Math 250b: Higher Algebra
Invariants of binary quadrics, etc.

**Proposition.** Let $T$ be a linear transformation of a finite-dimensional vector space $W$, with eigenvalues $c_i$. Then the character of the induced action of $T$ on $\text{Sym}^nW$ is the coefficient of $z^n$ in the generating function $\prod_i (1 - c_i z)^{-1}$.

(Cf. PS2 #2; again this is easy when $T$ is diagonalizable, which is the only case we shall use.)

**Example:** Suppose $W$ is the defining two-dimensional representation $V$ of $\text{SL}_2$, and $c_1 = \lambda$, $c_2 = \lambda^{-1}$. Then our generating function is

$$
\frac{1}{(1 - \lambda z)(1 - \lambda^{-1} z)} = \frac{1}{\lambda - \lambda^{-1}} \left( \frac{\lambda}{1 - \lambda z} - \frac{\lambda^{-1}}{1 - \lambda^{-1} z} \right)
$$

so the coefficient of $z^n$ is $(\lambda^{n+1} - \lambda^{-(n+1)})/(\lambda - \lambda^{-1})$, which we recognize as the character of $T$ acting on $\text{Sym}^nW$.

In general, if $W$ is a representation of $\text{SL}_2$, we can use this to get at the ring of $\text{SL}_2$-invariant polynomials in $W$ by computing the dimension of the space of invariants of degree $n$. Recall that for any representation of $\text{SL}_2$ the dimension of the invariant subspace is the constant coefficient of the polynomial obtained by multiplying the character of $\text{diag}(\lambda, \lambda^{-1})$ by $1 - \lambda^2$. For instance, for $\text{Sym}^nV$ we get

$$
(1 - \lambda^2) \frac{\lambda^{n+1} - \lambda^{-(n+1)}}{\lambda - \lambda^{-1}} = \lambda^n - \lambda^{n+2},
$$

confirming that the only invariant polynomials are the constants.

For a slightly more complicated example, let $W$ be the adjoint representation $\text{Sym}^2V$. Then $c_1, c_2, c_3 = \lambda^2, 1, \lambda^{-2}$; to reduce notational clutter let $u = \lambda^2$. Then the partial-fraction decomposition of our generating function is

$$
\frac{1 - u}{(1 - uz)(1 - z)(1 - u^{-1} z)} = \frac{u^3}{1 - u^2} \frac{1}{1 - uz} + \frac{1}{1 - u^2} \frac{1}{1 - uz} - \frac{u}{1 - u} \frac{1}{1 - z}
$$

so the dimension of $\text{SL}_2$ invariants in $\text{Sym}^nW$ is the constant coefficient of

$$
\frac{u^{n+3}}{1 - u^2} + \frac{u^{-n}}{1 - u^2} - \frac{u}{1 - u} = \frac{(u^{n+1} - 1)(u^2 - u^{-n})}{1 - u^2}.
$$

The first or second factor of the numerator is a multiple of $(1 - u^2)$ in $\mathbb{C}[u, u^{-1}]$ according as $n$ is odd or even. Canceling this common factor, we obtain the decomposition of $\text{Sym}^n(W)$ into irreducibles, as asked in Exercise 11.14 (page 153):

$$
\text{Sym}^n(\text{Sym}^2V) = \bigoplus_{\alpha = 0}^{\lfloor n/2 \rfloor} \text{Sym}^{2n - 4\alpha}V.
$$
In particular, the space of $\text{SL}_2$-invariant polynomials of degree $n$ has dimension 1 or 0 according as $n$ is even or odd. We already know for each even $n$ an invariant of degree $n$, namely the $(n/2)$nd power of the discriminant $\Delta$ of a quadratic polynomial. Hence $\mathbb{C}[\Delta]$ is the full ring of invariants. [This is the first instance of a celebrated theorem on the ring of invariant polynomials of a simple Lie group acting on its adjoint representation. Can you describe for each $\alpha$ an $\text{SL}_2$-invariant map from $\text{Sym}^3 V$ to $\text{Sym}^{2n-4\alpha} V$, and thus solve the plethysm problem associated to Ex. 11.14?]

When $W$ is of yet higher dimension, we need a more systematic way to keep track of the constant coefficients of our generating functions. For instance, consider the case $W = \text{Sym}^4 V$. We already know that the space of invariants in $\text{Sym}^2 W$ is $\mathbb{C}\Delta_2$ for some quadratic polynomial whose vanishing detects quartics with tetrahedral symmetry. One likewise computes that the invariants in $\text{Sym}^3 W$ are again one-dimensional, say $\mathbb{C}\Delta_3$ (see Exercise 11.25 on page 158); as it turns out the vanishing of $\Delta_3$ detects quartics like $x^4 - y^4$ with dihedral symmetry. Now $\Delta_2, \Delta_3$ are algebraically independent: else they would be proportional to the square and cube of an invariant linear polynomial, which does not exist. We shall show that $\mathbb{C}[\Delta_2, \Delta_3]$ is the full ring of invariants by proving that the dimension of invariants in $\text{Sym}^n W$ is the $z^n$ coefficient of the generating function $1/(1-z^2)(1-z^3)$.

Our method is most easily described in terms of complex analysis of one variable (though it can be framed algebraically too). We want the $u^0$ coefficient of

$$F(u, z) := \frac{1-u}{(1-u^2 z)(1-uz)(1-z)(1-u^{-1}z)(1-u^{-2}z)}$$

as a power series in $z$. That coefficient is $(2\pi i)^{-1} \oint_{|u|=1} F(u, z) \ du / u$. For small nonzero $z$, the contour contains three simple poles: at $u = z$ and $u = \pm \sqrt{z}$. The residues are $-z/(1-z)(1-z^2)(1-z^3)$ and $1/2(1-z)(1 \pm z^{3/2})(1-z^3)$. Since $1/(1-z^{3/2}) + 1/(1+z^{3/2}) = 2/(1-z^3)$, the sum of the residues is $(1-z)/(1-z)(1-z^2)(1-z^3)$, which simplifies to $1/(1-z^2)(1-z^3)$ as promised.

Can you show similarly that the ring of invariants of $\text{Sym}^3 V$ is $\mathbb{C}[\Delta_4]$, where $\Delta_4$ is the discriminant of a cubic polynomial? (A symbolic algebra package may be useful here, as well as for the adventure of extending such computations to $\text{Sym}^5 V$, $\text{Sym}^6 V$, and beyond.)