1. Lecture I: The Basics

1.1. Introduction.

1.1.1. Why derived algebraic geometry?

a) Fiber products.

b) Deformation theory.

c) Naturally defined moduli problems.

1.1.2. What we want to do?

a) Want to develop the formalism of quasi-coherent (rather ind-coherent) sheaves that will include D-modules.

b) Explain why the tangent space of a group is a Lie algebra (when one can’t write formulas).

c) Develop some basic infinitesimal geometry (e.g., differential operators).

1.1.3. Key concepts.

a) Inf-schemes will be our geometric objects. They will be accessed via their tangent complexes.

b) Ind-coherent sheaves on an inf-scheme will be the derived (rather, DG) category that we will study.

c) Group inf-schemes will be related to Lie algebras via the Koszul-Quillen duality.

1.2. Deformation Theory. The material here is taken from [Def].

1.2.1. Push-outs of schemes.

a) Push-outs of affine schemes (Sect. 1.1).

b) Push-outs of a closed nil-isomorphism (Lemma 1.4.2).

1.2.2. Deformation theory via push-outs.

a) The notion of convergence for prestacks.

b) Take Prop. 7.2.5 as the definition of admitting deformation theory.

c) Explain that schemes admit deformation theory.

d) Say that Artin stacks admit deformation theory (Sect. 7.4).
1.2.3. (Pro)-cotangent spaces.
    a) Split square-zero extensions (Sect. 2.1).
    b) The condition of admitting a pro-cotangent space (Defn. 2.2.3).
    c) Deformation theory implies admitting pro-cotangent spaces (an exercise).
    d) Extension as a functor QCoh(S) \rightarrow Vect (Sects. 2.2.4 and 2.2.5).
    e) Cotangent spaces (Defn 2.2.9).

1.2.4. (Pro)-cotangent complex.
    a) Functoriality of pro-cotangent spaces (Sects. 4.1.1-4.1.3).
    b) The condition of admitting a pro-cotangent complex (Defn. 4.1.4).

1.2.5. Deformation theory via infinitesimal cohesiveness (tutorial).
    a) Square-zero extensions (Sects. 5.1, 5.4, 5.5).
    b) Infinitesimal cohesiveness (Sects. 6.1, 6.2, Cor. 7.2.3).

1.3. Inf-schemes. The material here is taken from [InfSch].

1.3.1. Notion of laft prestack.
    a) The definition of the laft property.
    b) Laft prestacks as functors on \langle^0\text{DGSch}_{aff}^\text{ft}.

1.3.2. Notion of inf-scheme.
    a) Definition of inf-schemes (Defn. 3.1.2).
    a') Comment on relation to ind-schemes.
    b) Example of formal completion.
    c) Example of de Rham prestack.
    c') Relative de Rham prestack.
    d) Preview: Bg.

1.3.3. Exhibiting inf-schemes as colimits.
    a) Cofinality statement: Prop. 4.3.3 and Cor. 4.3.4.
    b) Construction of inf-schemes (Prop. 4.4.5).

1.3.4. Inf-schemes vs formal schemes (tutorial).
    a) The notions of ind-scheme and formal scheme (Sect. 1.1 and 1.8).
    b) An ind-schemes is recovered from its closed subschemes (Prop. 1.5.4 and Cor. 1.8.6).
    c) A deformation theory criterion for being an ind-scheme (Cor. 3.3.4).

1.4. Ind-coherent sheaves on inf-schemes.
1.4.1. **Recap on IndCoh.** The material here is taken from [Sch] and [Corr].
   a) IndCoh on a scheme ([Sch, Sect. 1]).
   b) Direct image ([Sch, Sect. 2]).
   c) !-pullback ([Sch, Sect. 5.1]).
   d) Base change ([Sch, Secta. 5.2 and 5.3]).
   e) Formalism of correspondences ([Corr, Sect. 2]).

