)du$ and $f'(u)$ is invertible as $f$ is separable over the residue field since the residue field has characteristic 0.

In particular, it follows that for every unit $u$ in $B$ there is some element $u' \in m_B$ such that $d(u - u') = 0$.

The next observation is that since $\Omega_{B/A}$ is torsion, it must be annihilated by some power of $t_y$.

Now let $A \subset A'$ be the subring of elements of $B$ whose differential is 0. We claim that $A'[t_y]$ generates $\Omega_{B/A}$ under the map $d$. Indeed, we know that for every $u \in B$ there is some $v_0$ with $u - v_0t_y \in m_B$. Writing $u - v_0t_y = t_yu_1$ we continue the process until we approximate $u$ by something in $A'[t_y]$ up to $m_B^k$ but for high enough $k$, $m_B^k$ projects to 0 in $\Omega_{B/A}$.

So it suffices to assume $B = A'[t_y]$ - in which case we immediately see that $\Omega_{B/A}$ is generated as a $B$-module by
$dt_y$. In that case note that $ut_y^m = t_x$ so that $d(u t_y^m) = 0$ where $u$ is a unit. Again we can write $u = u_0 t_y u_1 + \ldots + t_y u_r$ (up to some high power of $m_B$) where $u_i \in A'$ and then we deduce that $u_0 t_y^m t_x + \ldots + t_y u_r (m + r) t_y^{m + r - 1} (t_y^m) = 0$ i.e. $t_y^{m - 1} (m_0 t_y + t_y (m + 1) u_1 + \ldots) (t_y^m) = 0$. As $m_0 + t_y (m + 1) u_1 + \ldots$ is a unit ($m \notin m_B$ because $k$ has infinite characteristic), we find $t_y^{m - 1} = 0$ hence $\Omega_{B/A}$ is a quotient of $B[t]/t^m$ which has length $m$, as desired. □

04/20/2010

Let $X$ be a scheme of finite type over $k$. Recall that $K_X \in D_{QCoh(X)}^-(X)$ and if $X$ is smooth, $K_X = \Omega^n_X [n]$ (remember smooth⇒regular⇒CM). We also defined the functor $\mathbb{D}_X$ by $\mathbb{D}_X(\mathcal{F}) = R \mathcal{H}om(\mathcal{F}, K_X)$ and $\mathbb{D}_X^2 = \text{Id}$.

If $X \overset{i}{\to} Y$, $f^! : D_{QCoh}(Y) \to D_{QCoh}(X)$, $f^! K_Y = K_X$.

If $X$ is proper over $k$, then

$$\text{Hom}_{\mathbb{D}(X)}(\mathcal{F}, K_X) = \text{Hom}_{k-\text{Vect}}(R \Gamma(X, \mathcal{F}), k)$$

Recall that if $X$ is smooth, $K_X = \mathcal{L}[n]$ where $\mathcal{L}$ is a line bundle ($\Omega^1_X$) and we can see that at any point $x \in X$, $\mathcal{L}_x \cong \Lambda^{top}(T^*_x X)$. Here is how to see it:

Consider the embedding $\text{Spec}(k) \hookrightarrow X$ corresponding to $x$, then $R^i (K_X) = 0$, but on the other hand since we have regular embedding, it is also $R^i (\mathcal{L})[n] = \Lambda^{top}(T^*_x X) \otimes L x^* \mathcal{L}$ (note that $T_x X$ is $N_{X/\text{Spec}(k)}$ in this case) hence $\Lambda^{top}(T^*_x X) \otimes \mathcal{L}_x \cong 0$ so $\mathcal{L}_x \cong \Lambda^{top}(T^*_x X)$.

**Proposition.** Let $X$ be integral, projective, of dimension $\geq$. Then $X$ is $S2$ if and only if for some/any locally free $\mathcal{M}$ and an ample line bundle $\mathcal{L}$,

$$H^1(X, \mathcal{M} \otimes \mathcal{L}^{-k}) = 0 \text{ for } k \gg 0$$

Proof: Like before, the criterion is equivalent by Serre duality to $H^{-1}(X, K_X) = 0$.

