1. (Another construction of the trace and norm) Let $K/F$ be a finite field extension with $[K:F] = n$. For each $a \in K$, we may consider the map $M_a : K \to K, x \mapsto ax$ as a linear operator on $K$ considered as a vector space over $F$.

   i) Check that $a \mapsto M_a$ is a homomorphism from $K$ to $\text{End}_F(K)$, the algebra of $F$-linear operators on $K$.

   The trace and norm of $a$ (relative to the extension $K/F$) are the trace and determinant of $M_a$. These are denoted $\text{Tr}_{K/F}(a)$ and $\text{N}_{K/F}(a)$, or simply $\text{Tr}(a)$ and $\text{N}(a)$ if $K/F$ is understood.

   ii) Check that Tr is an $F$-linear map from $K$ to $F$, and that the norm is multiplicative: $N(ab) = N(a)N(b)$ for all $a, b \in K$. If $F = \mathbb{R}$ and $K = \mathbb{C}$, what are the trace and norm of $a = x + iy$? What are the eigenvalues of $M_a$?

   It is a fundamental result in linear algebra that a linear operator $T$ on a finite-dimensional vector space satisfies $P(T) = 0$ where $P(\lambda) = \det(\lambda I - T)$ is the characteristic polynomial of $T$. This gives an explicit construction of a monic polynomial (NB not always the minimal such polynomial!) satisfied by an element of $K$.

   iii) Suppose $K = F(u)$ where $u$ is a root of the irreducible polynomial $P(X) = X^n + \sum_{j=0}^{n-1} a_j X^j$ in $F[X]$. Determine the matrix of $M_u$ relative to the $F$-basis $\{1, u, u^2, \ldots, u^{n-1}\}$ of $K$, and check directly that $P$ is the characteristic polynomial of this matrix.

2. (An application of part (iii) of the last problem; cf. problem 2 of Jacobson 4.1) Let $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, and $a = \sqrt{2} + \sqrt{3} \in K$. Determine $n = [K:F]$. Prove that $K = F(a)$. (Hint: what can $[K:F(a)]$ be?) Choose a basis for $K$ as a $F$-vector space, and determine the matrix of $M_u$ relative to this basis. Use this to compute the minimal polynomial of $a$ over $F$. Check directly that this polynomial vanishes at $a$.

3. (Problem 7 of Jacobson 4.1) A field extension $L/F$ is said to be algebraic if every element of $L$ is algebraic over $F$. Suppose $L/F$ is algebraic and $K \subseteq L$ is an $F$-subalgebra, i.e., a subring containing $F$ (equivalently, an $F$-vector subspace containing $F$ and closed under multiplication). Prove that $K$ is a field.

4. (Problem 8 of Jacobson 4.1) Let $L/F$ be the transcendental extension $F(u)$. Suppose that $K$ is a subfield of $L$ properly containing $F$. Prove that $u$ is algebraic over $K$.

5. (Problem 2 of Jacobson 4.3) Construct a splitting field $K$ of $x^5 - 2$ over $\mathbb{Q}$, and determine $[K : \mathbb{Q}]$. (You may assume the irreducibility of $x^5 - 2$ over $\mathbb{Q}$, and of the polynomial $(x^5 - 1)/(x - 1)$ over $\mathbb{Q}(\sqrt{2})$; we shall learn later how to prove such results.)

6. (Problem 4 of Jacobson 4.3) Let $L/F$ be a splitting field over $F$ of some polynomial $f(X)$, and let $K$ be any subfield of $L$ containing $F$. Suppose $\iota : K \to L$ is a homomorphism whose restriction to $F$ is the identity. Prove that $\iota$ can be extended to an isomorphism of $L$.

7. (Problem 4 of Jacobson 4.4) Let $F$ be a field of characteristic $p$ that is not perfect. Thus there are elements of $F$ not contained in $F^p$; let $a$ be any such element. Prove that the polynomial $X^p - a$ is irreducible for every nonnegative integer $e$.

8*. (Curious behavior of an inseparable extension) Let $k$ be a field, $K = k(X,Y)$ the field of rational functions in two variables, and $F = k(X^p,Y^p)$ for some prime $p$. Show that $[K:F] = p^2$. Now assume $k$ is an infinite field of characteristic $p$. (For instance we may have $k = k_0(T)$ with $k_0 = \mathbb{Z}/p\mathbb{Z}$.) Prove that there are infinitely many intermediate fields $E$ between $K$ and $F$ (necessarily with $[K:E] = [E:F] = p$).

We shall see that this cannot happen if $K/F$ is a separable extension of finite degree.

Problem set is due in class Monday, October 4th.