Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #7 (π Day (March 14), 2003):
Differential forms, chains, integration, and more exterior algebra

so₆C ≅ sl₄C
— Fulton and Harris, *Representation Theory: a First Course* (Springer, 1991), page 282 and elsewhere. This is one version of the “ultimate explanation” of the identification of the lines in a four-dimensional vector space with the points of a quadric in a five-dimensional projective space over the same field.

1. Let $E \subset \mathbb{R}^2$ be the punctured plane $\mathbb{R}^2 - \{0\}$. Recall that we have constructed a closed but not exact 1-form $d\theta$ on $E$. Show that any closed 1-form $\omega$ on $E$ can be written uniquely as $\phi + c d\theta$ where $\phi$ is an exact 1-form and $c \in \mathbb{R}$. [Thus “the first (deRham) cohomology $H^1(E, \mathbb{R})$ is one-dimensional”, since it is generated by the class of $d\theta$.] Give a formula for $c$ in terms of $\omega$.

2. Prove that every closed affine 1-chain in a convex set $E \subseteq \mathbb{R}^n$ is the boundary of some affine 2-chain in $E$.

3. Let $E \subseteq \mathbb{R}^n$ be a convex set, and $\gamma : [a, b] \rightarrow E$ any $C^m$ curve. Construct a $C^m$ 2-chain in $E$ whose boundary is the difference between $\gamma$ and the affine 1-simplex $[\gamma(a), \gamma(b)]$. [Hint: rather than working directly with 2-simplices it will be easier to use a 2-cell and then apply Exercise 17.] Conclude from this and the previous problem that every closed $C^m$ 1-chain in a convex set is a boundary of a $C^m$ 2-chain in the same convex set.

Since we have shown that $\partial^2 = 0$, this means that a 1-chain in a convex set is closed if and only if it is a boundary. The same is true for $k$-chains; the proof uses the same basic ideas, but requires rather more bookkeeping. For both your and Andrei’s sake I’ll leave the details to a future course in algebraic or differential topology.

The next two problems from Rudin construct and investigate closed but not exact $(n-1)$-forms on $\mathbb{R}^n - \{0\}$, generalizing our form $d\theta$ on the punctured plane. These are a key ingredient in the proof of the Brouwer fixed-point theorem and related results (such as the “ham sandwich theorem” and its generalization to $\mathbb{R}^n$), at least for sufficiently differentiable functions.


Finally, a sorbet of exterior algebra:

7. i) Let $V$ be a finite-dimensional real inner product space. Prove that there is a unique inner product on $\wedge^d V$ such that $\langle(v_1 \wedge \cdots \wedge v_d), (v'_1 \wedge \cdots \wedge v'_d)\rangle = \det(\langle v_i, v'_j \rangle)_{i,j=1}^d$ for any $v_1, \ldots, v_d, v'_1, \ldots, v'_d \in V$.

ii) Now let $V$ have dimension 4 and $W = \wedge^2 V$. Fix a generator $\delta$ of $\wedge^4 V$ such that $\langle \delta, \delta \rangle = 1$. (We may take $\delta = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ where $e_1, \ldots, e_4$ is an orthonormal basis for $V$.) We then have a bilinear pairing $(\cdot, \cdot) : W \times W \to \mathbb{R}$ defined by $w \wedge w' = (w, w') \delta$. In the last problem set we showed in effect that this pairing is nondegenerate, and thus identifies $W$ with $W^*$. But now that $V$ has an inner product structure we have another such pairing, $\langle \cdot, \cdot \rangle$, and thus another identification of $W$ with its dual. Composing one of these two identifications with the other’s inverse yields a map $\iota : W \to W$ characterized by $\langle w, w' \rangle = (\iota w, w') \delta$. In the last problem set we showed in effect that this pairing is nondegenerate, and thus identifies $W$ with $W^*$. 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