More generally:

**Lemma.** $X$ is $S_k$ if and only if $H^i (K_X) = 0$ for $i \in [-k + 1, 0]$

Proof: $K_X = \mathbb{D}_X(\mathcal{O}_X)$, now embed $X \overset{i}{\to} Y$, $i, K_X = \mathbb{D}_X(i_* (\mathcal{O}_X))$. So we reduce to $M$ a module over a regular $k$-algebra $A$ of dimension $n$. $\mathbb{D}_Y(M) = R \mathcal{H}om(M, \Omega^n_Y)[n]$,

Because we are interested in what cohomologies exist, we can replace $\Omega^n_Y$ by any line bundle for example $\mathcal{O}_Y$. So we reformulate:

**Lemma.** $M$ is $S_k$ if and only if $\text{Ext}^i (M, A) = 0$ for $n - k + 1 \leq i \leq n$.

Proof of lemma 2: this criterion says that projection is $\geq n - k$, but $\text{depth} + \text{p.d.} = n$ hence depth at any localization is $\leq k$, as desired. □

**Corollary.** $X$ connected, and normal (hence irreducible) and projective. Let $Y \subset X$ the support of an ample Cartier divisor i.e. $I_Y$ is a line bundle.

$0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ and $\mathcal{O}(Y) = I_Y^{-1}$ is ample. Then $Y$ is connected.

Proof: We look at $0 \to \mathcal{O}(-n Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ where $Y_n$ is a different subscheme structure. Then $H^0(X, \mathcal{O}_X) \to H^0(Y_n, \mathcal{O}_{Y_n}) \to H^1(X, \mathcal{O}(-n Y))$ and the last module is (eventually) $0$ since $X$ is projective. while the first is $k$ because the scheme is connected and integral. It follows that for $n \gg 0$ $k \to H^0(Y_n, \mathcal{O}_{Y_n})$.

If $Y$ were disconnected it were to have at least 2 components and $H^0(Y_n, \mathcal{O}_{Y_n})$ would have had dimension $\geq 2$.

□

Let $X$ be a projective scheme. We look at $H^0(X, K_X) \cong (H^0(X, \mathcal{O}_X))^\vee$. Consider the element in $H^0(X, \mathcal{O}_X)$ corresponding to 1 (recall that $k$ maps to $H^0(X, \mathcal{O}_X)$), then it corresponds to a map $H^0(X, K_X) \to k$ which will be denoted by $\int_X$. Note that if $X$ is integral, the maps are isomorphisms.

Consider $H^1(X, \mathcal{F}) = R^1 \mathcal{H}om(\mathcal{O}_X, \mathcal{F})$. Then Serre Duality gives a perfect pairing:
Consider $U \xrightarrow{i} X \xrightarrow{j} U$ a CO pair. Recall that $i_*R^i(\mathcal{F}) = Cone(\mathcal{F} \to Rj_*j^*(\mathcal{F}))$ i.e. there is a distinguished triangle

$$i_*R^i(\mathcal{F}) \to \mathcal{F} \to Rj_*j^*(\mathcal{F}) \to i_*R^i(\mathcal{F})$$

corresponding to the admissible triple

$$D^+_{QCoh,Y}(X) \xrightarrow{i_*} D^+_{QCoh}(X) \xrightarrow{j^*} D^+_{QCoh}(U)$$

Consider $x \in X$ and $pt \xrightarrow{i} X$. We define the functor

$$R\Gamma_{\{x\}}(X, \mathcal{F}) := R\Gamma(X, i_*R^i(\mathcal{F}))$$

**Theorem.** i) $R\Gamma_{\{x\}}(X, K_X)$ lives only in degree 0.
ii) There exists a canonical "residue" map $Res_x : R\Gamma_{\{x\}}(X, K_X) \to k$
iii) The following commutative diagram holds

$$\oplus R^0\Gamma_{\{x\}}(X, K_X) \xrightarrow{R^0\Gamma(X, K_X)} \oplus_{Res_x} k \xleftarrow{f_X}$$

Remark: for curves, $Res(f(t)dt)$ is the coefficient of $t^{-1}$ in $f$.

Assume $X = Spec(A)$ is of finite type over $k$. It can be shown, using an embedding into affine space, that $K_X = Hom_k(A, k) = A^\vee$, and with respect to this the map dual to $k \to A$ is precisely integration $A^\vee \xrightarrow{\int} k$.

