GROTHENDIECK R-POINTS

Recall that given a map of commutative rings $\phi : A \to B$, we have a map $\Phi : \text{Spec } A \to \text{Spec } B$ given by taking pre-images of prime ideals.

**Proposition 1.1** Let $\phi : A \to B$ be a map of commutative rings such that $B$ is finitely generated as an $A$-module. Then $\Phi$ is a closed map.

**Proof:** Let $V(J) \subset \text{Spec } B$ be a closed set. We know from PS 4 that $\Phi(V(J)) = V(I)$, where $I = \phi^{-1}(J)$. We want to show that $\Phi(V(J))$ is closed, i.e. $\Phi(V(J)) = V(I)$. Equivalently we want the far left map

$$
\begin{array}{ccc}
\text{Spec } (B/J) & \to & \text{Spec } B \\
\downarrow & & \downarrow \Phi \\
\text{Spec } (A/I) & \to & \text{Spec } A
\end{array}
$$

to be surjective. Here we are identifying $V(I)$ with $\text{Spec } (A/I)$ and $V(J)$ with $\text{Spec } (B/J)$. Note that by definition, $A/I \hookrightarrow B/J$ is injective. Thus, we are reduced to showing that if $A \hookrightarrow B$ then $\text{Spec } B \twoheadrightarrow \text{Spec } A$. Let $\mathfrak{p} \in \text{Spec } A$. Then consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec } B_{\mathfrak{p}}/B_{\mathfrak{p}} & \to & \text{Spec } B \\
\downarrow & & \downarrow \\
\text{Spec } A_{\mathfrak{p}}/A_{\mathfrak{p}} & \to & \text{Spec } A
\end{array}
$$

By Nakayama’s Lemma, $B_{\mathfrak{p}}/B_{\mathfrak{p}} \neq 0$, so $\text{Spec } B_{\mathfrak{p}}/B_{\mathfrak{p}}$ is non-empty. Since $A_{\mathfrak{p}}/A_{\mathfrak{p}}$ is a field, $\text{Spec } A_{\mathfrak{p}}/A_{\mathfrak{p}}$ has one point. Therefore, the far left map is surjective. This completes the proof because $\mathfrak{p} \in \text{Spec } A$ has horizontal pre-image $B_{\mathfrak{p}}$, which has horizontal pre-image 0. By commutativity of the diagram, we obtain a pre-image in $\text{Spec } B$.

In what follows, let $k$ be algebraically closed, and let $A$ be a finitely generated $k$-algebra. Recall that $\text{Specm } A$ denotes the set of maximal ideals in $A$. Consider the natural $k$-algebra structure on $\text{Funct}(\text{Specm } A, k)$. We have a map

$$
A \to \text{Funct}(\text{Specm } A, k)
$$

which comes from the Weak Nullstellensatz as follows. Maximal ideals $\mathfrak{m} \subset A$ are in bijection with maps $\varphi_{\mathfrak{m}} : A \to k$ where $\ker(\varphi_{\mathfrak{m}}) = \mathfrak{m}$, so we define $a \mapsto [\mathfrak{m} \mapsto \varphi_{\mathfrak{m}}(a)]$. If $A$ is reduced, then this map is injective because if $a \in A$ maps to the zero function, then $a \in \cap \mathfrak{m} \Rightarrow a$ is nilpotent $\Rightarrow a = 0$.

**Definition 1.1** A function $f \in \text{Funct}(\text{Specm } A, k)$ is called algebraic if it is in the image of $A$ under the above map. (Alternate words for this are polynomial and regular.)

Let $A$ and $B$ be finitely generated $k$-algebras and $\phi : A \to B$ a homomorphism. This yields a map $\Phi : \text{Specm } A \to \text{Specm } B$ which comes from the Weak Nullstellensatz as follows. Maximal ideals $\mathfrak{m} \subset A$ are in bijection with maps $\varphi_{\mathfrak{m}} : A \to k$ where $\ker(\varphi_{\mathfrak{m}}) = \mathfrak{m}$, so we define $a \mapsto [\mathfrak{m} \mapsto \varphi_{\mathfrak{m}}(a)]$. If $A$ is reduced, then this map is injective because if $a \in A$ maps to the zero function, then $a \in \cap \mathfrak{m} \Rightarrow a$ is nilpotent $\Rightarrow a = 0$.
Specm $B \to$ Specm $A$ given by taking pre-images (see PS4 problem 7).

