Warning. These are rough notes that were taken during the lecture.

1. Introduction

1.1. Let $X$ be . . .

1.2. . . . That’s not how I’m supposed to begin the lecture.

I should give motivation, and to try to explain why the thing is interesting and if you actually want to be listening to it. Feel free to walk away! There is some aesthetic beauty – I learned a few things while developing it, but I’m not sure that it’s worth the time investment.

1.3. This story is about derived algebraic geometry.

In classical algebraic geometry, the geometric objects of study are $\text{Spec}(A)$, where $A$ is a commutative ring.

In derived algebraic geometry, we take $\text{Spec}(A)$ for $A$ a commutative DG algebra with $H^i(A) = 0$ for $i > 0$.

Remark 1.1. In this case, $H^0(A)$ is a usual commutative ring, and you should think about the "underlying topological space" of $\text{Spec}(A)$ as that of $\text{Spec}(H^0(A))$. Note that nilpotents don’t matter in that space, so you should think of the extra data of $A$ vs. $H^0(A)$ as kinds of higher nilpotents.

1.4. Why should you bother with derived algebraic geometry, and not just stick to usual commutative rings?

There are three reasons, which are somewhat disjoint.

- The first reason is base-change. Suppose you have a Cartesian square:

$$
\begin{array}{ccc}
X \times Z & \xrightarrow{\psi} & Z \\
\downarrow \varphi & & \downarrow f \\
X & \xrightarrow{g} & Y.
\end{array}
$$

There’s a base-change map:

$$f^*g_* \rightarrow \psi_*\varphi^*$$

in the derived category of quasi-coherent sheaves on $Z$. It’s an isomorphism if either $f$ or $g$ is flat, but not always (e.g., for $0 \leftrightarrow \mathbb{A}^1 \leftrightarrow 0$).

However, this map is always an isomorphism if we take the fiber product in derived algebraic geometry. The difference is that we remember the lower Tors that arose in forming the fiber product.

Date: June 27, 2014.
• Derived algebraic geometry gives an interpretation of the Grothendieck-Illusie-Quillen cotangent complex. E.g., the dual to the $i$th cohomology of the fiber of the cotangent complex of $X$ at a point $x$ is the set of maps:

$$\text{Spec}(k[\varepsilon]) \longrightarrow X$$

where $\varepsilon$ is a square-zero elements $\varepsilon^2 = 0$ sitting in cohomological degree $-i$. This generalizes the usual definition of tangent vectors in terms of the dual numbers.

• The next important topic is ind-coherent sheaves on inf-schemes. Inf-schemes are important creatures in derived algebraic geometry.

One reason to work with inf-schemes is that if you simultaneously want to work with quasi-coherent sheaves and $D$-modules (which one certainly always does in the classical theory of $D$-modules), you’re lead to the notation of inf-scheme.

Inf-schemes are also closely related to DG Lie algebras. Lie algebras are nice creatures in representation theory, and if you want to relate them to geometry (as in Beilinson-Bernstein type localization theorems), the best setting to do it in is the inf-scheme setting.

1.5. Now I’ll turn around, and those of you who want to leave can leave. It’s a nice city with nice weather.

2. Lecture 1

2.1. We now work in the setting of derived algebraic geometry.

We let $\text{ComAlg}$ denote the ($\infty$-)category of commutative DG algebras, suppressing DG in what follows (i.e., everything is derived). We then let $\text{ComAlg}_{\text{conn}}$ denote the subcategory of connective commutative (DG) algebras, i.e., those commutative algebras $A$ for which $H^i(A) = 0$ for $i > 0$.

We then define $\text{AffSch}$ as the opposite category to $\text{ComAlg}_{\text{conn}}$.

2.2. Pushouts in the category of affine schemes. To form a pushout diagram:

$$\begin{array}{ccc}
\text{Spec}(A_3) & \longrightarrow & \text{Spec}(A_2) \\
\downarrow & & \downarrow \\
\text{Spec}(A_1) & \longrightarrow & \text{Spec}(A_1) \coprod_{\text{Spec}(A_3)} \text{Spec}(A_2) \\
\end{array}$$

just take $\text{Spec}(\tau^{\leq 0}(A_1 \times_{A_3} A_2))$.

Remark 2.1. A technical remark: here we’re taking the truncation just to ensure that we stay in the world of connective commutative algebras. It’s not necessary if each of our maps are closed embeddings, i.e., surjective on $H^0$.

Example 2.2. The pushout $\mathbb{A}^1 \coprod_0 \mathbb{A}^1$ is $\text{Spec}(k[x, y]/xy)$.

When is this actually a pushout diagram in $\text{Sch}$?
Counterexample 2.3. The pushout of the diagram:

\[
\begin{array}{ccc}
\{0\} \times (\mathbb{A}^1 \setminus 0) & \longrightarrow & \mathbb{A}^1 \times (\mathbb{A}^1 \setminus 0) \\
\downarrow & & \uparrow \in \text{Sch} \\
\{0\} \times \mathbb{A}^1 & \longrightarrow & 
\end{array}
\]

is probably $\mathbb{A}^2 \setminus 0$, but that’s not an affine scheme (you get some mess if you try to do the pushout in \text{AffSch}).

However, pushouts are well-behaved when each of the maps is a closed embedding. In particular, they stay as pushouts in the category of schemes.

2.3. **Digression: higher categorical stuff.** We’re going to work with $\infty$-categories now. It’s not reasonable to expect that everyone has seen this material before, but also it’s not reasonable to give a complete treatment. Here are some hints how to work with them.

First of all, $\infty$-categories have objects, morphisms, ways of identifying two morphisms, ways of identifying two ways of identifying two morphisms, and so on.

Mostly, you work with them like usual categories, except that you can no longer write formulae. So functors should be given by categorical operations (like co/limits), not by just writing formulae for objects and morphisms and checking something about compositions.