Consider the closed subschemes $Y_n = Spec(O_{X,x}/m_{x,n}^n)$ $\xrightarrow{i_n} X$ and we have the direct system

$$Y_1 \xrightarrow{i_{1,2}} Y_2 \xrightarrow{i_{2,3}} \ldots \subset X$$

For all $n$, we look at

$$(R)\Gamma(X, i_nR^i_n(K_X)) = \Gamma(Y_n, R^i_n(K_X))$$

(or any $\mathcal{F}$ instead of $K_X$, for the matter)

**Lemma.**

$$R\Gamma_{\{x\}}(X, \mathcal{F}) = \lim_n \Gamma(Y_n, R^i_n(\mathcal{F}))$$

Proof: This is local, so we can assume $X = Spec(A)$. We can replace $\mathcal{F}$ by an injective resolution. Then what it is saying that $\{f \in \mathcal{F} \mid \exists n, m^n \cdot f = 0\} = \lim_n (\mathcal{F}m^n)$ which is obvious. □

Applying this lemma to $\mathcal{F} = K_X$ we deduce

$$R\Gamma_{\{x\}}(X, K_X) = \lim_n \Gamma(Y_n, K_{Y_n})$$
We also claim that this limit is compatible with the integration maps.
Indeed, the integration maps are maps $(\mathcal{O}/m^n)_{\geq i} \to k$ which are equivalent to $k \to \mathcal{O}/m^n$ corresponding to $1$, which are compatible.

To prove the theorem, we want to show that the following diagram commutes

$$
\begin{array}{ccc}
\Gamma_{\{x\}}(X, K_X) & \longrightarrow & H^0(X, K_X) \\
\downarrow \quad \text{Res}_x & & \downarrow \sim \\
& & k
\end{array}
$$

It is enough to construct commutative diagrams for each member of the direct limit.
We have a natural map $(i_n)_*(i_n^!(K_X)) \to K_X$ which yields the upper map in the following diagram

$$
\begin{array}{ccc}
\Gamma(X, (i_n)_*(i_n^!(K_X))) & \longrightarrow & \Gamma(X, K_X) \\
\downarrow f_{Y_n} & & \downarrow f_X \\
& & k
\end{array}
$$

and we claim that it commutes.
More generally, if $Y \xrightarrow{f} X$ proper, $K_Y \cong f^!K_X$. By adjunction there is a map (called the trace map) $f_*K_Y \to K_X$
If $Y \xrightarrow{f} X \xrightarrow{g} Z$ proper then

**Lemma.** The following diagram commutes

$$
\begin{array}{ccc}
g_*(f_*(K_Y)) & \longrightarrow & g_*(K_X) \\
\downarrow \sim & & \downarrow id \\
(g \circ f)_*(K_Y) & \longrightarrow & K_Z
\end{array}
$$

The commutativity follows directly by adjointness. Now taking $Z = k$, the above diagram follows.

Let $X$ be a smooth, projective connected curve over $k$.

**Lemma.** For any $x \in X$, $\mathcal{O}(x)$ is ample.

Prove: all higher cohomologies (of which there can be only one since the curve is regular of dimension 1), $H^1(X, \mathcal{O}(nx)) = 0$ for $n \gg 0$ since the degree of $\mathcal{O}(nx) \otimes \mathcal{F}$ will be $\gg 0$, and by a previous proposition it follows that its first cohomology is zero. \(\square\)

**Lemma.** Let $\mathcal{M}$ be locally free on a curve $X$, $x \in X$. Then $\Gamma(X, \mathcal{M}(-nx)) = 0$ for all $n \gg 0$.

Proof: First assume $\mathcal{M}$ is a line bundle. Choose $n > deg(M)$ and we are done.
If $\mathcal{M}$ has a finite filtration with quotients line bundles then we are done as well by the line bundle case.
Now we claim that anything has such a filtration by line bundles, and this will finish the proof.

**Corollary.** If $X$ is a proper curve, then $X$ is projective, and for any $x \in X$, $X - \{x\}$ is affine. Indeed, we choose $\mathcal{O}(x)$ ample on $X$, then by Sherry’s project it produces an embedding into $\mathbb{P}^n$ and $X \cap \mathbb{P}^{n-1} = x$ thus $X - x \in \mathbb{A}^n$ thus it is affine.