**Definition 1.2** A map $\Phi : \text{Specm } B \to \text{Specm } A$ is called algebraic if it comes from a homomorphism $\phi$ as above.

To demonstrate how these definitions relate to one another we have the following proposition.

**Proposition 1.2** A map $\Phi : \text{Specm } B \to \text{Specm } A$ is algebraic if and only if for any algebraic function $f \in \text{Funct}(\text{Specm } A, k)$, the pullback $f \circ \Phi \in \text{Funct}(\text{Specm } B, k)$ is algebraic.

Proof: [$\Rightarrow$] Suppose that $\Phi$ is algebraic. It suffices to check that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Funct}(\text{Specm } A, k) & \xrightarrow{- \circ \Phi} & \text{Funct}(\text{Specm } B, k) \\
\uparrow & & \uparrow \\
A & \xrightarrow{\phi} & B
\end{array}
\]

where $\phi : A \to B$ is the map that gives rise to $\Phi$.

[$\Leftarrow$] Suppose that for all algebraic functions $f \in \text{Funct}(\text{Specm } A, k)$, the pull-back $f \circ \Phi$ is algebraic. Then we have an induced map, obtained by chasing the diagram counter-clockwise:

\[
\begin{array}{ccc}
\text{Funct}(\text{Specm } A, k) & \xrightarrow{- \circ \Phi} & \text{Funct}(\text{Specm } B, k) \\
\uparrow & & \uparrow \\
A & \xrightarrow{\phi} & B
\end{array}
\]

From $\phi$, we can construct the map $\Phi' : \text{Specm } B \to \text{Specm } A$ given by $\Phi'(m) = \phi^{-1}(m)$. I claim that $\Phi = \Phi'$. If not, then for some $m \in \text{Specm } B$ we have $\Phi(m) \neq \Phi'(m)$. By definition, for all algebraic functions $f \in \text{Funct}(\text{Specm } A, k)$, $f \circ \Phi = f \circ \Phi'$ so to arrive at a contradiction we show the following lemma:

Given any two distinct points in $\text{Specm } A = V(I) \subset k^n$, there exists some algebraic $f$ that separates them. This is trivial when we realize that any polynomial function is algebraic, and such polynomials separate points. ■

**Definition 1.3** A space (or functor) $X$ is an assignment of every ring $R$ to a set $X(R)$ such that for any homomorphism $\alpha : R \to R'$, there exists a map of sets $X(\alpha) : X(R) \to X(R')$. Furthermore,

(i) If $\alpha = \text{id}$, then $X(\alpha) = \text{id}$.

(ii) If $\alpha : R \to R'$ and $\beta : R' \to R''$ then $X(\beta \circ \alpha) = X(\beta) \circ X(\alpha)$.

Example: Any ring $A$ gives rise to a space Spec $A$ defined as follows:

\[ (\text{Spec } A)(R) := \text{Hom}_{k-\text{alg}}(A, R) \]

**Definition 1.4** Let $X$ and $Y$ be spaces. A map of spaces (or natural transformation) $\Phi : X \to Y$ is an
assignment for any $R$, $\Phi : X(R) \to Y(R)$ such for any homomorphism $\alpha : R \to R'$ the following diagram commutes:

$$
\begin{array}{c}
X(R) \xrightarrow{\Phi_R} Y(R) \\
\downarrow{X(\alpha)} & \downarrow{Y(\alpha)} \\
X(R') \xrightarrow{\Phi_{R'}} Y(R')
\end{array}
$$

Example: Let $\varphi : A \to B$ be a ring homomorphism. This yields a map of spaces from $\text{Spec } B \to \text{Spec } A$ by pre-composition. It satisfies the axioms since the following diagram commutes.

$$
\begin{array}{c}
\text{Hom}(B, R) \xrightarrow{-\circ \varphi} \text{Hom}(A, R) \\
\downarrow{\circ -} & \downarrow{\circ -} \\
\text{Hom}(B, R') \xrightarrow{-\circ \varphi} \text{Hom}(A, R')
\end{array}
$$

It turns out that such maps of spaces are the only ones from $\text{Spec } B \to \text{Spec } A$. More precisely,

**Proposition 1.3** *(Yoneda’s Lemma)* For two $k$-algebras $A$ and $B$, there is a natural bijection between maps of $k$-algebras from $A \to B$ and maps of spaces $\text{Spec } B \to \text{Spec } A$, given by pre-composition.