From now on, we’ll just call $\infty$-categories by *categories*. Similarly, we let $\text{Gpd}$ denote the category of ($\infty-$)groupoids.

**Question 2.4.** What then distinguishes higher category theory from usual category theory, if we just treat things as though they are usual category theory? The answer is the looping/delooping formalism in $\text{Gpd}$, which does not hold for $n$-groupoids ($n < \infty$). This will be very relevant to our later material on Lie algebras and formal moduli problems!

2.4. By definition, a *prestack* is a functor $\text{AffSch}^{op} \to \text{Gpd}$.

**Definition 2.5.** A prestack $\mathcal{Y}$ admits deformation theory if:

- $\mathcal{Y}$ is *convergent*.
- For every pushout diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{f} & S'' \\
\downarrow & & \downarrow \\
S' & \longrightarrow & \bigsqcup_S S''
\end{array}
\]

in $\text{AffSch}$ such that $f$ is a closed *nil-isomorphism*, the induced map:

\[
\mathcal{Y}(\bigsqcup_S S'') \to \mathcal{Y}(S') \times_{\mathcal{Y}(S)} \mathcal{Y}(S'')
\]

is an isomorphism. Here a *closed nil-isomorphism* is a map that is surjective on $H^0$ and whose kernel on $H^0$ is a nilpotent ideal.

---

1Roughly, this means that $\mathcal{Y}$ can be recovered from its values on *eventually cocomplete* commutative rings, where these are those commutative rings $A$ such that $H^i(A) = 0$ for $i$ sufficiently small.
Remark 2.6. Push-outs along closed nil-isomorphisms are also pushouts in the category of (not necessarily affine) schemes.

Remark 2.7. This may not look much like deformation theory that you’ve seen before! It is actually a neat repackaging of the derived version of Schlessinger’s axioms. We’ll see that later.

2.5. **Square-zero extensions.** Suppose that $A \in \text{ComAlg}$ and $M \in A\text{-mod}$. Then we can form the square-zero extension $A \oplus M$ by taking $M$ as a non-unital $A$-algebra equipped with $0$-multiplication, and freely adjoining a unit (in the world of $A$-algebras).

**Exercise 2.8.** Check that this gives the usual description of square-zero extension, when the above notions are taken to be underived.

For $A \in \text{ComAlg}_{\text{conn}}$, $S \in \text{Spec}(A)$ and $M \in A\text{-mod}^{\leq 0} = \text{QCoh}(S)^{\leq 0}$, we let $S_M$ be the spectrum $\text{Spec}(A \oplus M)$ of the square-zero extension.

2.6. **Pro-cotangent spaces.** Note that for every $S \in \text{AffSch}$ and every exact triangle $M_0 \to M_1 \to M_2 \overset{+1}{\to} \text{QCoh}(S)^{\leq 0} \subseteq \text{QCoh}(S)$, we have:

$$S_{M_1 \coprod_{S_{M_2}} S} \simeq S_{M_0}. \tag{2.6.1}$$

**Definition 2.9.** Let $S$ be an affine scheme equipped with a map $x : S \to Y$. We say that $Y$ admits a pro-cotangent space at $x$ if for every datum as above the map:

$$\mathcal{Y}(S_{M_0}) \times_{\mathcal{Y}(S)} \{x\} \to \mathcal{Y}(S_{M_1}) \times_{\mathcal{Y}(S_{M_2})} \{y\}$$

is an isomorphism.

We say that $Y$ admits pro-cotangent spaces (alias: pro-cotangent complex) if it admits pro-cotangent spaces for every $S$ and $x$ as above. Equivalently, for every $S$, the map:

$$\mathcal{Y}(S_{M_1 \coprod_{S_{M_2}} S}) \to \mathcal{Y}(S_{M_1}) \times_{\mathcal{Y}(S_{M_2})} \mathcal{Y}(S) \tag{2.6.2}$$

is an isomorphism.

**Example 2.10.** If $Y$ admits deformation theory, then it admits pro-cotangent spaces.

We’ll see the relationship to the usual cotangent complex in the next lecture.

3. Lecture 2

3.1. In the last lecture, we said that we’re going to work with higher categories, and that higher categories are a lot like usual categories.

What I want to explain is how to actually work in this $\infty$-world, and what the differences are. Today we’ll show how to use the stuff, with the hope of conveying some of the intuition.

3.2. **Homotopy fiber products.** Fix a point $\star$ in the circle $S^1$. What is the fiber product $\star \times_* \star_{S^1}$?

Of course, the fiber product in the category of topological spaces is just the point. However, we really mean to ask about the homotopy fiber product, the version which is homotopy invariant (i.e., the fiber product in $\text{Gpd}$).

We should therefore think of maps $Y \to \star \times_* \star$ as giving a pair of maps $Y \to \star$ (which is no data), and an identification of the induced maps $Y \to S^1$. Thinking in terms of topological spaces,
you can think of this as a path between the corresponding trivial points, i.e., it’s a map to the loop space \( \Omega S^1 \).

Since \( S^1 \) is a \( K(\mathbb{Z}, 1) \) space, this loop space is just \( \mathbb{Z} \), so we see that this fiber product (as computed in \( \text{Gpd} \)) is the discrete groupoid \( \mathbb{Z} \), where we think of \( \mathbb{Z} \) as arising as the fundamental group of \( S^1 \).

3.3. A remark about the square-zero construction from last time.

Exercise 3.1. For \( S \) an affine scheme, convince yourself that the square-zero functor:

\[
\text{QCoh}(S) \mod \mathcal{E}^0 \to \text{ComCoalg}(\text{AffSch}_{/S})
\]
is an equivalence. Here \( \text{AffSch}_{/S} \) is the category of affine schemes equipped with a map from \( S \), and we consider as a symmetric monoidal category using pushouts.

Now find the corresponding precise statement in Lurie’s *Higher algebra*. Hint: a key word is “stabilization.”

