It is a mathematical fact that the casting of this pebble from my hand alters
the centre of gravity of the universe.
— Thomas Carlyle (1795–1881), who apparently knew his John Donne (“if
a clod be washed away by the sea, Europe is the less”\(^1\)) but was not as clear
on Newton’s Laws...

1. (More about separability) Let \(X\) be a metric space, and \(S\) any subspace. Prove that
if \(X\) is separable than so is \(S\). Prove that the following are equivalent:
   i) \(S\) is separable;
   ii) For each \(\epsilon > 0\) there is a countable set of \(\epsilon\)-neighborhoods in \(X\) whose union
       contains \(S\);
   iii) For each \(\epsilon > 0\) there is a countable set of \(\epsilon\)-neighborhoods in \(S\) whose union
       contains \(S\).

2. i) Show directly (i.e., without Weierstrass approximation or Fourier analysis) that
\(C([0,1])\) is separable. (As noted in class, the separability of \(L_2([0,1])\) follows.)
ii) The space \(L_2(\mathbb{R})\) is defined as the completion of the space of continuous func-
tions on \(\mathbb{R}\) with compact support, relative to the usual inner product
\((f,g) := \int_{-\infty}^{\infty} f(x)g(x)\,dx\). [Since \(f,g\) have compact support, this is actually an integral
over a bounded interval, so it makes sense.] Prove that \(L_2(\mathbb{R})\) is separable, and
thus isometric with \(l_2\). Can you find an explicit ontb for \(L_2(\mathbb{R})\)?

3. (Compact subsets of \(l_2\)):
   i) Let \(r_1,r_2,r_3,\ldots\) be a sequence of positive real numbers. Prove that the “box”
\[
\{x \in l_2 : |x_1| \leq r_1, |x_2| \leq r_2, \ldots\}
\]
in \(l_2\) is compact if and only if it is bounded, that is, \(\text{iff } \sum_{i=1}^{\infty} r_i^2 < \infty\).
   ii) Prove that the “ellipsoid” \(\{x \in l_2 : \sum_{i=1}^{\infty} |x_i/r_i|^2 \leq 1\}\) is compact \(\text{iff } r_i \to 0\) as
   \(i \to \infty\).

A linear map \(L : V \to W\) between complete normed spaces \(V,W\) is said to be compact
if the closure in \(W\) of the image of the closed unit ball \(\hat{N}_1(0)\) in \(V\) is a compact subset
of \(W\). Clearly a compact map must be bounded (why?). Thus the last part of Problem 3
shows that the linear map \(l_2 \to l_2\) defined by \((x_1,x_2,x_3,\ldots) \mapsto (r_1x_1,r_2x_2,r_3x_3,\ldots)\)
is compact \(\text{iff } r_i \to 0\).\(^2\) Note that this linear map is self-adjoint, and the unit vectors

---

\(^1\)From Donne’s Meditation XVII, which more famously is the source of “no man is an island” and
“never send to know for whom the bell tolls; it tolls for thee”. The Carlyle quote is from Sartor
Resartus, according to several sources on the Web. No, this has no specific connection with Hilbert
space, separability, or Fourier analysis.

\(^2\)Here \(\{r_i\}\) is any bounded sequence of scalars, which need not be positive reals.
constitute an orthonormal topological basis of eigenvectors. As was true in the finite dimensional case, there is a spectral theorem for compact self-adjoint linear transformations $L$ of an infinite-dimensional separable Hilbert space $\mathcal{H}$, which states that any such $L$ has an ontb of eigenvectors and thus is equivalent to the linear map above under a suitable identification of $\mathcal{H}$ with $l_2$ (i.e. using those eigenvectors as the unit vectors of $l_2$). The proof is outlined in the next two problems, under the slightly stronger hypothesis that $L(\bar{N}_1(0))$ [rather than $\bar{L}(\bar{N}_1(0))$] is compact. You may assume that $\mathcal{H}$ is a real Hilbert space; the complex case is essentially the same, but with a few extra wrinkles.

4. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space, and $L: \mathcal{H} \to \mathcal{H}$ a self-adjoint linear map such that $L(\bar{N}_1(0))$ is compact. Since $K = \{L(x) : x \in \mathcal{H}, |x| \leq 1\}$ is compact, it contains some vector $L(x_1)$ of maximal norm. Prove that (unless $L$ is identically zero, in which case we’re done already) $|x_1| = 1$ and $x_1$ is an eigenvector of $L^2$. Show also that if $x$ is any vector of $\mathcal{H}$ orthogonal to $x_1$ then $L^2(x)$ is also orthogonal to $x_1$, i.e. that $L^2$ restricts to a linear map from the orthogonal complement of $\langle x_1 \rangle$ to itself.

5. Note that this restriction is itself compact and self-adjoint. Using the result of the previous problem, obtain orthonormal eigenvectors $x_2, x_3, \ldots$ of $L^2$, and use these to prove the spectral theorem for compact self-adjoint linear operators $L$ on $\mathcal{H}$.

6. Show that for each $x \in [0,1]$ the functional $f \mapsto \int_0^x f(t) \, dt$ on $C([0,1])$ extends continuously to a functional $\phi_x$ on $L_2([0,1])$. What is the norm of $\phi_x$? Show that for any $f \in L_2([0,1])$ the map $Tf : x \mapsto \phi_{1-x}(f)$ is continuous. Prove that $f \mapsto Tf$ is a continuous linear transformation from $L_2([0,1])$ to $C([0,1])$, and thus to $L_2([0,1])$. Prove further that $T$, considered as a linear operator on $L_2([0,1])$, is self-adjoint and compact. Find an ontb of eigenvectors of $T$.

This abbreviated problem set is due Friday, April 4 in class.