Proof: This was problem 4 on PS6, so we omit the proof here. ■

**Proposition 1.4** Let $X$ be a space. Then we have $\text{Hom}_{\text{spaces}}(\text{Spec } R, X) = X(R)$.

Proof: Let $\Phi$ be a map of spaces, so we have an assignment $\Phi_R : (\text{Spec } R)(R) \to X(R)$. Since $(\text{Spec } R)(R) = \text{Hom}(R, R)$ we can take $\Phi_R(\text{id}) \in X(R)$. Conversely, suppose we are given an element $x \in X(R)$. We want for each $R'$ a map from $\text{Hom}(R, R') \to X(R')$. We define such a map as follows. If $\varphi : R \to R'$ then

$$
\varphi \mapsto X(\varphi)(x) \in X(R')
$$

It is trivial to check that this is indeed a map of spaces, and that the two constructions are inverses of each other. ■

**Proposition 1.5** *(Cayley-Hamilton Theorem)*

Proof: ?
HOMOLOGICAL ALGEBRA

Let \( R \) be a commutative ring.

**Definition 2.1** A complex \( M^\bullet \) is a sequence of \( R \)-modules \( \{M^i\} \) with maps \( d^i : M^i \to M^{i+1} \)

\[
\cdots \to M^{-2} \xrightarrow{d^{-2}} M^{-1} \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \xrightarrow{d^2} \cdots
\]

such that \( d^i \circ d^{i-1} = 0 \), i.e. \( \text{Im} \, d^{i-1} \subset \ker \, d^i \).

**Definition 2.2** The \( i \)-th cohomology is the quotient module

\[
H^i(M^\bullet) := \ker d^i / \text{Im} \, d_{i-1}^i
\]

A complex is called acyclic if it is exact at each index, i.e. \( H^i(M^\bullet) = 0 \) for all \( i \).

**Definition 2.3** We define \( \text{Hom}_R(M^\bullet, N^\bullet) \) to be the set of maps of complexes from \( M^\bullet \to N^\bullet \). Such a map is an element \( \{\varphi^i\} \in \prod_i \text{Hom}_R(M^i, N^i) \) such that for all \( i \), the following diagram is commutative.

\[
\begin{array}{ccc}
M^i & \xrightarrow{d^i_M} & M^{i+1} \\
\varphi^i & \downarrow & \varphi^{i+1} \\
N^i & \xrightarrow{d^i_N} & N^{i+1}
\end{array}
\]

**Proposition-Construction 2.1** A map of complexes \( \varphi : M^\bullet \to N^\bullet \) induces a map of cohomologies \( H^i(M^\bullet) \to H^i(N^\bullet) \) for all \( i \).

Proof: We define the map by restricting \( \varphi^i \) to \( \ker d^i_M \). Since each square is commutative, \( \varphi^i \) maps \( \ker d^i_M \to \ker d^i_N \) and \( \text{Im} \, d_{i-1}^M \to \text{Im} \, d_{i-1}^N \). Thus, the induced map is well-defined on \( H^i(M^\bullet) \). □

**Definition 2.4** A map of complexes is a quasi-isomorphism if it induces an isomorphism of cohomologies.

**Definition 2.5** Let \( \varphi \) and \( \psi \) be maps of complexes from \( M^\bullet \to N^\bullet \). A homotopy from \( \varphi \) to \( \psi \) is an element \( \{h^i\} \in \prod_i \text{Hom}_R(M^i, N^i) \) such that

\[
\varphi^i - \psi^i = h^{i+1} \circ d^i_M + d^i_{N} \circ h^i
\]

**Lemma 2.1** If \( \varphi \) and \( \psi \) are homotopic, then their induced maps of cohomologies coincide.

Proof: Let \( m \in \ker(d^i_M) \). Then

\[
\varphi^i(m) - \psi^i(m) = h^{i+1} \circ d^i_M(m) + d^i_{N} \circ h^i(m) = d^i_{N} \circ h^i(m) \in \text{Im}(d^i_{N-1})
\]

which is zero in the cohomology \( H^i(N^\bullet) \). □
Proposition 2.1 If we have a short exact sequence of complexes $0 \rightarrow M^1_\bullet \rightarrow M^2_\bullet \rightarrow M^3_\bullet \rightarrow 0$, this induces a long exact sequence of cohomologies:

$$\cdots \rightarrow H^{i-1}(M^3_\bullet) \rightarrow H^i(M^1_\bullet) \rightarrow H^i(M^2_\bullet) \rightarrow H^{i+1}(M_1) \rightarrow \cdots$$

Proof: This was problem 1(b) on PS7, so we omit the proof here. ■

Definition 2.6 A map is null-homotopic if it is homotopic to the zero map.

