MATH 221, PROBLEM SET 1, DUE: SEPT. 29.

Problems marked with (*) are more difficult, but still mandatory.

1*. Let $M$ be an $R$-module. Show that the following conditions are equivalent:
(a) Any ascending sequence $M_1 \subset M_2 \subset \ldots$ of submodules such that $\bigcup_i M_i = M$, stabilizes.
(b) For any set of $R$-modules $N_i, i \in I$, for any map $M \to \bigoplus_{i \in I} N_i$ there exists a finite subset $I_0 \subset I$ such that the above map factors through
\[ \bigoplus_{i \in I_0} N_i \subset \bigoplus_{i \in I} N_i, \]
i.e., the map
\[ \bigoplus_{i \in I} \text{Hom}_R(M, N_i) \to \text{Hom}_R(M, \bigoplus_{i \in I} N_i) \]
is an isomorphism. Note that the above map always injective.

Hint: let $j \mapsto (M_j \subset M)$ be a chain of submodules of $M$, such that $\bigcup_j M_j = M$. Can you map $M$ to $\bigoplus_j M/M_j$? When will such a map (if it exists) be a finite sum of maps to some particular $M/M_j$’s?

2. Let $\phi : A \to B$ be a homomorphism of rings with $A$ left-Noetherian.
(a) Assume that $\phi$ is surjective. Show that $B$ is left-Noetherian.
(b) Assume that $\phi$ makes $B$ a f.g. left $A$-module. Show that $B$ is left-Noetherian.

3. Let $k$ be a field and let $R = k[x_1, x_2, \ldots]$ be the polynomial algebra on $\mathbb{N}$-many generators. By definition, it is isomorphic to $\lim_{\rightarrow n} k[x_1, \ldots, x_n]$. Consider the homomorphism of $k$-algebras $\phi : R \to k$ that sends all $x_i$ to 0. Show that $\ker(\phi)$ is not finitely generated.

Hint: pass from $R = k[x_1, x_2, \ldots]$ to a quotient algebra by killing $x_i^2$ for all $i$.

4*. Let $A$ be a ring, and consider the ring of formal power series $A[[x]]$. Modify the proof of Hilbert’s basis theorem to show that if $A$ is Noetherian, then so is $A[[x]]$.

In the rest of the PS, all rings are commutative.

5. Let $A \to B \to C$ be homomorphisms of rings. Assume that $B$ is finite over $A$ and $C$ if finite over $B$. Show that $C$ is finite over $A$.

6. Let $A \to B$ be a homomorphism, with $A$ Noetherian.
(a) Let $b \in B$ be integral over $A$. Show that for the corresponding homomorphism $\phi : A[x] \to B$, any element in $\text{Im}(\phi)$ is integral over $A$.
(b) Let $b_1, \ldots, b_n \in B$ be integral over $A$. Show that for the corresponding homomorphism $\phi_n : A[x_1, \ldots, x_n] \to B$, any element in $\text{Im}(\phi)$ is integral over $A$.
(c) Show that the set of elements in $B$ that are integral over $A$ forms a subring.

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7. Let \( A \to B \) be a homomorphism. We say \( B \) is integral over \( A \) if every element \( b \in B \) is integral over \( A \).

Assume that \( A \) is Noetherian. Show that \( B \) is finite over \( A \) if and only if it is integral and finitely generated as an \( A \)-algebra.

8. Let \( A \to B \) be injective, with \( A \) Noetherian and \( B \) integral over \( A \). Assume that neither \( A \) nor \( B \) have zero divisors.
   (a) Show that \( A \) is a field then so is \( B \).
   (b) Deduce that a field \( k \) is algebraically closed (i.e., every polynomial has a root) if and only for every finite field extension \( k \subset k' \) (i.e., \( k' \) is f.d. as a \( k \)-vector space) we have \( k = k' \).
   (c) Show that if \( B \) is a field, then so is \( A \).

9. Let \( k \) be an arbitrary field (not necessarily algebraically closed). Recall that the Weak Nullstellensatz says that every field extension \( k \subset k' \), such that \( k' \) is f.d. as a \( k \)-algebra, is finite. Deduce from it the following statements:
   (a) Every maximal ideal in \( k[x_1, \ldots, x_n] \) is the kernel of a surjective \( k \)-algebra homomorphism \( \phi : k[x_1, \ldots, x_n] \to k' \), where \( k' \) is a finite field extension of \( k \). Show that for any two choices \((k'_1, \phi_1)\) and \((k'_2, \phi_2)\) there exists a unique \( k \)-algebra homomorphism \( \psi : k'_1 \to k'_2 \) such that \( \psi \circ \phi_1 = \phi_2 \).
   (b) For every maximal ideal in \( m \subset k[x_1, \ldots, x_n] \) there exists a field extension \( k' \) and a point \((c'_1, \ldots, c'_n)\in (k')^n\) such that \( m = m(c'_1, \ldots, c'_n) \cap k[x_1, \ldots, x_n] \subset k'[x_1, \ldots, x_n] \). Give an example, how for the same field extension \( k' \) two different choices of \((c'_1, \ldots, c'_n)\) give rise to the same ideal \( m \).