**04/23/2010.**

Let $X$ be projective, smooth, connected curve over $k$ algebraically closed field.
Recall the genus of $X$, defined to be $\dim_k H^1(X, \mathcal{O}_X)$. We also recall that $K(X) \xrightarrow{\sim} \mathbb{Z} \oplus Pic(X)$ given by $[\mathcal{F}] \mapsto rk(\mathcal{F}), det(\mathcal{F})$ and the inverse map given by $(1, 0) \to [\mathcal{O}_X]$ and $(0, \mathcal{L}) \to [\mathcal{L}] - [\mathcal{O}_X]$.
For example if $D \subset X$, $[\mathcal{O}_D] \sim (0, \mathcal{O}(D))$
The map $Pic(X) \xrightarrow{deg} \mathbb{Z}$ so we get a map $K(X) \to \mathbb{Z} \oplus \mathbb{Z}$
In particular, $\mathcal{O}(D)$ gets mapped to $(0, \deg(D))$ and $\deg(D) = \dim_K \Gamma(X, \mathcal{O}(D)/\mathcal{O}_X)$.

We also have the Euler characteristic map $\chi: K(X) \to \mathbb{Z}$

$$\chi([\mathcal{F}]) - \dim_k(H^0(X, \mathcal{F})) - \dim_k H^1(X, \mathcal{F})$$

Via this map $(1, 0)$ goes to $1 - g$ and we claim that $(0, \mathcal{L})$ goes to $\deg(\mathcal{L})$.

Indeed, this follows from Riemann-Roch that says that $\chi(\mathcal{L})$ is $1 - g + \deg(\mathcal{L})$, and so $\deg(\mathcal{O}(D)) = \deg(D)$.

Remark: the conventions about $\mathcal{O}(D)$ differ. For example, Hartshorne uses $\mathcal{O}(D)$ to denote what we call $\mathcal{O}(-D)$: if $D > 0$, he has the short exact sequence $0 \to \mathcal{O} \to \mathcal{O}(D)$ and $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$.

For curves, Serre duality goes as follows: if $\mathcal{M}$ is locally free on $X$, then

$$(H^1(X, \mathcal{M}))^\vee \cong H^{-1}(X, \mathcal{M}^\vee \otimes K_X) \cong H^{1-i}(X, \mathcal{M}^\vee \otimes \Omega_X)$$

as $K_X = \Omega_X[-1]$.

**Corollary.** i) $\dim_k H^0(X, \Omega) = g$

ii) $\deg(\Omega_X) = 2g - 2$

iii) $\deg(\mathcal{L}) > 2g - 2$ then $H^1(X, \mathcal{L}) = 0$

iv) Any line bundle of positive degree is ample.

v) If $x_1, \ldots, x_n$ are points of $X$, and $\tilde{X}$ is the complement then $\tilde{X} \to X$ is affine.

Proof: The first part follows from Serre duality for $\mathcal{M} = \mathcal{O}$ and $i = 1$. The second part follows from Serre duality for $\mathcal{M} = \mathcal{O}$ and $i = 0$ and Riemann-Roch for $\Omega_X$, since $\chi(\Omega_X) = g - 1$. The last part follows from Serre duality for $\mathcal{L}$ since then $\Omega_X \otimes \mathcal{L}^{-1}$ has negative degree therefore it does not have global sections: global sections line bundle $\mathcal{O}(D)$ are elements $f \in K$ such that $(f) + D > 0$ but $(f)$ has degree 0 so if $D$ has negative degree there are no such sections. The last two parts have been proven before. □

Now consider $j: \tilde{X} \to X$ the open embedding, and let $\mathcal{M}$ be a line bundle.

We then have the short exact sequence $0 \to \mathcal{M} \to j_* j^* \mathcal{M} \to \mathcal{O}(\infty_{x_i}) \to 0$

Because $\mathcal{M} \to X$ is affine, by Leray $j_* j^* M$ has trivial higher cohomologies, and therefore the LES collapses to

$$0 \to H^0(X, \mathcal{M}) \to \Gamma(\tilde{X}, M) \to \oplus \Gamma(\mathcal{M}(\infty_{x_i})/M) \to H^1(X, \mathcal{M}) \to 0$$

Now say $\tilde{X} \xrightarrow{\sim} X$ a dominant map of curves. We claim that is is flat. We need to show that the dimension of the fiber of every closed point is $\leq 0$ as any irreducible component is 0-dimensional, but that is clear.

