Suppose $X$.

For any set $X$ define the discrete metric on $X$ by $d(p, q) = 0$ if $p = q$ and $d(p, q) = 1$ if $p \neq q$. Prove that this is indeed a metric. With this metric, which subsets of $X$ are open? Which are closed? Which are dense?

Let $(X, d)$ be a metric space. Define $d_0(x, y) := d(x, y)/(1 + d(x, y))$ for all $x, y \in X$.

i) Prove that $d_0$ is also a metric on $X$.

ii) Prove that a subset of $X$ is open under the metric $d$ if and only if it is open under $d_0$. [Thus $(X, d)$ and $(X, d_0)$ are the same as “topological spaces”, but generally not isometric (identical as metric spaces); see Problem 6 below.]

iii) Show that the metric space $(X, d_0)$ is always bounded, even though $(X, d)$ might not be.

Which of the following defines a metric on $R$? Explain.

i) $d_1(x, y) := (x - y)^2$

ii) $d_2(x, y) := \sqrt{|x - y|}$

iii) $d_3(x, y) := |x^2 - y^2|$

iv) $d_4(x, y) := |x^3 - y^3|$

v) $d_5(x, y) := |x - 2y|$

vi) $d_6(x, y) := |x - y|/(1 + |x - y|)$

Suppose $X$ is a set and $d : X \times X \rightarrow R$ is a function satisfying all the distance axioms except that $d(p, q) = 0$ need not imply $p = q$.

i) Check that the following is an example of such a pair $(X, d)$: let $X = R^3$ and

$$d((x_1, x_2, x_3), (x'_1, x'_2, x'_3)) := \max(|x_1 - x'_1|, |x_2 - x'_2|).$$

NB You should solve parts (ii)-(iv) for any such $(X, d)$, not just this example with $X = R^3$.

ii) For $p, q \in X$ define $p \sim q$ to mean $d(p, q) = 0$. Prove that this is an equivalence relation: $p \sim p$ for all $p \in X$, $p \sim q \Rightarrow q \sim p$, and $p \sim q, q \sim r \Rightarrow p \sim r$ [Rudin, Definition 2.3, p.25].

iii) Show that if $p \sim p'$ and $q \sim q'$ then $d(p, q) = d(p', q')$.

iv) Let $\tilde{X}$ be the set of equivalence classes, i.e., subsets of $X$ of the form $[p]$, defined as $[p] := \{p' \in X : p \sim p'\}$. [NB $[p] = [p'] \iff p \sim p'$] Part (iii) showed that

$$\tilde{d}([p], [q]) = d(p, q)$$

is a well-defined function $\tilde{d} : \tilde{X} \times \tilde{X} \rightarrow R$; that is, for all $P, Q \in \tilde{X}$ the value of $\tilde{d}(P, Q)$ does not depend on the choice of representatives of the equivalence classes $P, Q$. Prove that $\tilde{d}(. , .)$ satisfies the axioms of a metric.

v) Part (iv) makes $\tilde{X}$ a metric space. What is this metric space for our above example with $X = R^3$?

Problems 1 through 4 are due Monday, 26 September, at the beginning of class.
The following problems concern isometries between metric spaces. Recall that an isometry between metric spaces \( X, Y \) is a bijection \( i : X \to Y \) such that

\[
d_Y(i(x_1), i(x_2)) = d_X(x_1, x_2)
\]

for all \( x_1, x_2 \in X \).

5. Prove that:
   
   i) The identity map on a metric space is always an isometry.
   
   ii) If \( i : X \to Y \) is an isometry, then so is the inverse map \( i^{-1} : Y \to X \).

   [Note that \( j \circ i \) is the correct order, not \( i \circ j \). One sometimes expresses facts (i) and (iii) by saying that metric spaces and isometries between them form a “category”. Parts (i),(ii),(iii) together, applied in the special case \( X = Y = Z \), are expressed by saying that the isometries from \( X \) to itself constitute a “group”.

   The remaining two parts determine this group in the special case of the metric space \( \mathbb{R} \).

   iv) For \( X = Y = \mathbb{R} \), the function \( i(x) = -x \) is an isometry, as is \( j_a(x) = x + a \) for any \( a \in \mathbb{R} \).

   v) Every isometry from \( \mathbb{R} \) to itself is either \( j_a \) or \( i \circ j_a \) for some \( a \).

