1. Let \( J_i = [a_i, b_i] \subset [0, 1] \) be a sequence of disjoint, closed intervals. Prove that \( [0, 1] \neq \bigcup_1^\infty J_i \).

2. Let \( X \) be a topological space. We say \( A \subset X \) is a \( G_\delta \)-subset if \( A \) can be expressed as the intersection of countably many open sets, \( A = \bigcap_1^\infty U_i \).

(a) Show that every closed subset of a metric space is a \( G_\delta \).

(b) Show that if \( f : X \to \mathbb{R} \) is continuous and \( Z = f^{-1}(0) \) is its zero set, then \( Z \) is a \( G_\delta \).

(c) Show that if \( A = \{p\} \) consists of a single point in \( X = 2^\mathbb{R} \) in the product topology, then \( A \) is not a \( G_\delta \).

3. Let \( F \subset C([0, 1]) \) be the space of continuous functions such that \( |f(x)| \leq 1 \) for all \( x \). Which of the following subsets of \( F \) are compact? Explain your answers.

(a) The set \( F \) itself.

(b) The set of \( f \in F \) such that \( |f(x) - f(y)| \leq |x - y|^{1/2} \) for all \( x, y \).

(c) The set of \( f \) such that \( \sup_{x \neq y} |f(x) - f(y)|/|x - y| < \infty \).

(d) The set of \( f \) such that \( f(x) = \sum_0^\infty a_n x^n/n! \) for some real numbers \( a_n \) with \( \sup |a_n| \leq 1 \).

4. Let \( A \) and \( B \) be disjoint closed subsets of a metric space \((X, d)\). Construct (using \( d \)) an explicit continuous function \( f : X \to [0, 1] \) such that \( f|A = 1 \) and \( f|B = 0 \).

5. Let \( K \) be a compact Hausdorff space. Prove that any real-linear map \( \phi : C(K) \to \mathbb{R} \) that satisfies, for all \( f, g \in C(K) \),
\[
\phi(1) = 1 \quad \text{and} \quad \phi(fg) = \phi(f)\phi(g),
\]
is given by \( \phi(f) = f(x) \) for a unique \( x \in K \).

(Hint: Let \( M = \{f : \phi(f) = 0\} \). Suppose \( f_i \in M \) has no zeros on \( U_i \), and \( \bigcup U_i = K \). Then \( g = \sum f_i^2 \in M \) has no zeros, so \( 1/g \in C(K) \). But then \( 1 = (1/g) \cdot g \in M \), contradicting the fact that \( \phi(1) = 1 \).)

6. Let \( \beta(\mathbb{N}) \) be the Stone–Čech compactification of \( \mathbb{N} \) with the discrete topology. Show the sequence \( a_n = n \in \mathbb{N} \subset \beta(\mathbb{N}) \) has no convergent subsequence.

7. Let \( X \) be a normal space, and let \( C \) be the set of all continuous maps \( f : X \to [0, 1] \). Prove that the map \( \iota : X \to [0, 1]^C \), given by \( \iota(x)f = f(x) \), is a homeomorphism to its image. (The target is given the product topology.)