Discrete Fourier analysis provides a good framework for “Gauss sums” and “Jacobi sums” which are ubiquitous in number theory. For our purposes we shall define them as follows. Fix a prime number $p$, and let $\zeta$ be the primitive $p$th root of unity $e^{2\pi i/p}$. A character is a homomorphism from the multiplicative group $(\mathbb{Z}/p)^* = (\mathbb{Z}/p) - \{0\}$ of order $p-1$ to the $(p-1)$-st roots of unity. The Gauss sum associated to a character $\psi$ is
\[ G(\psi) := \sum_{n=1}^{p-1} \zeta^n \psi(n). \]

The Jacobi sum associated to a pair $\psi_1, \psi_2$ of characters is defined by
\[ J(\psi_1, \psi_2) := \sum_{n=2}^{p-1} \psi_1(n) \psi_2(1-n). \]

For instance, if $p = 5$ and $\psi$ takes $\pm1$ to $1$ and $\pm2$ to $-1$ then $G(\psi) = \zeta - \zeta^2 - \zeta^3 + \zeta^4 = \sqrt{5}$ and $J(\psi, \psi) = -1 + 1 - 1 = -1$.

1. Let $\psi$ be a nontrivial character (the “trivial character” sends every element of $(\mathbb{Z}/p)^*$ to $1$), and extend it to a complex-valued function on $\mathbb{Z}/p$ by defining $\psi(0) = 0$. The Gauss sum $G(\psi)$ is one value of the discrete Fourier transform of this function. Determine its discrete Fourier transform at all elements of $\mathbb{Z}/p$.

2. Prove that
\[ J(\psi_1, \psi_2) = G(\psi_1)G(\psi_2)/G(\psi_1 \psi_2) \]
provided none of $\psi_1$, $\psi_2$, $\psi_1 \psi_2$ is the “trivial character” sending every element of $(\mathbb{Z}/p)^*$ to $1$. What happens if one or more of these characters is trivial? [Hint: remember our formula for $B(x, y)$ and its proof.]

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1 Körner, *Fourier Analysis*, p.296 (in Chapter 60). By now I hope none of our 55b class believes this; keep in mind (also for the take-home final) that I do not believe it either!
3. Prove that $|G(\psi)|^2 = p$ for all nontrivial $\psi$. If moreover $\psi$ is “real” (sends each element of $(\mathbb{Z}/p)^*$ to either $+1$ or $-1$), show that $G(\psi) = \pm \sqrt{p}$ if $p \equiv 1 \mod 4$ and $G(\psi) = \pm i \sqrt{p}$ if $p \equiv -1 \mod 4$. Numerically compute $G(\psi)$ for nontrivial real characters $\psi \mod p$ for enough values of $p$ that you detect a pattern in the choices of sign.

I do not ask you to prove this pattern; this sign problem occupied Gauss for years! Our last problem on these sums may suggest one way to solve it:

4. Let $N$ be any positive integer and $\zeta = e^{2\pi i/N}$. Let $M$ be the $N \times N$ matrix whose $(a,b)$ entry is $\zeta^{ab}$; that is, $M$ is the matrix for the discrete Fourier transform mod $N$. Show that $M^4 = N^2 I_n$. Deduce that each eigenvalue of $M$ is $\pm N^{1/2}$ or $\pm i N^{1/2}$. Conclude that there exist integers $r_N, s_N$ such that $\sum_{a=1}^{N} \zeta^a = (r_N + i s_N) \sqrt{N}$. Again, compute $r_N, s_N$ for enough small $N$ until you can guess a pattern. How much of this pattern can you prove?

Here’s one of many applications of Poisson summation:

5. Fix $c > 0$ and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(x^2 + c^2)$. For $t \in \mathbb{R}$ define $F(t) := \sum_{n \in \mathbb{Z}} f(t + 2\pi n)$. Prove that $F$ is a differentiable function of period $2\pi$, and thus can be regarded as a differentiable function on $\mathbb{T}$. Determine its Fourier series (NB we know $\hat{f}$), and deduce the value of $F(0) = \sum_{n \in \mathbb{Z}} f(2\pi n)$. [Your answer should agree with your result for the first problem of the tenth assignment.] Generalize.\(^2\)

The final batch of problems develops a proof of the memorable theorem of Müntz, which answers the following question: Fix an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of nonnegative real numbers and let $V$ be the $\mathbb{R}$-vector space of functions generated by $f^{n_i}, i = 0, 1, 2, \ldots$ For what $\{n_i\}$ is $V$ dense (A) in $L^2([0,1])$, (B) in $C[0,1]$? Müntz’s Theorem asserts:

A) $V$ is dense in $L^2([0,1])$ iff $\sum_{i=1}^{\infty} 1/n_i$ diverges.