Therefore the map $\pi^* = L\pi^*$ descends to a map $K(X) \to K(\tilde{X})$.

If the degree of the map is $n$, then diagram chasing as results proven before (namely $\deg(\pi^* D) = n \cdot \deg(D)$) yield the following commutative diagram:

$$\begin{array}{ccc}
K(\tilde{X}) & \xrightarrow{\sim} & \mathbb{Z} \oplus \text{Pic}(\tilde{X}) \\
\pi^* & & (id \oplus \pi^*) \\
\downarrow & & (id, n) \\
K(X) & \xrightarrow{\sim} & \mathbb{Z} \oplus \text{Pic}(X)
\end{array}$$

Moreover, we claim $n$ is also $\deg(K_X/K_X)$ and $rk \pi_* \mathcal{O}_X$ as a locally free sheaf over $X$. (NB: the map is finite as it is proper and quasi-finite).

Proof: If $T$ is torsion then $\deg_X(T) = \dim_k \Gamma(X, T)$. We want to compare it with $\deg(\pi^*(T)) = \dim_k \Gamma(\tilde{X}, \pi^* T)$.

But $\Gamma(\tilde{X}, \pi^* T) = \Gamma(\mathcal{X}, \pi_* \pi^* T)$ which by the projective formula equals $\Gamma(X, T \otimes \pi_* \mathcal{O}_X)$ and since $\pi_* \mathcal{O}_X$ is locally free of rank $n$, the latter is $n \dim_k \Gamma(X, T)$ □

It remains to verify what the map $\text{Pic}(X) \to \text{Pic}(\tilde{X})$ is.

Note that for a torsion sheaf $\tilde{T}$ on $\tilde{X}$, $\deg(\tilde{T}) = \dim_k \Gamma(\tilde{X}, \tilde{T}) = \dim_k \Gamma(\mathcal{X}, \pi_* (\tilde{T})) = \deg(\pi_* \tilde{T})$ and from here we deduce that $\deg(\mathcal{F}) = \deg(\pi_* \mathcal{F})$. 151
We then have a diagram as follows:

\[
\begin{array}{ccc}
Pic(X) & \xrightarrow{\text{deg}_X} & \mathbb{Z} \\
N & \downarrow{id} & \\
Pic(X) & \xrightarrow{\text{deg}_X} & \mathbb{Z}
\end{array}
\]

**Theorem.** The map \( N \) is \( \mathcal{L} \rightarrow \det(\pi_*\mathcal{L}) \otimes \det(\pi_*\mathcal{O}_X)^{\otimes -1} \)

The theorem follows by unwinding the definitions. \( \square \)

Also if \( \mathcal{L} = \mathcal{O}(-\hat{D}) \) we have the short exact sequence \( 0 \rightarrow \mathcal{O}_{\hat{X}}(-\hat{D}) \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_{\hat{D}} \rightarrow 0 \)

**Lemma.** \( N(\mathcal{O}(\hat{D})) = \mathcal{O}(\pi_*(\hat{D})) \)

Proof: Applying the exact functor \( \pi_* \) to the above short sequence and noting that \( \pi_*\mathcal{O}_{\hat{D}} = \mathcal{O}(\pi_*(\hat{D})) \) we take determinants and obtain \( \det(\pi_*\mathcal{O}(-\hat{D})) = \det(\pi_*\mathcal{O}_X) \otimes \mathcal{O}(\pi_*(\hat{D}))^{-1} \) and use the previous theorem. \( \square \)

Now assume that \( \hat{X} \xrightarrow{\pi} X \) is generally separable, thus generically etale, so that \( \Omega_{\hat{X}/X} \) is torsion. Let \( \hat{x} \in \hat{X}, x = \pi(\hat{x}) \).

We obtain a DVR map \( \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\hat{X},\hat{x}} \), let \( t \) be the image of the uniformizer of \( \mathcal{O}_{X,x} \) is \( \mathcal{O}_{\hat{X},\hat{x}} \).

We then deduce that \( \text{length}_{\hat{x}}(\Omega_{\hat{X}/X}) \geq v_\hat{x}(t) - 1 \).