Definition 2.7 A map $\varphi : M^\bullet \rightarrow N^\bullet$ is a homotopy equivalence if there exists some $\psi : N^\bullet \rightarrow M^\bullet$ such that

$$\text{id}_{N^\bullet} \simeq \varphi \circ \psi$$
$$\text{id}_{M^\bullet} \simeq \psi \circ \varphi$$

where $\simeq$ denotes homotopy.

Lemma 2.2 A homotopy equivalence is a quasi-isomorphism.

Proof: This follows directly from the definition.

Example: Not every quasi-isomorphism is a homotopy equivalence. Consider the complex

$$\cdots \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0 \rightarrow \cdots$$

so $H^0 = \mathbb{Z}/2\mathbb{Z}$ and all cohomologies are 0. We have a quasi-isomorphism from the above complex to the complex

$$\cdots \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

but no inverse can be defined (no map from $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$).

Definition 2.8 If $M^\bullet$ is a complex then for any integer $k$, we define a new complex $M^\bullet[k]$ by shifting indices, i.e. $(M^\bullet[k])^i := M^{i+k}$.

Definition 2.9 If $f : M^\bullet \rightarrow N^\bullet$ is a map of complexes, we define a complex $\text{Cone}(f) := \{N^i \oplus M^{i+1}\}$ with differential

$$d(n^i, m^{i+1}) := (d^N n_i) + (-1)^i \cdot f(m^{i+1}, d^M m^{i+1})$$

Remark: This is a special case of the total complex construction to be seen later.

Proposition 2.2 A map $f : M^\bullet \rightarrow N^\bullet$ is a quasi-isomorphism if and only if $\text{Cone}(f)$ is acyclic.

Proof: Note that by definition we have a short exact sequence of complexes

$$0 \rightarrow N^\bullet \rightarrow \text{Cone}(f) \rightarrow M^\bullet[1] \rightarrow 0$$
so by Proposition 2.1, we have a long exact sequence

\[ \cdots \to H^{i-1}(\text{Cone}(f)) \to H^i(M) \to H^i(N) \to H^i(\text{Cone}(f)) \to \cdots \]

so by exactness, we see that \( H^i(M) \cong H^i(N) \) if and only if \( H^{i-1}(\text{Cone}(f)) = 0 \) and \( H^i(\text{Cone}(f)) = 0 \). Since this is the case for all \( i \), the claim follows. \( \blacksquare \)

**Definition 2.10** Let \( M^\bullet \) and \( N^\bullet \) be complexes. We define the inner Hom complex \((\text{Hom}(M^\bullet, N^\bullet))^\bullet\) as:

\[
(\text{Hom}(M^\bullet, N^\bullet))^i := \prod_n \text{Hom}(M^n, N^{n+i})
\]

with differential \( d \varphi(m^n) := d_N^{n+1} \circ \varphi^n(m^n) + (-1)^{i+1} \cdot \varphi^{n+1} \circ d_M^n(m^n) \).

**Remark:** From the definition of the inner Hom complex, we have that \( \ker(d^0) = \text{Hom}(M^\bullet, N^\bullet) \), the usual maps of complexes. Similarly, \( \text{Im}(d^{-1}) \) are those maps that are null-homotopic. Thus, the cohomology \( H^0((\text{Hom}(M^\bullet, N^\bullet))^\bullet) \) can be thought of as maps of complexes, up to homotopy. This is denoted \( h\text{Hom}(M^\bullet, N^\bullet) := H^0((\text{Hom}(M^\bullet, N^\bullet))^\bullet) \).