3.4. Digression: looping and delooping. Let \( \text{Gpd}_{/*} \) denote the category of pointed groupoids, i.e., groupoids equipped with a distinguished object, which we label \( * \).

There is a looping functor \( \Omega : \text{Gpd}_{/*} \to \text{Gpd} \) sending \( * \to \mathcal{S} \) \( \in \text{Gpd}_{/*} \) to \( \Omega \mathcal{S} := * \times * \). This functor upgrades: it is easy to see that \( \Omega(\mathcal{S}) \) has a monoid structure (“composition of loops”) and that \( \pi_0(\Omega(\mathcal{S})) \) is a group with respect to this structure.

Remark 3.2. \( \Omega(\mathcal{S}) \) should be thought of as the group of automorphisms of the object \( * \in \mathcal{S} \).

We define the category of (higher) groups to be the category of monoids in \( \text{Gpd} \) whose \( \pi_0 \) is a group in the usual sense. Let \( \text{Group} \) denote the category of groups.

Then the functor \( \Omega \) upgrades to a functor:

\[
\Omega : \text{Gpd}_{/*} \to \text{Group}.
\]

Here is a key feature that distinguishes higher categories from usual categories:

**Theorem 3.3.** The functor \( \Omega : \text{Gpd}_{/*} \to \text{Group} \) is an equivalence when restricted to the category of pointed and connected groupoids, i.e., those (pointed) groupoids for which every object is (possibly non-uniquely) isomorphic.

We denote the inverse functor \( \mathbb{B} \). This is the functor of classifying space of a group.

Remark 3.4. This is a higher categorical version of the statement that groupoids with a distinguished object and with only one object up to isomorphism is the same as a group. However, in the form of Theorem \( \square \) it does not hold for 1-categories: for \( A \) an abelian group, the 1-groupoid \( \mathbb{B}A \) admits a group structure, but it is certainly not obtained by looping some other 1-groupoid. Indeed, you need 2-morphisms for that to be possible. But then to deloop 2-groupoids, you need 3-groupoids, and so on. So we are led to see that Theorem \( \square \) leads us into the world of \( \infty \)-categories.

3.5. Pro-objects. We’ll need the following formal digression in what follows.

Let \( \mathcal{C} \) be a category with fiber products. We define \( \text{Pro}(\mathcal{C}) \subseteq \text{Funct}(\mathcal{C}, \text{Gpd})^{op} \) to be the subcategory of functors \( \mathcal{C} \to \text{Gpd} \) sending fiber products to fiber products.

**Lemma 3.5.** \( F \in \text{Funct}(\mathcal{C}, \text{Gpd}) \) lies in \( \text{Pro}(\mathcal{C}) \) if and only if there is a functor \( \varphi : I \to \mathcal{C} \) with \( I \) filtered such that:

\[
F(X) = \operatorname{colim}_{i \in I^{op}} \operatorname{Hom}(\varphi(i), X).
\]
In this case, $F = \lim_{i \in I} \varphi(i) \in \text{Pro}(\mathcal{C})$.

Remark 3.6. The lemma tells us that we can think about pro-objects as formal filtered projective limits. But the definition given using the Yoneda yoga provides a way of saying this without appealing to specific choices of how we write a given pro-object in such a form.

**Lemma 3.7.** Suppose that $\mathcal{C}$ is a DG category equipped with a $t$-structure, and let $\mathcal{C}^-$ be the subcategory of bounded below objects.

Then the canonical functor:

$$\text{Pro}(\mathcal{C}^{\leq 0}) \to \text{Pro}(\mathcal{C}^-)$$

is fully-faithful with essential image the subcategory of functors $\mathcal{C}^- \to \text{Gpd}$ whose restriction to $\mathcal{C}^{\leq 0}$ consists of those functors such that, for every $F \in \mathcal{C}^{\leq 0}$, the pullback square:

$$
\begin{array}{ccc}
F & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & F[1]
\end{array}
$$

maps again to a pullback square.

Remark 3.8. In other words, the lemma says that diagram:

$$
\begin{array}{ccc}
\mathcal{C}^{\leq -1} & \xrightarrow{\Omega} & \mathcal{C}^{\leq 0} \\
\downarrow & & \downarrow \\
\text{Gpd}_{*/} & \xrightarrow{\Omega} & \text{Gpd}_{*/}
\end{array}
$$

should commute.

3.6. Suppose we are in the setting of §2.6 with $x : S \to \mathcal{Y}$ as in loc. cit.

Remark 3.9. Suppose that $\mathcal{F} \in \text{Qcoh}(S)^{\leq 0}$. Then because the diagram:

$$
\begin{array}{ccc}
\mathcal{F} & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{F}[1]
\end{array}
$$

is a pullback diagram in $\text{Qcoh}(S)$, the induced square:

$$
\begin{array}{ccc}
S_{\mathcal{F}[1]} & \to & S \\
\downarrow & & \downarrow \\
S & \to & S_{\mathcal{F}}
\end{array}
$$

is a pushout square in $\text{AffSch}$ of the type considered in (2.6.1).
3.7. Now suppose that $\mathcal{Y}$ admits a pro-cotangent space at $x : S \to \mathcal{Y}$ in the sense of §2.6.

In this case, we have a canonical object $T^*_x : \mathcal{Y} \to \text{Pro}(\text{QCoh}(S)^-)$, which we call the pro-cotangent space of $\mathcal{Y}$ at $x$.

Indeed, thought of as a functor $\text{QCoh}(S)^{\leq 0} \to \text{Gpd}$, it sends $f \in \text{QCoh}(S)^{\leq 0}$ to the space of maps from $S \to \mathcal{Y}$ extending the given map $x$. That the functor defines an object of $\text{Pro}(\text{QCoh}(S)^-)$ follows from combining Lemma 3.7 and Remark 3.9.