B) $V$ is dense in $C[0,1]$ iff $n_0 = 0$ and $\sum_{i=1}^{\infty} 1/n_i$ diverges.

Note that the $L^2([0,1])$ and $C[0,1]$ versions of the Weierstrass Approximation Theorem are the special case $n_i = i$ of Müntz; we assume the Weierstrass theorem in the following proof.

6. For any vectors $x_1, x_2, \ldots, x_m$ in a real Hilbert space $\mathcal{H}$, let $\Delta_m(x_1, \ldots, x_m)$ be the determinant of the $m \times m$ matrix whose $ij$th entry is the inner product of $x_i$ with $x_j$. Recall that if the $x_i$ are linearly independent then this matrix is positive definite, so in particular $\Delta_m(x_1, \ldots, x_m)$ is positive. In this case let $V_m$ be the

\(^2\)This generalizes in many directions; e.g., if $f : \mathbb{Z}/12\mathbb{Z} \to \mathbb{C}$, what is $f(0) + f(3) + f(6) + f(9)$ in terms of $f$?
m-dimensional subspace spanned by the $x_i$, and show that for any vector $y \in \mathcal{H}$ the distance from $y$ to the nearest point of $V_m$ (that is, the norm of the projection of $y$ to the orthogonal complement $V_m^\perp$) is the square root of the ratio

$$\Delta_{m+1}(x_1, \ldots, x_m, y) / \Delta_m(x_1, \ldots, x_m).$$

[Note that this problem only uses the finite-dimensional space generated by $y$ and the $x_i$’s; the full Hilbert space $\mathcal{H}$ is needed only for what follows.]

7. Taking $\mathcal{H} = L_2([0,1])$, $x_{i+1} = t^{n_i}$ and $y = t^k$ in problem 7 we find determinants $\Delta_m, \Delta_{m+1}$ of the form $\det(1/(a_i + b_j))_{i,j=1}^M$. Prove that, for any real numbers $a_1, \ldots, a_M; b_1, \ldots, b_M$ such that none of the $a_i + b_j$ vanishes, the value of this determinant is

$$D_M(a_1, \ldots, a_M)D_M(b_1, \ldots, b_M) / \prod_{i=1}^M \prod_{j=1}^M (a_i + b_j)$$

where $D_M(r_1, \ldots, r_M) = \prod_{1 \leq i < j \leq M} (r_i - r_j)$. Use this to compute the $L_2$ distance from $x_k$ to the space $V_m$ spanned by $x^{n_i}$, $0 \leq i < m$. (Why are these $m$ vectors linearly independent?)

8. Conclude that, provided $k$ is not one of the $n_i$, the $L_2([0,1])$ closure of $V = \bigcup_{m=1}^\infty V_m$ contains $x^k$ if and only if $\sum_{i=1}^\infty 1/n_i$ diverges. Use this to deduce part A of Müntz’s Theorem.

9. The “only if” half of part B is now easily accessible: prove that if $n_0 > 0$ or $\sum_{i=1}^\infty 1/n_i < \infty$ then $C[0,1]$ contains functions not in the closure of $V$. To get the reverse implication we need one more trick: for any $f \in L_2([0,1])$ define $\int f : [0,1] \to \mathbb{R}$ by $\int f(x) = \int_0^x f(t) \, dt$, i.e., the inner product of $f$ with the characteristic function of $[0,x]$. As part of last problem of PS8, we showed in effect that $\int$ is a continuous linear map of norm $\leq 1$ from $L_2([0,1])$ to $C[0,1]$. Use this map to finish the proof of Müntz’s Theorem.

This problem set is due Friday, May 2 in class.