1.4.2. **Existence of \( f_!^{IndCoh} \).** The material here is taken from [IndonSch, Sect. 4].
   a) Existence of the left adjoint to \( f_! \) for an inf-schematic nil-isomorphisms (Prop. 3.1.2).
      a’) Example: Lie algebra homology.
      a”) Induced D-modules.
   b) Construction of direct image as a left Kan extension (Sect. 4.3).

1.4.3. **Extension to correspondences.** The material here is taken from [IndonSch, Sect. 5].
   a) Want we want to achieve.
   b) Verifying the conditions.

1.4.4. **Applications to crystals.** The material here is taken from [Crys].
   a) The functor of de Rham direct image (Sect. 1.4).
   b) D-mod as a functor from the category of correspondences (Sect. 2).

2. **Lecture II: Lie algebras and group inf-schemes**

2.1. **Formal moduli problems à la Lurie.** The material here is taken from [Form].

2.1.1. **Groupoids and groups.** The material here is taken from [Form, Sect. 2.1].
   a) The notion of groupoid.
   b) Quotient by a groupoid.
   c) Groups.

2.1.2. **Groupoids in formal geometry.** The material here is taken from [Form, Sects. 2.2-2.3].
   a) The notion of formal moduli problem.
   b) The procedure of taking the quotient by a groupoid—the main theorem.
   c) Some examples.
   d) The case of groups.

2.1.3. **Sketch of proof.** The material here is taken from [Form, Sects. 2.4-2.5].
   a) Construction of the quotient.
   b) Verification of deformation theory.

2.2. **Review of Quillen duality.** The material here is taken from [Lie].
2.2.1. *Lie algebras and the Chevalley complex.*

a) Lie algebras in a symmetric monoidal DG category (Sect. 1.1).

b) The Chevalley complex and its enhancement (Sect. 1.4).

c) The functor of primitives (Sect. 1.5).

d) Koszul-Quillen duality (Sect. 1.6).

e) Primitives of the symmetric co-algebra (Sect. 1.7).

2.2.2. *Looping Lie algebras.*

a) Group Lie algebras (Sects. 2.1-2.2).

b) Chevalley complex of a group Lie algebra (Sects. 2.3-2.4).

c) Relation to the universal enveloping algebra (Sect. 4.1)

d) Relation to the PBW theorem.

2.3. *The exponential construction.*

2.3.1. *Digression:* tangent spaces. The material here is taken from [Def, 3.3].

a) Laft objects of Pro(QCoh(\(X\))\(^-\)).

b) Relation to the laft condition on prestacks.

c) Duality with IndCoh(\(X\)).

d) The notion of tangent space and tangent complex.

2.3.2. *Formal moduli problems and co-commutative co-algebras.* The material here is taken from [Exp, Sects. 1 and 2].

a) Co-commutative co-algebras associated to formal moduli problems (Sects. 1.1-1.2).

b) The functor of inf-spectrum (Sects. 1.3-1.4).

c) Inf-affineness (Sect. 2).

2.3.3. *Lie algebras and formal groups.* The material here is taken from [Exp, Sects. 3 and 4].

a) From Lie algebras to formal groups: the exponential construction (Sect. 3.1).

b) The inverse functor (Sect. 3.2).

c) Summary: Lie algebras vs formal groups vs formal moduli problems (Sects. 3.3 and 3.6).

d) Sketch of the proof (Sect. 4).
References

[Sch] Part II.1: Ind-coherent sheaves on schemes
[Corr] Part II.2: The $!$-pullback and base change
[Def] Part III.1: Deformation Theory
[InfSch] Part III.2: Ind-inf-schemes
[IndonSch] Part III.3: Ind-coherent sheaves on ind-inf-schemes
[Crys] Part III.4: An application: crystals
[Lie] Part IV.1: Lie algebras and co-commutative co-algebras
[Form] Part IV.2: Formal moduli
[Exp] Part IV.3: Formal groups and Lie algebras
[Alg] Part IV.4: Lie Algebroids