The ring of entire functions $f : \mathbb{C} \to \mathbb{C}$ is denoted $\mathcal{O}(\mathbb{C})$.

1. Prove that for any sequences $a_n, b_n \in \mathbb{C}$ with $a_n \to \infty$, there exists an entire function such that $f(a_n) = b_n$. (Hint: write $f(z) = \sum g_n(z)$ where $g_n$ is a polynomial constructed by induction, such that $g_n(a_i) = 0$ for $i < n$, $g_n(a_n) = b_n - \sum_{i=1}^{n-1} g_i(a_n)$, and $|g_n(z)| < 2^{-n}$ when $|z| < |a_n|/2$.)

2. Show that if $f, g \in \mathcal{O}(\mathbb{C})$ have no common zeros, then $(f, g) = (1)$; i.e. show there exist $r, s \in \mathcal{O}(\mathbb{C})$ such that $fr + gs = 1$.

(First reduce to the case where $f$ and $g$ have simple zeros, using the fact that $(f, g) = (f + ag, g)$; then use Problem 1.)

3. Let $I = (f_1, f_2, \ldots) \subset \mathcal{O}(\mathbb{C})$ be the ideal generated by the sequence of functions $f_n(z) = \sin(z/n)/z$. Prove that $I$ is not contained in any proper principal ideal. Conclude that $\mathcal{O}(\mathbb{C})$ is not a PID and that $\mathcal{O}(\mathbb{C})$ contains maximal ideals that are not of the form $(z - a)$.

4. Recall that the conformal radius of $(U, p)$ is given by $|f'(0)|$ for any holomorphic bijection $f : \Delta \to U$ satisfying $f(0) = p$. What are the conformal radii of the following pointed regions?

   (a) $(\mathbb{H}, i)$;
   (b) $(\{z : -1 < \text{Re } z < 1\}, 0)$;
   (c) $(\{z \in \mathbb{H} : -\pi < \text{Re } z < \pi\}, i)$;
   (d) $(\mathbb{C} - [-2, 2], \infty)$; and
   (e) $(S_\alpha, r)$, where $r > 0$ and $S_\alpha = \{z : \arg(z) \in (-\alpha, \alpha)\}$.

5. Let $f_n : U \to \mathbb{C}$ be a sequence of analytic functions converging uniformly on compact sets to a nonconstant function $f : U \to \mathbb{C}$. Assume the mappings $f_n(z)$ are at most $d$-to-1. Show the same is true of $f(z)$.

6. Given $1 \leq e < d$, construct a sequence of proper maps $f_n : \Delta \to \Delta$ of degree $d$ such that $f_n \to g$ uniformly on compact sets and $g : \Delta \to \Delta$ is proper of degree $e$.

7. Prove that for any $p \geq 1$, the analytic functions

$$\mathcal{F} = \left\{ f : \Delta \to \mathbb{C} : \int |f(z)|^p |dz|^2 \leq 1 \right\}$$

form a normal family.
8. Given $f : U \to \mathbb{C}$ analytic, let $N(f) = \frac{f''(z)}{f'(z)} \, dz$ as a meromorphic 1-form. (a) Show $N(f) = 0$ iff $f(z) = az + b$ for some $a, b$. (b) Show $N(f \circ g) = N(g) + g^* N(f)$.