Math 55b: Honors Advanced Calculus and Linear Algebra

Homework Assignment #3 (Valentine’s Day (Feb.14), 2003):
More univariate calculus, and Stone-Weierstrass

Fejér discovered his theorem\(^1\) at the age of 19, Weierstrass published [his Polynomial Approximation Theorem] at 70. With time the reader may come to appreciate why many mathematicians regard the second circumstance as even more romantic and heart warming than the first.\(^2\)

More about power series:

1. Our two proofs of formula (5) on p.173 (termwise differentiation of power series inside the circle of convergence) used special properties of calculus over \(\mathbb{R}\): the Mean Value Theorem and the Fundamental Theorem of Calculus. Give a direct proof that applies equally well to power series over \(\mathbb{C}\) or the field \(\mathbb{Q}_p\) of \(p\)-adic numbers.

2. For \(p\)-adic numbers \(a_n\) \((n = 1, 2, 3, \ldots)\), prove that \(\sum_{n=1}^{\infty} a_n\) converges if and only if \(a_n \to 0\) in \(\mathbb{Q}_p\). For which \(x \in \mathbb{Q}_p\) does the exponential series \(E(x) = \sum_{n=1}^{\infty} x^n/n!\) converge? Which \(a \in \mathbb{Q}_p\) can be written as \(E(x)\) for some \(x \in \mathbb{Q}_p\) such that the sum for \(E(x)\) converges?

Some integration techniques. First we show how to integrate an arbitrary rational function:

3. [Partial fractions\(^3\)] Let \(k\) be an algebraically closed field. Let \(K = k(x)\), the field of rational functions in one variable \(x\) with coefficients in \(k\). Show that the following elements of \(K\) constitute a basis for \(K\) as a vector space over \(k\): \(x^n\) for \(n = 0, 1, 2, 3, \ldots\), and \(1/(x-x_0)^n\) for \(x_0 \in K\) and \(n = 1, 2, 3, \ldots\). (Linear independence is easy. To prove that the span is all of \(K\), consider for any polynomial \(Q \in k[x]\) the subspace \(V_Q := \{P/Q : P \in k[x], \deg(P) < \deg(Q)\}\) of \(K\), and compare its dimension with the number of basis vectors in \(V_Q\).)

4. Prove that \(\tan(x) := \sin(x)/\cos(x)\) is an increasing function on \((-\pi/2, \pi/2)\) mapping this interval bijectively to \(\mathbb{R}\). Prove that the inverse map \(\tan^{-1}(x)\) has derivative \(1/(x^2+1)\). Use this to determine \(\int_0^{\pi/4}(x-x^2)^4dx/(x^2+1)\). What does this tell you about \(\pi\)?

5. Prove that the integral of any \(f \in \mathbb{R}(x)\) is a rational function plus a linear combination of functions of the form \(\log|x-x_0|, \log((x-x_0)^2+c)\), and \(\tan^{-1}(ax+b)\) \((x_0, a, b, c \in \mathbb{R}, c > 0)\).

Next we derive some classical product formulas and integrals. Be careful about justifying all steps!

6. Prove that \(\int_0^{\pi/2}\cos^n x \, dx = \frac{n-1}{n} \int_0^{\pi/2}\cos^{n-2} x \, dx\) for all \(n \geq 2\). Deduce that

\[
\int_0^{\pi/2}\cos^n x \, dx = \begin{cases} \frac{2}{3} \frac{4}{5} \cdots \frac{n-1}{n}, & \text{if } n \text{ is odd}; \\ \frac{1}{2} \frac{1}{2} \cdots \frac{n-1}{n}, & \text{if } n \text{ is even}. \end{cases}
\]

\(^1\)On Fourier series; see Rudin, pages 199–200 for a sneak preview.

\(^2\)Körner, Fourier Analysis, p.294 (conclusion of Chapter 59: “Weierstrass’s proof of Weierstrass’s theorem”).

\(^3\)The decomposition of any \(f \in K\) as a linear combination of the basis elements described in this problem is called the “partial fraction decomposition” of \(f\).
7. It follows that
\[ \frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \cdot \frac{\int_{0}^{\pi/2} \cos^{2m} x \, dx}{\int_{0}^{\pi/2} \cos^{2m+1} x \, dx}. \]
Show that
\[ 1 < \frac{\int_{0}^{\pi/2} \cos^{2m} x \, dx}{\int_{0}^{\pi/2} \cos^{2m+1} x \, dx} < \frac{\int_{0}^{\pi/2} \cos^{2m-1} x \, dx}{\int_{0}^{\pi/2} \cos^{2m+1} x \, dx} = 1 + \frac{1}{2m}, \]
and therefore
\[ \frac{\pi}{2} = \lim_{m \to \infty} \left( \frac{2}{1} \frac{2}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} \right). \]
[This is usually written as the “infinite product”
\[ \frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{5} \cdots, \]
attributed to Wallis.]

8. Use the formulas of the previous problem to prove that
\[ \lim_{n \to \infty} \int_{0}^{\sqrt{n\pi}/2} \cos^n \left( \frac{x}{\sqrt{n}} \right) \, dx = \sqrt{\pi/2}. \]
Now show that \( \lim_{n \to \infty} \cos^n \left( \frac{x}{\sqrt{n}} \right) = \exp(-x^2/2) \) for any \( x \geq 0 \), and use this to prove that
\[ \int_{0}^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi/2}. \]
Finally, some (Stone-)Weierstrass stuff:

9. i) Suppose \( f : [a, b] \to \mathbb{R} \) is a continuous function such that \( \int_{a}^{b} f(x) x^n \, dx = 0 \) for each \( n = 0, 1, 2, 3, \ldots \). Prove that \( f \) is the zero function. [This is problem 20 on page 169; it also appeared — without the hint provided there — on a Putnam exam many years ago.]
ii) Suppose \( \alpha, \beta : [0, 1] \to \mathbb{R} \) are increasing functions such that there exists \( n_0 \) with \( \int_{0}^{1} x^n d\alpha(x) = \int_{0}^{1} x^n d\beta(x) \) for each integer \( n \geq n_0 \). Prove that \( \alpha_+ - \beta_+ \) and \( \alpha_- - \beta_- \) are constant functions on \( [0, 1) \) and \( (0, 1] \) respectively, where \( \alpha_{\pm}(x) := \lim_{x \to x_{\pm}} \alpha(t) \) and \( \beta_{\pm} \) is defined in the same way.
iii) Solve Problem 21 on page 169.

This problem set due Friday, 21 February, at the beginning of class.

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As noted in class, it is remarkable that this ubiquitous definite integral can be evaluated in closed form, considering that the indefinite integral \( \int \exp(cx^2) \, dx \) cannot be simplified. We shall give another proof of this result when we come to the change of variable formula for multiple integrals.