3.8. Given a diagram:

$$
\begin{array}{ccc}
S' & \xrightarrow{f} & S \\
\downarrow{x'} & & \downarrow{x} \\
\mathcal{Y} & & \\
\end{array}
$$

one has a canonical map:

$$T^*_x : \mathcal{Y} \to \text{Pro}(f^*)(T^*_{x'} : S' \to \text{Pro}(\text{QCoh}(S)^-)). \quad (3.8.1)
$$

Definition 3.10. $\mathcal{Y}$ admits a pro-cotangent complex if the map (3.8.1) is an isomorphism for every datum as above.

Exercise 3.11. If $\mathcal{Y}$ admits deformation theory, then $\mathcal{Y}$ admits a pro-cotangent complex.

Remark 3.12. One can show that admitting a pro-cotangent complex and satisfying Schlessinger’s infinitesimal cohesiveness axiom is equivalent to admitting deformation theory.

3.9. Cotangent complex. We say that $\mathcal{Y}$ admits an $n$-connective cotangent complex if it admits a pro-cotangent complex, and for every $x : S \to \mathcal{Y}$ the pro-complex $T^*_x$ lies in $\text{QCoh}(S)^{\leq n} \subseteq \text{Pro}(\text{QCoh}(S)^-)$. 

Exercise 3.13. Show that any scheme admits a 0-connective cotangent complex. Show that an Artin stack admits a 1-connective cotangent complex.

Remark 3.14. The cotangent complex of a scheme coincides with the Illusie construction. Note that derived algebraic geometry gives it a more vivid meaning.

4. Lecture 3

4.1. Today we’re going to introduce the notion of inf-scheme.

4.2. Convergence. For $\mathcal{Y}$ a prestack, we say that $\mathcal{Y}$ is convergent if for every $S \in \text{AffSch}$, $\mathcal{Y}(S) \xrightarrow{\sim} \lim_{\leq 0} \mathcal{Y}(\leq S)$. Here for $S = \text{Spec}(A)$, we have set $\leq S := \text{Spec}(H^0(A))$.

Roughly, this condition says that the prestack $\mathcal{Y}$ can be recovered from its values on eventually coconnective affine schemes. It’s a technical condition in derived algebraic geometry without a classical counterpart, but it happens to be satisfied in all examples coming from geometry.

4.3. Finiteness conditions. First we’ll need some technical material on finiteness conditions in the derived setting.

Recall the following lemma.

Lemma 4.1. Let $A$ be a classical commutative $k$-algebra (i.e., $A = H^0(A)$). Then $A$ is finitely generated if and only if $\text{Hom}(A, -)$ commutes with filtered colimits.

We want to extend this definition to the derived setting.
Definition 4.2. An $n$-truncated $k$-algebra $A \in \ComAlgConn$ (i.e., $A = \tau^{\geq -n} A$) is finite type if $\Hom(A, -)$ commutes with filtered colimits (as a functor from the category of $n$-truncated $k$-algebras to the category of groupoids).

A connective $k$-algebra $A$ is almost finite type if $\tau^{\geq -n} A$ is finite type for every $n \geq 0$.

Notation 4.3. We'll need the following notation for categories of $k$-algebras in what follows.

We let $\leq_n \ComAlgConn \subseteq \ComAlgConn$ denote the category of $n$-truncated $k$-algebras. We let $\ComAlgConn$ then denote the subcategory of eventually coconnective $k$-algebras, i.e., $\ComAlgConn := \bigcup_{\leq n} \ComAlgConn \subseteq \ComAlgConn$.

For the opposite categories having to do with $\AffSch$, we use the same notation but using a superscript instead.

Now for each of the above categories, we impose the superscript $aft$ to mean the subcategory of almost finite type subobjects.

Exercise 4.4. Show that $\Spec(A)$ is almost of finite type if and only if $H^0(A)$ is finitely generated and $H^i(A)$ is a finitely generated $A$-module for each $i$.

Exercise 4.5. Give an example of $A \in \ComAlgConn$ such that the functor $\Hom(A, -) : \ComAlgConn \to \Gpd$ does not commute with filtered colimits. (This is the reason for “almost” in the definition).

4.4. We now give a version of these definitions for prestacks.

Definition 4.6. A prestack $\mathcal{Y}$ is locally almost of finite type if $\mathcal{Y}$ is convergent, and for every $n$ the restriction of $\mathcal{Y}$ to a functor $\leq_n \ComAlgConn \to \Gpd$ preserves filtered colimits.

Exercise 4.7. Show that a convergent prestack $\mathcal{Y}$ is locally almost of finite type if and only if the following condition holds for every $n$: the functor $\mathcal{Y}|_{\leq_n \ComAlgConn} : \leq_n \ComAlgConn \to \Gpd$ is the left Kan extension of its restriction to the subcategory $\leq_n \ComAlgConn$ of almost finite type $k$-algebras.

Remark 4.8. Roughly, the reader should understand these technical conditions as follows: convergence of $\mathcal{Y}$ means that $\mathcal{Y}$ can be recovered from its values on $n$-truncated $k$-algebras, and the second condition then further means that $\mathcal{Y}$ from its values on $\ComAlgConn$.

However, the two recovery procedures are different: for convergence, it is a right Kan extension, while for the finite type condition, it is a left Kan extension (computed for each $n$ as above).

4.5. Inf-schemes. The following notion is key in joint work with Nick Rozenblyum.

Definition 4.9. An inf-scheme is prestack $\mathcal{Y}$ locally almost of finite type that admits deformation theory and such that there exists a scheme $Y$ with functorial identifications $Y(S^{red}) = \mathcal{Y}(S^{red})$ defined for every $S \in \AffSch$. (Here for $S = \Spec(A) \in \AffSch$, we define the reduced scheme underlying $S$ as $\Spec(H^0(A)^{red})$).

Remark 4.10. The last condition should be interpreted as saying that the reduced part $Y^{red}$ of $\mathcal{Y}$ is a scheme. (I.e., any precise meaning for these words in any context will be equivalent to the above definition).

