Complex Analysis: Homework 2

You may collaborate with others on this and future homework, but you should cite your collaborators as well as any references you use.

1. Let $u(n)$ be the number of partitions of $n$ into unequal parts. (E.g. $u(7) = 5$ because 7 can be written as 7, 1 + 2 + 4, 1 + 6, 2 + 5 and 3 + 4.) (i) Show that $g(z) = \prod_{n=1}^{\infty} (1 + q^n) = \sum u(n)q^n$ for $|q| < 1$. (ii) Show that $g(z)$ cannot be analytically continued beyond the unit disk. (This means if $f(z)$ is analytic on a connected domain $U \supset \Delta$, and $f(z) = g(z)$ on $\Delta$ then $U = \Delta$.)

2. The distributional derivative $(d/dz)(1/z)$ is a constant multiple of the $\delta$-function at the origin; that is, for any compactly supported smooth function $\phi$, we have

$$\int_{\mathbb{C}} \frac{1}{z} \frac{d\phi}{dz} dx dy = C\phi(0).$$

Prove this and evaluate $C$.

3. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $r_1, \ldots, r_d$. Show that $\sum 1/p'(r_i) = 0$.

4. Prove that the automorphisms of $\hat{\mathbb{C}}$ preserving the spherical metric coincide natural with $\text{SU}(2)$, the group of linear maps of $\mathbb{C}^2$ preserving $|z|^2 = |z_1|^2 + |z_2|^2$.

5. Give necessary and sufficient conditions on $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C})$ such that $T$ is in $U(1,1)$, i.e. such that $T$ preserves $|z|^2 = |z_1|^2 - |z_2|^2$. Then show every element of $\text{PU}(1,1) = U(1,1)/S^1$ is represented by a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$, $|a|^2 - |b|^2 = 1$.

6. Which curves on $\hat{\mathbb{C}}$ are geodesics with respect to the spherical metric $\rho = 2 |dz|/(1 + |z|^2)$?

7. Pick a basis for the Lie algebra of $\text{SL}_2(\mathbb{C})$, and show how each basis element can be canonically interpreted as a holomorphic vector field on $\hat{\mathbb{C}} = \mathbb{P}\mathbb{C}^2$. Check that Lie bracket corresponds to bracket of vector fields.