Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #6 (7 March 2003):
Interlude on convexity; introduction to differential forms

If you see a good move, don’t play it: look for a better one!
Edward Lasker (1868–1941), mathematician and World Chess Champion.

In other words, if you see a complicated way to solve a problem, stop — before carrying it through to completion, check whether a simpler approach is available. In the long run this will save you time and reduce the probability of error.

First, some basic facts about convex functions. Recall that a subset $E$ of a real vector space $V$ is said to be convex if $x, y \in E \Rightarrow px + qy \in E$ for all $p, q \in [0, 1]$ such that $p + q = 1$. If $E$ is convex, an (upward) convex function on $E$ is a function $f : E \to \mathbb{R}$ such that $f(px + qy) \leq pf(x) + qf(y)$ for all $x, y \in E$, $p, q \in [0, 1]$ with $p + q = 1$; equivalently, $f$ is convex if \[ \{(x, t) \in V \oplus \mathbb{R} : t > f(x)\} \] is a convex subset of $V \oplus \mathbb{R}$.

1. i) Show that any convex function on a convex open set in $\mathbb{R}^k$ is continuous.
   ii) Let $U$ be a convex open set in $\mathbb{R}^k$, and fix $B \in (0, \infty)$ and a compact subset $K \subset U$. Let $C$ be the set of all convex functions $f : U \to [-B, B]$. Prove that the restriction of $C$ to the space of continuous functions on $K$ is equicontinuous.

2. [Jensen’s inequalities] Let $f$ be a convex function on a convex set $E$ in some real vector space.
   i) If $x_i \in E$, $p_i \geq 0$, and $\sum_{i=1}^{n} p_i = 1$, prove that $x := \sum_{i=1}^{n} p_i x_i$ is in $E$ and $f(x) \leq \sum_{i=1}^{n} p_i f(x_i)$. (This contains many classical inequalities as special cases; e.g., the inequality on the arithmetic and geometric means is obtained by taking $E = (0, \infty)$, $f(x) = -\log x$, and $p_i = 1/n$.)
   ii) If $\phi : [a, b] \to E$ is a continuous function and $\alpha : [a, b] \to \mathbb{R}$ is an increasing function such that $\alpha(b) - \alpha(a) = 1$, prove that $x := \int_{a}^{b} \phi(t) d\alpha(t)$ is in $E$ and $f(x) \leq \int_{a}^{b} f(\phi(t)) d\alpha(t)$.

3. The logarithmic convexity of $\Gamma(x)$, or more generally of any function of the form $f(x) = \int (\alpha(t))^x \beta(t) dt$, can be interpreted as the nonnegativity of the determinant of a symmetric $2 \times 2$ matrix. Generalize this to larger determinants. For instance, prove that for any positive reals $a_1, \ldots, a_n$ the determinant of the $n \times n$ matrix with entries $\Gamma(a_i + a_j)$ is nonnegative, as is the determinant with entries $(a_i + a_j)^{-k}$ for any $k > 0$. [Hint for this last part: what is $\int_{0}^{\infty} t^{x-1} e^{-ct} dt$?]

The next problem reviews exterior algebra:
4. Let $V$ be a vector space over some field $k$, and let $U$ be a subspace of finite dimension $d$. For any basis $v_1, \ldots, v_d$ of $U$, consider $\omega := v_1 \wedge v_2 \wedge \cdots \wedge v_d \in \bigwedge^d V$.

i) Show that $\omega \neq 0$, and that any other choice of basis would yield a nonzero scalar multiple of $\omega$. Moreover, if $U' \subset V$ is a subspace of dimension $d$ for which the same procedure yields a scalar multiple of $\omega$ then $U' = U$.

ii) Now suppose $V$ is of dimension 4 over a field $k$ not of characteristic 2, and let $W$ be the $k$-vector space $\bigwedge^2 V$ of dimension 6. Fix a generator $\delta$ of the one-dimensional space $\bigwedge^4 V$. Then for each $\omega \in W$ we have $\omega \wedge \omega = Q(\omega)\delta$ for some $Q(\omega) \in k$. Prove that $\omega \mapsto Q(\omega)$ is a nondegenerate quadratic form, and that $Q(\omega) = 0$ if and only if $\omega = v_1 \wedge v_2$ for some $v_1, v_2 \in V$. (This, combined with (i), identifies the lines in $V$ with a quadric in the 5-dimensional projective space $(W - 0)/k^*$. If $k = \mathbb{R}$, what is the signature of $Q$?

Simplices, boxes, etc.:


Finally, a more-or-less explicit approach to the simplest case of “closed on convex is exact”, and an application to a fundamental fact about complex analysis:

7. Solve exercises 24, 25 on page 296. [Such arguments are ubiquitous in differential geometry and algebraic topology.]

8. [Conjugate harmonic functions] In problem 5 of the 4th set we obtained the “Cauchy-Riemann equations” for the real and complex parts $u, v$ of a differentiable function $u + iv$ on an open subset $E$ of $\mathbb{C}$.

i) Rewrite these partial differential equations as formulas for the exact differentials $df, dg$ in terms of $g, f$ respectively. Use this to show that, if $E$ is convex, for any harmonic function $f \in \mathcal{C}^2(E)$ there exists $g$, unique up to additive constants, such that $f, g$ satisfy the Cauchy-Riemann equations; and conversely that given a harmonic $g$ there exists $f$. (Such $f, g$ are called “conjugate harmonic functions” on $E$.)

ii) Now let $E$ be the non-convex set $\mathbb{C} - \{0\}$. Give an example of a harmonic $\mathcal{C}^\infty$ function $f : E \rightarrow \mathbb{R}$ that has no conjugate harmonic function $g : E \rightarrow \mathbb{R}$.

This problem set due $\pi$ Day, March 14, at the beginning of class.