We now provide the basic litany of examples.

Example 4.11. A scheme $X$ almost of finite type is an inf-scheme.

Example 4.12. Let $\mathcal{Y}$ be a prestack. Define the de Rham prestack $\mathcal{Y}_{dR} \in \PreStk$ of $\mathcal{Y}$ by $\mathcal{Y}_{dR}(S) := \mathcal{Y}(S^{red})$.

Then for $X$ a scheme almost of finite type, $X_{dR}$ is an inf-scheme.

Note that there is an obvious map $\mathcal{Y} \to \mathcal{Y}_{dR}$.
Exercise 4.13. If \( X \hookrightarrow Y \) is a closed embedding of almost finite type schemes, define the formal completion \( \hat{X} \in \text{PreStk} \) as \( X_{dR} \times_{Y_{dR}} Y \). Convince yourself that this actually is a reasonable definition of the formal completion.

Example 4.14. The formal completion \( \hat{X} \) is an inf-scheme.

Example 4.15. Generalizing Example 4.14: the fibre product of any two inf-schemes is again an inf-scheme.

Remark 4.16. The definition of \( \hat{X} \) makes sense for any map \( X \to Y \); it is again the fibre product \( X_{dR} \times_{Y_{dR}} Y \). We could also call this the \emph{relative de Rham} construction and denote it \( X_{dR/Y} \), since it is a prestack over \( Y \) and its fibre at \( y \in Y(k) \) is de Rham of the fiber.

This is rather remarkable! It is indicative of the looping/delooping story for formal moduli problems if you view \( X_{dR/Y} \) as the quotient of \( X \) by vertical vector fields along the map \( X \to Y \).

Example 4.17. The quotient \( X/G \) of an almost finite type scheme \( X \) by the action of a formal group \( G \) is an inf-scheme.

Here the notion of formal group (in the derived setting) will be introduced later.

Exercise 4.18. Let \( G \) be an algebraic group and let \( G^\wedge \) be the associated formal group (its formal completion at the identity). Then \( G/G^\wedge = G_{dR} \).

4.6. Preview of the next lecture. Our topic for the next lecture is to show that the following categories are equivalent:

- Inf-schemes with reduced part the point (i.e., inf-schemes \( Y \) with \( Y(A) = \{ * \} \) for every reduced \( A \)).
- Formal groups.
- DG Lie algebras.
- All functors defined on local Artinian (DG and connective) commutative rings with residue field \( k \) satisfying some deformation theory conditions.

4.7. Coherent sheaves. Let \( X \) be a scheme locally almost of finite type.

If \( X = \text{Spec}(A) \) is affine, we define the DG category \( \text{Qcoh}(X) \) as the category of modules for \( A \). In general, we define \( \text{Qcoh}(X) \) by gluing.

Remark 4.19. Note that \( \text{Qcoh}(X) \) has a \( t \)-structure defined so that if \( X \) is affine the forgetful functor \( \Gamma : \text{Qcoh}(X) \to \text{Vect} \) is \( t \)-exact. Note that \( \mathcal{O}_X \) lies in the heart of the \( t \)-structure if and only if \( X \) is classical.

Definition 4.20. We say \( \mathcal{F} \in \text{Qcoh}(X) \) is \emph{coherent} if \( \mathcal{F} \) is bounded in the \( t \)-structure and locally on \( X \), \( H^i(\mathcal{F}) \) is finitely generated over \( H^i(\mathcal{O}_X) \).

Example 4.21. Note that \( \mathcal{O}_X \) is coherent if and only if \( X \) is eventually coconnective.

We denote the resulting subcategory of \( \text{Qcoh}(X) \) by \( \text{Coh}(X) \), and then let \( \text{IndCoh}(X) \) denote the ind-completion of \( \text{Coh}(X) \).

4.8. Here are the basic structures on \( \text{IndCoh} \).

There is a nice \( t \)-structure on \( \text{Coh}(X) \), since coherent sheaves are preserved under truncations in \( \text{Qcoh}(X) \). This \( t \)-structure extends in an obvious way to \( \text{IndCoh}(X) \).

There is a functor \( \Psi_X : \text{IndCoh}(X) \to \text{Qcoh}(X) \) given by applying ind-extension (i.e., left Kan extension) to the functor \( \text{Coh}(X) \to \text{Qcoh}(X) \). One can show that this functor is an equivalence on bounded below derived categories.
For $f : X \to Y$, there is a unique continuous functor $f^\text{IndCoh}_* : \text{IndCoh}(X) \to \text{IndCoh}(Y)$, characterized by (continuity and) the commutative diagram:

\[
\begin{array}{ccc}
\text{IndCoh}(X) & \xrightarrow{f^\text{IndCoh}_*} & \text{IndCoh}(Y) \\
\downarrow \psi_X & & \downarrow \psi_Y \\
\text{QCoh}(X) & \xrightarrow{f_*} & \text{QCoh}(Y). \\
\end{array}
\]

Now, suppose that $f$ is proper. Note that $f_* : \text{QCoh}(X) \to \text{QCoh}(Y)$ does not preserve compact objects, e.g., Serre tells us that this is false if $f$ is the embedding of a singular point in a variety.

However, $f^\text{IndCoh}_*$ does preserve compact objects: this is just the usual fact that (derived) push-forward along proper maps preserves coherence.

Therefore, $f^\text{IndCoh}_*$ admits a continuous right adjoint $f^!$ for $f$ proper. One can construct the morphism $f^!$ for general maps $f$: basically it’s characterized by that identity, and the identity $f^!$ is the left adjoint $f^*_*,f^\text{IndCoh}$ to $f^\text{IndCoh}_*$ for $f$ an open embedding. But we’ll explain later how to really give the right functoriality for this functor.