We say that \( \pi \) is tamely ramified at \( \hat{x} \) if \( v_\hat{x}(t) \) is not divisible by \( \text{char}(k)_\hat{x} \).

It is easy to show that \( \text{length}_{\hat{x}}(\Omega_{\hat{X}/X}) = v_\hat{x}(t) - 1 \) if and only if \( \pi \) is tamely ramified at \( x \) - the result immediately follows from the fact that \( \Omega_X \) is free generated by \( du \) where \( u \) is the uniformizer.

Consider now the Frobenius map \( x \rightarrow x^p \) from \( k \) to \( k \). It yields a map \( \text{Spec}(k) \xrightarrow{\text{Frob}} \text{Spec}(k) \). Similarly, if \( A \) is a \( k \)-algebra, we get a map \( \text{Spec}(A) \rightarrow \text{Spec}(A) \).

More generally, for \( X \) over \( \text{Spec}(k) \) we have a map \( X \xrightarrow{\text{Frob}_X} X \) that produces the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Frob}_X} & X \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{\text{Frob}} & \text{Spec}(k)
\end{array}
\]

Note that this is NOT a map of schemes over the ground field i.e. the map \( X \rightarrow \text{Spec}(k) \) changes.

To make it one, we base change:

We take \( X^1 \rightarrow X \) the base change of \( \text{Spec}(k) \xrightarrow{\text{Frob}} \text{Spec}(k) \) then \( X^1 \rightarrow X \) and we map \( X \rightarrow X^1 \) by mapping \( X \rightarrow X \) as the identity and \( X \rightarrow \text{Spec}(k) \) the base map. We will call this map \( X \rightarrow X^1 \) as \( \text{Frob}_1 \).

\[
\begin{array}{ccc}
X & \xrightarrow{\text{Frob}} & X^1 \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \xrightarrow{\text{Frob}} & \text{Spec}(k)
\end{array}
\]

For example consider \( X = \mathbb{A}^1_k \). It is easy to see that \( X^1 = \mathbb{A}^1_k \) and the map \( k[t] \rightarrow k[t] \) is given by \( t \rightarrow t, a \rightarrow a^p \).

**Theorem.** Over an algebraically closed field \( k \), there exists and equivalence of categories between
i) Projective connected smooth curves over \( k \)
ii) Field extensions \( K/k \) with \( \text{tr.deg.}(K/k) = 1 \).

The map is given by sending each curve to its fraction field.
We also know that for each field extension $K'/K$ there is an intermediate maximal separable extension $K^{s'}/K$ determined by $K^{s'}/K$ - separable, $K'/K$ - purely inseparable.

For example $X \overset{Frob}{\longrightarrow} X^1$ induces a purely inseparable map of fraction fields.

**Proposition.** If $X \rightarrow Y$ is a purely inseparable map then $Y = X^{(n)}$ i.e. the map factors as

$$X \rightarrow X^1 \rightarrow (X^1)^1 \rightarrow \ldots \cong Y$$

For a proof of this theorem, see Hartshorne IV.2.

Recall the normalization: if $A$ is an integral domain, $A^n$ is a its integral closure inside the fraction field $K_A$. It localizes, so (integral) schemes can also be localized.

**Theorem.** i) If $X$ is a curve, then $X^n$ is also a curve.

ii) The category of smooth (complete) proper curves is equivalent to extensions $K/k$ of transcendence degree 1.

iii) Every curve is quasi-projective (if it’s complete it’s projective)

iv) Every non-complete curve is affine.

Proof: i) We only need to show that the map is of finite type as dimension cannot increase. This follows from a famous theorem of Noether that if $A$ is f.g. algebra over $k$ then $A^n$ is f.g.

ii) The functor sends a curve to its field of fractions.

First we show fully-faithfulness.

Faithfulness: we need to show that $Maps(X,Y) \subset Maps(K_Y,K_X)$. Indeed, locally on affines $Spec(A) \rightarrow Spec(B)$ it follows from the fact that that $A$ can be canonically embedded into $K_Y$ (and $B$ into $K_X$).

For fullness, we claim that if we have a map $K(Y) \rightarrow K(X)$ then there is an open $\tilde{X}$ such that we can build $\tilde{X} \rightarrow Y$. Indeed, pick an affine $Spec(A) = \tilde{Y} \subset Y$.

