Math 250a: Higher Algebra
Problem Set #9 (6 December 2004):
Quaternion algebras

A bit more about Baer multiplication:

1. Let $A$ be an abelian group and $G$ any group acting on $A$. For any extension $1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$ consistent with this action, let $E^o$ be the extension $1 \to A \xrightarrow{-i} E \xrightarrow{\pi} G \to 1$ with the opposite embedding of $A$ in $E$. [Why do $E,E^o$ have the same $G$-action on $A$?] Prove that $E^o$ is the inverse of $E$ in two ways: by identifying $(E,E^o)/Q$ with the semidirect product $A \rtimes G$, and by showing that $E,E^o$ correspond to inverse elements of $H^2(G,A)$.

Note that the formula for $E^o$ is what one might expect from the special case $(G,A) = (\text{Gal}(L/k),L^*)$ and our results about the opposite of a central simple algebra.

In the next two problems, we describe generalized quaternion algebras over an arbitrary field $k$ not of characteristic 2.

2. i) For any (commutative) field $k$, define a map $x \mapsto \bar{x}$ on $M_2(k)$ by $\bar{x} = \text{Tr}(x) \cdot 1 - x$. Here $\text{Tr}(x)$ is the trace of $x$ as a $2 \times 2$ matrix, and $1$ is the $2 \times 2$ identity matrix, which is the unit element of $M_2(k)$. Prove that this map is an anti-involution, i.e., that it satisfies the identities $\bar{\bar{x}} = x$ and $\bar{x+y} = \bar{x} + \bar{y}$. [This can be done either by explicit computation or via a relation between $\bar{x}$ and the transpose of $x$.]

ii) Now suppose that $A/k$ is any central simple algebra with $\dim_k A = 4$. Define a map $x \mapsto \bar{x}$ on $A$ by $\bar{x} = \text{Tr}(x) \cdot 1 - x$, where $\text{Tr}(x)$ is the reduced trace of $x$ and $1$ is the unit element of $A$. Prove that this map is an anti-involution.

Let $A_0$ be the kernel of $\text{Tr}$; it is a $k$-vector subspace of $A$ of dimension $4 - 1 = 3$. Let $N : A \to k$ be the reduced norm, so $N(x) = x\bar{x}$. This is a quadratic form on $A$, and the associated bilinear form is

$$(x,y) = N(x+y) - N(x) - N(y) = xy \bar{y} + y\bar{x} = \text{Tr}(xy).$$

Note that if $x \in A_0$ then $N(x) = -x^2$.

3. i) Prove that if $x,y \in A_0$ with $N(x) = N(y) \neq 0$ then $x,y$ are conjugate in $A$.

ii) Prove that there exist $i \in A_0$ with $N(i)$ nonzero. Fix one such $i$, and let $c = N(i)$. Since also $N(-i) = N(i)$, by part (i) there exist invertible $z \in A$ such that $iz = -\bar{z}i$. Show that $iz = -\bar{z}i$, and hence that $ij = -ji$ where $j := z - \bar{z}$. Show that $j \in A_0$ and $j \neq 0$.

iii) Now let $k = ij = -ji$. Show that $k \in A_0$ and $ki = -ik = cj$. Let $d = N(j) = -j^2$, and determine $jk,kj,k^2$ in terms of $c,d,i,j,k$. In particular show that $i,j,k$ are pairwise orthogonal for the bilinear form $(\cdot,\cdot)$.

iv) If $A$ is a division algebra, prove that $i,j,k$ are linearly independent, and
thus that $A = k + ki + kj + kk$. What happens if $A = M_2(k)$?

v) Since we know the multiplication table of $\{1, i, j, k\}$, we have determined $A$.

Show that for any nonzero $c, d$ the algebra obtained in this way is a division ring if and only if there are no $(r, s, t) \in k^3$ such that $cr^2 + ds^2 + cdt^2 = 0$ other than $(r, s, t) = (0, 0, 0)$.

It can be shown that every nondegenerate quadratic form on $k^3$ is equivalent to a multiple of $cr^2 + ds^2 + cdt^2 = 0$ for some $c, d \in k^*$. These $c, d$ are not uniquely determined by the form, but the central simple algebras $A$ associated to the quadratic form is uniquely determined by the equivalence class of the quadratic form up to scaling, and vice versa. Starting from part (i) we can also identify $A^*/\{\pm 1\}$ with the group of $k$-linear transformations of $A_0$ of determinant 1 that preserve the bilinear form $(\cdot, \cdot)$. This generalizes the identification of $H^*/\{\pm 1\}$ with $SO_3(\mathbb{R})$. If we regard $cr^2 + ds^2 + cdt^2 = 0$ as a conic in the projective plane over $k$, we get the simplest example of a “Brauer-Severi variety” associated to a central simple algebra.

If $k = \mathbb{R}$ and $A$ is a division ring, then clearly $c, d > 0$; we may then scale $i, j$ by $c^{1/2}, d^{1/2}$ to identify $A$ with $H$. This completes the cohomology-free proof that $\mathbb{R}, \mathbb{C}$ and $H$ are the only division algebras of finite dimension over $\mathbb{R}$. Likewise it can be shown that for each $p$ there is a unique division algebra $H_p$ with center $\mathbb{Q}_p$ and of dimension 4 over $\mathbb{Q}_p$. For even $p$ we constructed $H_p$ in the seventh problem set. For our final problem, we treat the even case:

4. Find $c, d \in \mathbb{Q}_2^*$ that yield a division ring $H_2$ with center $\mathbb{Q}_2$ and of dimension 4 over $\mathbb{Q}_2$.

You won’t have to look very long for suitable $c, d$!

Problems 1–4 are due in class Monday, December the 13th.

5. Send me e-mail, or schedule a time to meet with me, to discuss your final paper topic.