5. Lecture 4

5.1. More on IndCoh. Last time, we defined IndCoh for a scheme. Here are more remarks.

Given $f : X \to Y$, recall that we have a functor $f^! : \text{IndCoh}(Y) \to \text{IndCoh}(X)$.

In particular, using the diagonal map $X \hookrightarrow X \times X$, we obtain a functor:

\[
\text{IndCoh}(X) \times \text{IndCoh}(X) \xrightarrow{\Box} \text{IndCoh}(X \times X) \to \text{IndCoh}(X)
\]

analogous to the tensor product functor:

\[
\text{QCoh}(X) \times \text{QCoh}(X) \xrightarrow{\Box} \text{QCoh}(X \times X) \to \text{QCoh}(X).
\]

This defines a symmetric monoidal structure on $\text{IndCoh}(X)$, and we denote the monoidal operation by $\Box$.

5.2. Lie algebras. Lie algebras make sense in any symmetric monoidal DG category: they are algebras over the classical DG Lie operad considered as an operad in Vect.

Example 5.1. In particular, for a scheme $X$, we have a category $\text{LieAlg}_X$ of Lie algebras in $\text{IndCoh}(X)$. For $X = \text{Spec}(k)$, we obtain the category $\text{LieAlg}_k$ of (DG) Lie algebras over $k$.

Remark 5.2. The Lie algebra axioms are non-trivial to work with in the derived setting! They come across as somewhat random, non-conceptual formulae. But some formal properties — in particular, the PBW theorem — survive and can be proved in this generality, and we can appeal to such things in lieu of working directly with the operad.

\[\text{This means commuting with colimits, equivalently filtered colimits}\]

\[\text{This follows from the above remarks about } t\text{-structures}.\]
Let $X$ be an affine scheme, and define $\text{FmlModuli}_X$ to be the category of an inf-scheme $Y$ over $X$ such that $Y^{\text{red}} \to X^{\text{red}}$ is an isomorphism. We can take group objects in this category, and we define those to be the category $\text{FmlGrp}_X$ of formal groups over $X$.

Our goal for today is to discuss the following theorem.

**Theorem 5.3.** The functor of taking the Lie algebra is an equivalence:

$$\text{FmlGrp}_X \to \text{LieAlg}_X$$

is an equivalence.

There’s a major problem though — how do you construct a Lie algebra out of a formal group? The Lie operad isn’t so obviously amenable to usual techniques in the derived setting.

5.4. **Some constructions with Lie algebras.** Let $\mathcal{C}$ be a symmetric monoidal category. We have adjoint functors:

$$\text{LieAlg}(\mathcal{C}) \xrightarrow{\text{Chev}^+} \mathcal{C} \xrightarrow{\text{triv}_{\text{Lie}}[-1]} \mathcal{C}_{\mathcal{E}c}.$$ 

Here $\text{triv}_{\text{Lie}}[-1]$ is the functor of forming a trivial Lie algebra with a shift incorporated, and $\text{Chev}^+$ is by definition its left adjoint (the reason for the notation is that classically first Lie algebra homology, i.e., the 0th homology of the shifted augmented Chevalley complex, is the abelianization).

5.5. We will now upgrade the Chevalley complex to have extra structure.

Consider $\text{LieAlg}(\mathcal{C})$ as a symmetric monoidal category under products, and consider $\mathcal{C}_{\mathcal{E}c}$ as a symmetric monoidal category using the structure induced from that of $\mathcal{C}$.

Then the functor $\mathcal{C}_{\mathcal{E}c} \to \text{LieAlg}(\mathcal{C})$ is right lax symmetric monoidal, so its left adjoint $\text{Chev} := \mathcal{C}_{\mathcal{E}c} \oplus \text{Chev}^+$ is naturally left lax symmetric monoidal. Actually, one can show that it is honestly symmetric monoidal.

Therefore, since every object $L \in \text{LieAlg}(\mathcal{C})$ is canonically a (co)commutative coalgebra (using the diagonal maps), the functor $\text{Chev}$ upgrades to a functor:

$$\xymatrix{ \text{LieAlg}(\mathcal{C}) \ar[r]^{\text{Chev}} \ar[d]_{\text{Chev}} & \text{CocomCoalg}(\mathcal{C}) \ar[d]^{\text{Oblv}} \ar[l]_{\text{Chev}} \ar[ld]_{\mathcal{C}}. \}$$

5.6. We have adjoint functors:

$$\mathcal{C} \xrightarrow{\text{triv}_{\text{CocomCoalg}^{\text{aug}}}} \text{CocomCoalg}^{\text{aug}}(\mathcal{C}).$$

A great result, which is impressively just a tautology in the above, is the following.

**Exercise 5.4.** Show that $\text{Chev} : \text{LieAlg}(\mathcal{C}) \to \text{CocomCoalg}^{\text{aug}}(\mathcal{C})$ admits a right adjoint, and the composite functor:

$$\xymatrix{ \text{CocomCoalg}^{\text{aug}}(\mathcal{C}) \ar[r]^{\text{the right adjoint}} & \text{LieAlg}(\mathcal{C}) \ar[r]^{\text{Oblv}} & \mathcal{C} \}$$

is computed by $\text{Prim}[-1]$. 
Therefore, we simply write this right adjoint as \( \text{Prim}[-1] \).

Quillen duality is about the adjoint functors:

\[
\begin{array}{ccc}
\text{LieAlg}(\mathcal{C}) & \xrightarrow{\text{Chev}} & \text{CocomCoalg}^{aug}(\mathcal{C}) \\
\text{Prim}[-1] & \xleftarrow{\text{Prim}} & 
\end{array}
\]

Unfortunately, the functors aren’t an equivalence! E.g., for a semisimple Lie algebra, one already sees that the Lie homology is too small. More precisely, neither of these functors is conservative (basically, they both send non-zero objects to zero).

