1. [Gauss multiplication formula] Let \( n \) be a positive integer, and define

\[
F(z) = \prod_{k=0}^{n-1} \Gamma\left(\frac{z+k}{n}\right).
\]

i) Show that \( F(z) \) has the same poles as \( \Gamma(z) \), and satisfies the functional equation \( F(z+1) = \frac{z}{n} F(z) \).

ii) This suggests that \( F(z) \) should be proportional to \( n^{-z} \Gamma(z) \). Prove that this is in fact the case, and determine the constant of proportionality.

2. Determine for each \( n = 0, 1, 2, \ldots \) the residue of \( \Gamma(z) \) at the pole \( z = -n \).

Use this to compute \( \int_\gamma \Gamma(s)x^s \, ds \) for all \( x > 0 \), where \( \gamma \) is the contour \( \{1 + it \mid t \in \mathbb{R}\} \).

3. Prove the integral formula

\[
\int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 a^{it} \, dt = 2\pi \Gamma(2\sigma) \frac{a^\sigma}{(1 + a)^{2\sigma}}
\]

for all real and positive \( a, \sigma \). For which complex \( a \) does this formula remain valid?

This result, together with the inversion formula for Fourier transforms, yields a closed form for the Fourier transform of \( (\text{sech} x)^{2\sigma} \); In particular, for \( \sigma = 1/2 \) we recover the formula for the Fourier transform of \( \text{sech}(x) \). For general \( \sigma > 0 \) we can also obtain the orthogonal polynomials for the weight function \( |\Gamma(\sigma + it)|^2 \) — that is, polynomials \( P_n(t) \) of degree \( n \) in \( t \) such that

\[
I(m, n) := \int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 P_m(t)P_n(t) \, dt
\]

vanishes unless \( m = n \). One nice way is to use the coefficients in the generating function

\[
(1 + ix)^{-\sigma-\text{i}t}(1 - ix)^{-\sigma + \text{i}t} = \sum_{n=0}^{\infty} P_n(t)x^n.
\]

Can you determine \( I(n, n) \), or more generally

\[
\int_{-\infty}^{\infty} |\Gamma(\sigma + it)|^2 P_l(t)P_m(t)P_n(t) \, dt
\]

for nonnegative integers \( l, m, n \)? These \( P_n(t) \) are the symmetric Meixner-Pollaczek polynomials ("symmetric" because one can more generally describe the orthogonal polynomials for the weight function \( |\Gamma(\sigma + it)|^2 e^{ct} \) for any constant \( c \) with \( |c| < \pi/2 \)).
Back to complex analysis: the remaining problems concern normal families and univalent functions.

4. Let

\[ f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \]

be a univalent meromorphic function on the open unit disc \( \Delta = \{ |z| < 1 \} \). Show that for each positive \( r < 1 \) the complement in \( \mathbb{C} \) of \( \{ f(z); |z| < r \} \) has area

\[ \pi \left( \frac{1}{r^2} - \sum_{n=0}^{\infty} n|a_n|^2 r^{2n} \right), \]

and thus that the complement of \( f(\Delta) \) has area

\[ \pi \left( 1 - \sum_{n=0}^{\infty} n|a_n|^2 \right). \]

[Recall from the zeroth problem set that the area enclosed by a simple closed analytic arc \( \gamma \) is \( (1/2i) \oint_{\gamma} \bar{z} \, dz \).] Conclude that the univalent functions \( 1/z + O(z) \) on \( \Delta \) constitute a normal family.

5. Let \( \mathcal{F} \) be the family of analytic functions \( f(\cdot) \) on \( \Delta \) such that \( f \) is univalent and normalized by \( f(0) = 0 \) and \( f'(0) = 1 \).

i) Show that if \( f \in \mathcal{F} \) then \( \mathcal{F} \) also contains a function \( g(\cdot) \) such that \( (g(z))^2 = f(z^2) \) for all \( z \in \Delta \).

ii) Apply the result of Problem 4 to conclude that \( \mathcal{F} \) is a normal family.

In particular it follows that for each \( n \) there is an upper bound \( B_n \) on the absolute value of the \( z^n \) coefficient in the Taylor expansion of any \( f \in \mathcal{F} \). The Bieberbach conjecture, finally proved in the mid-1980’s by L. de Branges, asserts that the least bound is \( n \), and specifies all functions attaining this bound. The next problem gives an easy first step in this direction, and some more information about \( \mathcal{F} \).

6. i) Show that \( |f''(0)| \leq 4 \) for all \( f \in \mathcal{F} \), with equality if and only if \( f(z) = z/(1-cz)^2 \) for some \( c \in \mathbb{C} \) with \( |c| = 1 \). [Such \( f \) turn out to be the only functions attaining the Bieberbach bound for any \( n \).]

ii) Prove that for each \( f \in \mathcal{F} \) there exists \( w \in \mathbb{C} \) such that \( w \notin f(\Delta) \) and \( |w| \leq 1 \).

This problem set is due Wednesday, November 5, at the beginning of class.