We want to map $\tilde{X} \rightarrow \tilde{Y}$, $A \rightarrow \Gamma(\tilde{X},\mathcal{O}_{\tilde{X}})$. Let $f_1,\ldots,f_n$ be generators of $A$, then they (in $K(X)$) go to rational functions that have finitely many poles - we remove them i.e. take $\tilde{X}$ that does not contain them, and because $X$ is regular the images will land inside $\Gamma(\tilde{X},\mathcal{O}_{\tilde{X}})$.

The map now extends to the entire $\tilde{X}$ by the valuative criterion of properness:

$$\begin{array}{ccc}
\tilde{X} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Spec(k)
\end{array}$$

as $Y \rightarrow Spec(k)$ is proper.

Essential surjectivity:

Let $A$ be an integral domain and $K' \supset K_A$.

**Definition.** The normalization of $A$ in $K'$ consists of all elements in $K'$ that are algebraic over $A$.

**Proposition.** If $A$ is finitely generated over a field and $K'$ is an algebraic extension of $K(A)$ then the normalization of $A$ inside $K'$ is also finitely generated, it is also normal and finite as an $A$-module.

Proof: Let $b_1,\ldots,b_m$ be a basis of $K$ over $K(A)$ consisting of elements that satisfy monic polynomials over $A$. Then $\tilde{A} = A[b_1,\ldots,b_m]$ is finite over $A$, and the normalization we seek is $\tilde{A}^n$ which satisfies all conditions required. $\square$

Note that this construction localizes so we can normalize schemes to "extend" $X$ to $\overline{X}$ with $\overline{X} \rightarrow X$ a finite map.
In particular, we apply this situation for the scheme \( \mathbb{P}^1 \) with residue field \( k(t) \) contained in some given field \( K \) of transcendence degree 1. The constructed normalization \( \overline{X} \) will be the required curve.

For part iii) of the theorem, assume \( X \) is complete and we want to show it is projective.

**Proposition.** Let \( X \to Y \). If \( Y \) is projective and \( \pi \) finite then \( X \) is projective.

Proof: \( X \) is complete. We need an ample line bundle. Take \( \mathcal{L} \) an ample line bundle on \( Y \) and use \( \pi^* (\mathcal{L}) \). If \( \mathcal{F} \) is a coherent sheaf then \( H^i (X, \mathcal{F} \otimes \mathcal{L}^\otimes n) \) by the projective formula (and the map being affine) equals \( H^i (Y, \pi_* (\mathcal{F}) \otimes \mathcal{L}^\otimes n) \) that eventually vanishes by \( \mathcal{L} \) being ample on \( Y \). \( \Box \)

This implies the assertion because every smooth complete curve can be mapped to \( \mathbb{P}^1 \) and the map will be finite - we just need to map \( k(t) \) into its fraction field and we know we can do it by the previous part.

We now do the non-smooth case. Take such \( X \) and we want to map it to \( \mathbb{P}^1 \). Let \( \overline{X} \) be the normalization of \( X \) then it can be mapped to \( \mathbb{P}^1 \) (why?).

\( X \) is generically smooth. Let \( x_1, \ldots, x_n \) be the points where it is not smooth and \( y_1, \ldots, y_n \) the preimages in \( \overline{X} \).

We have the following situation: \( A \) a Noetherian local ring of dimension 1, \( A' \) its normalization (semi-local), let \( m_1, \ldots, m_k \) be the maximal ideals of \( A \);

We claim that there exists some \( n \) such that \( m_1^n, \ldots, m_k^n \subset A' \) (inside \( A' \))

Indeed, let \( m \) be the maximal ideal of \( A \) then \( A'/mA' \) is Artinian which implies that eventually \( m_i^n \in A'm \). And we now note that \( A'/A \) is \( m \)-torsion which does what we want.

Now consider the points \( y_1, \ldots, y_n \). We take a rational section of the line bundle \( \mathcal{O}(-ny_1 - ny_2 - \ldots + D) \) for some fixed effective divisor \( D \) - note that such a section exists if we take \( n \) large enough.

This section \( t \) defines a map to \( \mathbb{P}^1 \) and \( y_1, \ldots, y_m \) go to 0. The map also factors through \( X \) because \( t \in A \). \( \Box \)