**Lemma 2.3** Let \( M^\bullet \) be an acyclic complex. Let \( P^\bullet \) be a complex of projective modules that is bounded from above, i.e. \( P^n = 0 \) for \( i > 0 \). Then the complex \( \text{Hom}(P^\bullet, M^\bullet) \) is acyclic.

**Proof:** This can be shown by a simple diagram chase. \( \blacksquare \)

**Corollary 2.1** Let \( M_1^\bullet \to M_2^\bullet \) be a quasi-isomorphism, and let \( P^\bullet \) be as in the lemma above. Then \( \text{Hom}(P^\bullet, M_1^\bullet) \to \text{Hom}(P^\bullet, M_2^\bullet) \) is a quasi-isomorphism (let us call this map \( \phi \)).

**Proof:** Consider the acyclic complex \( \text{Cone}(f) \). By the lemma, \( \text{Hom}(P^\bullet, \text{Cone}(f)) \) is acyclic. We want to show that \( \text{Cone}(\phi) \) is acyclic. I claim that the two complexes are isomorphic:

\[
\text{Hom}(P^\bullet, M_2^\bullet) \oplus \text{Hom}(P^\bullet, M_1^\bullet)^{i+1} \cong \prod_n \text{Hom}(P^n, M_2^{n+i}) \\
\text{Hom}(P^n, M_1^{n+i+1})
\]

which is true by the universal property of the direct sum. It can be checked that the differentials are the same. \( \blacksquare \)

**Proposition 2.3** Let \( M \) be an \( R \)-module.

(i) There exists a complex of projective modules called the **projective resolution** of \( M \):

\[
\begin{array}{cccccccc}
\cdots & \to & P^{-2} & \to & P^{-1} & \to & P^0 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & & & M & & & & \\
\end{array}
\]

such that \( H^0(P^\bullet) = M \) and \( H^i(P^\bullet) = 0 \) for \( i \neq 0 \).
(ii) If we have two such resolutions $P_1^\bullet$ and $P_2^\bullet$, then there exist unique (up to homotopy) maps of complexes $\alpha$ and $\beta$ such that $\alpha \circ \beta = \text{id}$, $\beta \circ \alpha = \text{id}$, and the triangle below commutes (up to homotopy):

![Diagram]

Proof: (i) Since free $R$-modules are projective, we can just take a free resolution, i.e. let $P^0$ be a free module surjecting onto $M$ with kernel $K^0$, $P^1$ a free module surjecting onto $K^0$ and so on.

(ii) For this, we consider $M$ as a complex:

$$\cdots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \cdots$$

Since $\text{Cone}(\phi)$ is acyclic, we have that $\phi$ is a quasi-isomorphism. In particular,

$$H^0(\text{Hom}(P_1^\bullet, P_2^\bullet)) \simeq H^0(\text{Hom}(P_1^\bullet, M))$$

The resolution gives us a map of complexes $P_1^\bullet \rightarrow M^\bullet$, i.e. an element of the right-hand side. The corresponding element of the left-hand side is $\alpha$. An analogous construction yields $\beta$, and they are inverses by uniqueness of the construction. □

**Definition 2.11** Let $M$ and $N$ be $R$-modules. Let $P^\bullet$ be a projective resolution for $M$. We define

$$\text{Tor}_i^R(M, N) := H^{-i}(P^\bullet \otimes_A N)$$

Remark: $\text{Tor}_0^R(M, N) = \text{coker}(P^{-1} \otimes N \rightarrow P^0 \otimes N) \simeq \text{coker}(P^{-1} \rightarrow P^0) \otimes N \simeq M \otimes N$, so $\text{Tor}$ can be seen as a generalization of the tensor product. Also, as a direct consequence of this definition, we see that $M$ is flat if and only if $\text{Tor}_1^R(M, N) = 0$ for all $R$-modules $N$.

**Proposition 2.4** Let $M$ and $N$ be $R$-modules. Then

(i) $\text{Tor}_i^R(M, N)$ is independent of the choice of projective resolution.

(ii) $\text{Tor}_i^R(M, N) \simeq \text{Tor}_i^R(N, M)$, despite the asymmetry in the definition.

