Existence and uniqueness of the Hoffman-Singleton Graph (outline)
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0. Let $G$ be a Moore graph of degree 7, i.e., a strongly regular graph with parameters $(50, 7, 0, 1)$, or equivalently a graph of degree 7 on 50 vertices with diameter 2 and girth 5. Such $G$ contains $50 \cdot 7 \cdot 6 \cdot 6 / 10 = 1260$ pentagons.

1. Let $A$ be the neighborhood of one of those pentagons, and let $B$ be its complement in $V(G)$. Then $V(G) = A \cup B$ is a partition of $V(G)$ into two 25-vertex parts such that every vertex of $G$ is adjacent to 2 in the same part and 5 in the other part. This is all we’ll use about $A, B$.

2. Of the 1260 pentagons, 1000 are of type $A B A x B A B$ and 250 of type $A A x B B A B$. This leaves only 10 of type $A A A A x$ and $B B B B x$. It soon follows that each of $A, B$ consists of five pentagons.

3. We now know what the $A A$ and $B B$ edges look like; the condition that $G$ have no 3- or 4-cycles will force the $A B$ edges uniquely up to automorphism of $A, B$. First, each $A$ vertex has its one of its five $B$ neighbors in each of the five $B$ pentagons and vice versa. Thus the restriction of $G$ to the union of an $A$ and a $B$ pentagon is a Petersen graph.

4. We can orient the $A$ and $B$ pentagons compatibly, i.e., label the 50 vertices $A_i(x), B_j(y)$ ($i, j, x, y \in F_5$) so that the pentagon edges are $\{A_i(x), A_i(x+1)\}$ and $\{B_j(y), B_j(y+1)\}$ and the $A B$ edges are $\{A_i(x), B_j(2x+c_{ij})\}$ for some $c_{ij}$. [A priori it might have been necessary to use $-2x + c_{ij}$ instead of $2x + c_{ij}$ for some $i, j$, but thanks to the girth condition we can always flip some pentagons, i.e., relabel $A_i(x), B_j(x)$ as $A_i(-x), B_j(-x)$ for certain $i, j$, to eliminate all the minus signs.]

5. To avoid 4-cycles it is now only necessary that $c_{ij} + c_{i'j'} \neq c_{i'j} + c_{ij'}$ ($i \neq i', j \neq j'$). This determines the $c_{ij}$ up to rotating the pentagons [i.e., relabeling $A_i(x)$ as $A_i(x - \delta_i)$, which translates the $c_{ij}$ by $\delta_i$, and likewise for $B_j(y)$] and permuting the $i$’s and $j$’s among themselves. For instance we may take $c_{ij} = i \cdot j$. 