   (This last is by far the hardest part of this problem; some mathematicians would say — after solving the problem — “the only nontrivial” instead of “by far the hardest”…)

6. Let \( (X, d) \) be a metric space, and \( (X, d_0) \) the bounded metric space of Problem 2 [with the same \( X \), and \( d_0 = d/(1 + d) \)].
   
   i) Prove that \( (X, d_0) \) is isometric with \( (X, d) \) if and only if \( X \) has at most one element. (Warning: this means you must prove that no map from \( X \) to itself is an isometry, not just that the identity map is not an isometry!)

   ii) Construct an example of an infinite metric space \( (X, d) \) and a map \( i : X \to X \) satisfying

   \[
d(i(x_1), i(x_2)) = d_0(x_1, x_2)
\]

   for all \( x_1, x_2 \in X \). [That is, \( i \) is an isometry between \( (X, d_0) \) and \( (i(X), d|_{i(X)}) \).]

Closures, etc.:

7. [Rudin, p.43, Ex.6] Let \( E \) be a subset of a metric space, and \( E' \) its set of limit points. Prove that \( E' \) is closed, and that \( E \) and \( E' \) have the same limit points. (Recall that \( E' \), the closure of \( E \), is defined by \( E' = E \cup E' \).) Is it true that \( E \) and \( E' \) have the same limit points for every \( E \)?

8. [Rudin, p.43-4, Ex.5,13]
   i) Construct a bounded closed subset of \( \mathbb{R} \) with exactly three limit points.

   ii) [This is rather trickier] Construct a bounded closed set \( E \subset \mathbb{R} \) for which \( E' \) is an (infinite) countable set.

9. [Rudin, p.43, Ex.7] Let \( A_1, A_2, A_3, \ldots \) be subsets of a metric space.
   
   i) If \( B_n = \bigcup_{i=1}^n A_i \), prove that \( \bar{B}_n = \bigcup_{i=1}^n \bar{A}_i \) (the closure of a finite union is the union of the closures).

   ii) If \( B = \bigcup_{i=1}^\infty A_i \), prove that \( \bar{B} \supseteq \bigcup_{i=1}^\infty \bar{A}_i \) (the closure of a countable union contains the union of the closures).

   iii) Give an example where this inclusion is proper (a.k.a. strict), that is, an example of \( \bar{B} \neq \bigcup_{i=1}^\infty \bar{A}_i \).
Two different notions of distance between subsets of a metric space:

10. [Distance between subsets of a metric space] For any two nonempty subsets $A, B$ of a metric space $X$, define the distance $d(A, B)$ between $A$ and $B$ by

$$d(A, B) := \inf \{d(x, y) : x \in A, y \in B\}.$$  

Prove that for any subsets $A, B, C$ of $X$ and any element $x \in X$ we have:

i) $d(\bar{A}, \bar{B}) = d(A, B)$ (where $\bar{A}, \bar{B}$ are the closures of $A, B$ respectively);

ii) $d(\{x\}, A) = 0$ if and only if $x \in \bar{A}$;

iii) $d(A, B \cup C) = \min\{d(A, B), d(A, C)\}$;

iv) $d(A, \{x\}) + d(\{x\}, B) \geq d(A, B)$.

Is it true that the triangle inequality $d(A, C) + d(C, B) \geq d(A, B)$ holds for all $A, B, C$?

11. [Minkowski distance between nonempty bounded closed subsets of a metric space] Recall that $N_r(x)$ is the radius-$r$ neighborhood of $x$, a.k.a. the open ball of radius $r$ about $x$. For a subset $A$ of a metric space $X$, and a positive real number $r$, define

$$N_r(A) := \bigcup_{x \in A} N_r(x).$$

One may visualize $N_r(A)$ as the radius-$r$ neighborhood of $A$. For instance, $N_r(\emptyset) = \emptyset$; $N_r(\{x\}) = N_r(x)$; $N_r(X) = X$; and $r' \geq r \Rightarrow N_{r'}(A) \supseteq N_r(A)$.

For two nonempty, bounded, closed subsets $A, B$ of a metric space $X$, define the Minkowski distance $\delta(A, B)$ between $A$ and $B$ by

$$\delta(A, B) := \inf \{r : N_r(A) \supseteq B \text{ and } N_r(B) \supseteq A\}.$$  

Prove that this defines a metric on the space of nonempty, bounded, closed subsets of $X$. (You may have noticed that the triangle inequality holds even without the requirement that our bounded nonempty subset be closed. Why then must $X$ consist only of the closed subsets?)

Problems 5 through 11 are due Friday, 30 September, at the beginning of class.