Proof: (i) If we take two different projective resolutions $P_1^\bullet$ and $P_2^\bullet$, then by proposition 2.3(ii), we have $\alpha$ and $\beta$ which induce isomorphisms on the cohomologies:

$$P_1^\bullet \otimes N \xrightarrow{\alpha} P_2^\bullet \otimes N$$

(ii) Let $P^\bullet$ be a projective resolution for $M$ and $Q^\bullet$ a projective resolution for $N$. Consider the bi-complex $P^\bullet \otimes Q^\bullet$ and define $\text{Tot}(P^\bullet \otimes Q^\bullet)$ complex with $n$-th term

$$\bigoplus_{i+j=n} P^i \otimes Q^j$$

and differential $d^{i,j}(m^{i,j}) := d^i_{m^{i,j}} + (-1)^i \cdot d^j_{m^{i,j}}$. From problem 3 of PS7, we see that this is indeed a complex, and there is a canonical quasi-isomorphism from $\text{Tot}(P^\bullet \otimes Q^\bullet)$ to $P^\bullet \otimes N$ and to $M \otimes Q^\bullet$. □
Corollary 2.2 If $0 \to M_1 \to M_2 \to M_3$ is a short exact sequence of $R$-modules, then for any $R$-module $N$, there exists a long exact sequence:

$$\cdots \to \text{Tor}_i^R(M_1, N) \to \text{Tor}_i^R(M_2, N) \to \text{Tor}_i^R(M_3, N) \to \text{Tor}_{i-1}^R(M_1, N) \to \cdots$$

Proof: Take a projective resolution $Q^\bullet$ for $N$. Since projective implies flat, we have a short exact sequence of complexes:

$$0 \to M_1 \otimes Q^\bullet \to M_2 \otimes Q^\bullet \to M_3 \otimes Q^\bullet \to 0$$

the result follows from applying the long exact cohomology sequence construction. ■

Definition 2.12 Let $M$ and $N$ be $R$-modules. Let $P^\bullet$ be a projective resolution for $M$. Consider the complex:

$$0 \to \text{Hom}(P^0, N) \to \text{Hom}(P^{-1}, N) \to \text{Hom}(P^{-2}, N) \to \cdots$$

We define $\text{Ext}_R^i(M, N)$ to be the $i$-th cohomology of this complex.

Remark: From the definition, $\text{Ext}_R^0(M, N) = \text{Hom}(M, N)$ and $M$ is projective if and only if $\text{Ext}_R^1(M, N) = 0$ for all $R$-modules $N$.

Definition 2.13 A module $I$ is injective if given an injection $L_1 \hookrightarrow L_2$ and a map from $L_1 \to I$, there exists a map from $L_2 \to I$ such that the following triangle commutes:

\[
\begin{array}{ccc}
L_1 & \longrightarrow & L_2 \\
\downarrow & & \downarrow \\
\quad & & I
\end{array}
\]

Proposition 2.5 Any module can be imbedded into an injective module.

Proof: This was problem 2 on PS7, so we omit the proof here. ■

Remark: This allows us to take injective resolutions $0 \to M \to I^0 \to I^1 \to \cdots$ that are unique up to homotopy (also shown on PS7).

Proposition 2.6 Let $M$ and $N$ be $R$-modules. Let $I^\bullet$ be a projective resolution for $M$. Consider the complex:

$$0 \to \text{Hom}(M, I^0) \to \text{Hom}(M, I^1) \to \text{Hom}(M, I^2) \to \cdots$$

We can define $\text{Ext}_R^i(M, N)$ as the $i$-th cohomology of this complex as well.

Proof: Use the same argument as for Tor symmetry (with the Tot complex).
Proposition 2.7  (i) If $0 \to N_1 \to N_2 \to N_3$ is a short exact sequence of $R$-modules, then for any $R$-module $M$, there exists a long exact sequence:

$$\cdots \to \text{Ext}^i_R(M, N_1) \to \text{Ext}^i_R(M, N_2) \to \text{Ext}^i_R(M, N_3) \to \text{Ext}^{i+1}_R(M, N_1) \to \cdots$$

(ii) If $0 \to M_1 \to M_2 \to M_3$ is a short exact sequence of $R$-modules, then for any $R$-module $N$, there exists a long exact sequence:

$$\cdots \to \text{Ext}^i_R(M_3, N) \to \text{Ext}^i_R(M_2, N) \to \text{Ext}^i_R(M_1, N) \to \text{Ext}^{i+1}_R(M_3, N) \to \cdots$$

Proof: Use the same argument as for the Tor long exact sequence.