And while no one actually knows what the precise version of Quillen duality says, here’s a conjecture:

**Conjecture.** (1) The adjunction map:

\[
\text{Chev} \circ \text{Prim}[-1] \rightarrow \text{Id}_{\text{CocomCoalg}^{aug}(\mathcal{C})}
\]

is an isomorphism on the essential image of Chev.

(2) The adjunction map:

\[
\text{Id}_{\text{LieAlg}(\mathcal{C})} \rightarrow \text{Prim}[-1] \circ \text{Chev}
\]

is an isomorphism on the essential image of \( \text{Prim}[-1] \).

5.7. **Looping Lie algebras.** Here’s a functor you might not have seen before: the forgetful functor \( \text{LieAlg}(\mathcal{C}) \rightarrow \mathcal{C} \) commutes with limits by construction, and therefore, for \( L \) a Lie algebra, \( \Omega(L) = L[-1] \) is canonically a Lie algebra.

It’s hard to write down an interesting bracket on it by formulae! Actually, it turns out that \( \Omega(L) \) is canonically commutative. This will turn out to be an incarnation of the PBW theorem.

We more easily have the following result:

**Lemma 5.5.** The functor \( \Omega \) is an equivalence:

\[
\text{LieAlg}(\mathcal{C}) \rightarrow \text{Group}(\text{LieAlg}(\mathcal{C})).
\]

5.8. By construction, Chev is left lax symmetric monoidal. Actually, one can show it is honestly symmetric monoidal.

So we obtain a functor:

\[
\begin{array}{ccc}
\text{LieAlg}(\mathcal{C}) & \xrightarrow{\Omega} & \text{Group}(\text{LieAlg}(\mathcal{C})) \\
& \xrightarrow{\text{Chev}} & \text{Alg}(\text{CocomCoalg}^{aug}(\mathcal{C})) =: \text{CocommHopfAlg}(\mathcal{C})
\end{array}
\]

computed by Chev(\( \Omega L \)) at the level of objects.

Here is another derived (and non-obvious) counterpart to the PBW theorem:

**Theorem 5.6.** Chev(\( \Omega L \)) is canonically identified with the enveloping algebra \( U(L) \) as a Hopf algebra.

More generally, one can show that Chev(\( \Omega^k L \)) is given by inducing from the (shifted) Lie operad to the \( E_k \)-operad.
5.9. A lemma on inf-schemes. The following theorem is very important in making the theory work.

**Theorem 5.7.** Let $X$ be an affine scheme. Then the functor:

$$
\{X \to Y \text{ an inf-scheme with } X^{\text{red}} \xrightarrow{\simeq} Y^{\text{red}}\} \to
$$

$\begin{align*}
\text{Contravariant functors: } & \{\text{Affine schemes with reduced part } X^{\text{red}}\} \to \text{Gpd} \\
& \text{sending pushouts along nilisomorphisms to fiber products}
\end{align*}$

is an equivalence.

**Remark 5.8.** There are two main parts to proving the theorem: proving that for an inf-scheme $Y$, the nil-isomorphisms $S \to Y$ with $S \in \text{AffSch}$ are cofinal among all maps from affine schemes, and proving that $Y$ is recovered as the left Kan extension of this restriction.

6. Lecture 5

6.1. Lie algebras and formal moduli problems. Let $\mathcal{C}$ be a symmetric monoidal DG category. Last time we constructed a functor $\text{Chev} : \text{LieAlg}(\mathcal{C}) \to \text{CocomCoalg}(\mathcal{C})$ with right adjoint $\text{Prim}[-1]$.

Here is an informal take on what our construction is meant to do. It is a naive version and will need to be corrected, so it should not be taken overly literally.

Let $x \in X(k)$ be a point and say $\hat{\mathcal{O}}_{X,x}$ is the formal completion of its ring of functions. This is a pro-vector space with a commutative algebra structure, so its dual $(\hat{\mathcal{O}}_{X,x})^\vee$ is a cocommutative coalgebra. Then $\text{Prim}[-1]((\hat{\mathcal{O}}_{X,x})^\vee)$ is known to be the shifted tangent complex $T_{X,x}[-1]$, which by the above carries a Lie algebra structure.

This Lie algebra structure can be understood by interpreting it as the Lie algebra of the formal group $\{\ast\} \times_X \{\ast\}$.

In fact, this gives a way how to construct the Lie algebra of a formal group: if $G$ is a group, then $T_{G,1}[-1]$ now picks up an additional structure of group object in the category of Lie algebras, and we have seen in the last lecture that these arise canonically as a looping, i.e., $T_{G,1}$ then inherits a Lie algebra structure.

6.2. $\text{IndCoh}$ for inf-schemes. The $\text{IndCoh}$ formalism extends as follows. First of all, $\text{IndCoh}$ makes sense for any locally almost finite type prestack by right Kan extension. Tautologically, this $\text{IndCoh}$ has contravariant functoriality, i.e., we have upper-! functors for any map of prestacks.

Then given $f : \mathcal{X} \to \mathcal{Y}$ a map of inf-schemes, we also have a functor $f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}) \to \text{IndCoh}(\mathcal{Y})$ satisfying base-change with upper-! functors.

**Example 6.1.** If $\mathcal{X} = X_{dR}$, $\mathcal{Y} = Y_{dR}$, then $f_*^{\text{IndCoh}}$ recovers the $D$-module direct image.

6.3. Let $S$ be an affine scheme and let $\mathcal{Y}$ be a pointed formal moduli problem over $S$, i.e., $\mathcal{Y}$ is an inf-scheme equipped with a projection $p : \mathcal{Y} \to S$ and a section $i : S \to \mathcal{Y}$, each of which become isomorphisms when evaluated on reduced commutative rings. We denote this category by $\text{InfSch}_{S-\text{ptd}}$.

For such $\mathcal{Y}$, we define the distributions $\text{Distr}(\mathcal{Y}) \in \text{IndCoh}(S)$ of $\mathcal{Y}$ as $p_*^{\text{IndCoh}} p^!(\omega(S)) = p_*^{\text{IndCoh}}(\omega_{\mathcal{Y}})$.

**Exercise 6.2.** Understand why it is called distributions by considering some examples.
By the adjunction $(p^{\text{IndCoh}}, p)$, $\text{Distr}(\mathcal{Y})$ is a cocommutative coalgebra in $\text{IndCoh}(S)$. Using the section $i$, this structure further upgrades to define an augmentation, i.e., $\text{Distr}(\mathcal{Y}) \in \text{CocomCoalg}^{\text{aug}}(\text{IndCoh}(S))$.

Therefore, $\text{Prim}[-1](\text{Distr}(\mathcal{Y}))$ inherits a Lie algebra structure.

6.4. **Digression:** $\text{Pro}(\text{Qcoh}(S)^{-})$ and $\text{IndCoh}$. Recall that for $\mathcal{Y}$ admitting a pro-cotangent complex at $x : S \to \mathcal{Y}$, we have an object $T^*_y(\mathcal{Y})$.

Suppose $\mathcal{Y}$ and $S$ are locally almost of finite type. Then tautologically, $\mathcal{Y}$ satisfies the property that for $\mathcal{F} \in \text{Qcoh}(S)^{-}$, the map:

$$\text{Hom}(T^*_y(\mathcal{Y}), \mathcal{F}) \xrightarrow{\sim} \colim_n \text{Hom}(\tau^{\geq -n}T^*_y(\mathcal{Y}), \mathcal{F})$$

is an isomorphism. Also, $\tau^{\geq -n}T^*_y(\mathcal{Y})$ satisfies the property that $\text{Hom}(\tau^{\geq -n}T^*_y(\mathcal{Y}))$ commutes with filtered colimits computed in the subcategory of $\text{Qcoh}(S)^-$ consisting of objects cohomologically bounded below by $-n$. Indeed, these two conditions exactly encode the two conditions in the definition of being locally almost of finite type.

We denote the subcategory of objects of $\text{Pro}(\text{Qcoh}(S)^{-})$ satisfying the analogous condition by $\text{Pro}(\text{Qcoh}(S)^{-})_{\text{laft}}$. One can show that Serre duality defines an equivalence:

$$\text{Pro}(\text{Qcoh}(S)^{-})_{\text{laft}} \xrightarrow{\mathcal{D}^\text{Serre}_{\text{red}}} \text{IndCoh}(S)^{\text{op}}.$$ 

We then define the tangent complex $T_{\mathcal{Y},x} \in \text{IndCoh}(\mathcal{Y})$ as the Serre dual of the pro-cotangent complex. Note that this operation loses no information!

**Remark 6.3.** There's also a more naive tangent complex $T^{\text{naive}}_{\mathcal{Y},x} := \frac{\text{Hom}_{\text{IndCoh}(S)}(T_{Y,Y}, \omega_S)}{\text{Hom}_{\text{IndCoh}(S)}(T_{Y,Y})}$, which is the thing which is usually considered. Here $\omega_S \in \text{IndCoh}(S)$ is the dualizing complex of $S$, which by definition is the object obtained by applying the upper-$!$ functor to $k \in \text{Vect} = \text{IndCoh}(\text{Spec}(k))$ along the projection $S \to \text{Spec}(k)$.

6.5. Now we return to our earlier setting of $\mathcal{Y} \in \text{InfSch}_{S-\text{ptd}}$.

**Definition 6.4.** $\mathcal{Y}$ is $\text{inf-affine}$ if the canonical map:

$$T_i(\mathcal{Y}) \to \text{Prim}(\text{Distr}(\mathcal{Y}))$$

is an isomorphism. Here $i$ is again the section map $i : S \to \mathcal{Y}$.

**Proposition 6.5.** Any group object of the category of formal moduliproblems over $S$ is $\text{inf-affine}$.

This allows us to make sense of the Lie algebra of a formal group by our earlier technique.

6.6. Now how do we pass in the opposite direction, from Lie algebras to formal groups?

**Remark 6.6.** Classically, people use the Campbell-Baker-Hausdorff formula, which is completely impossible in the derived setting. Even smarter ways of saying this passage still are generally insufficient derivedly.

6.7. Let $\mathfrak{g} \in \text{LieAlg}_{/S}$ be fixed. We want to define $\text{exp}(\mathfrak{g})$ as a formal group over $S$.

Let $T$ be a pointed scheme over $S$ with $T^{\text{red}} \simeq S^{\text{red}}$. We define:

$$\text{Hom}_{\text{ptd}/S}(T, \text{exp}(\mathfrak{g})) := \text{Hom}_{\text{CocomCoalg}^{\text{aug}}}(\text{Distr}(T), U(\mathfrak{g})).$$

By the pointed version of Theorem 5.7 (which follows readily from the cited result), this defines $\text{exp}(\mathfrak{g})$ once we check that some deformation theory conditions are satisfied here (i.e., some pushouts go to some pullbacks), and these are easily checked here.
6.8. Now I want to cite a few more general results. We have a looping functor \( \Omega : \text{InfSch}_{S-ptd} \to \text{FmlGrp}_S \).

**Theorem 6.7.** The functor \( \Omega : \text{InfSch}_{S-ptd} \to \text{FmlGrp}_S \) is an equivalence.

In other words, every pointed formal moduli problem over \( S \) arises as the classifying space of a formal group.

**Remark 6.8.** This kind of thing could only possibly happen derivedly. Indeed, consider the case of the formal completion of 0 in \( \mathbb{A}^1 \).

We now obtain the following picture:

\[
\text{InfSch}_{S-ptd} \cong \text{FmlGrp}_S^{\text{Lie}} \cong \text{LieAlg(IndCoh}(S)).
\]

When \( S = \text{Spec}(k) \), every formal moduli problem is automatically pointed, and in this way we recover the picture from Lurie’s DAG X.