Modular forms of classical groups and even unimodular lattices

Three lectures, Harvard. Joint work with Jean Lannes.

1. Even unimodular lattices

$\mathbb{R}^n$ euclidean space. $(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i y_i$.

$L \subset \mathbb{R}^n$ lattice. $L$ is even if $\forall x \in \mathbb{R}^n$, $x \cdot x \in 2\mathbb{Z}$ (integral)

unimodular if $\text{vol} \ L = 1$

$E_8 \subset \mathbb{Z}^n$ unim. not even, $D_m = \{ (x_1, \ldots, x_n) \in \mathbb{Z}^n, \sum x_i \equiv 0 \pmod{2} \}$ even

$e_i = \frac{1}{2} (1, \ldots, 1)$, $e = \frac{m}{4}$, $E_m = D_m + \mathbb{Z} e$ even unim. if $m \equiv 0 \pmod{8}$

(unique $e \in L$ up to norm).

Set $L_m = \text{set of eul. in } \mathbb{R}^n$

classical facts

$L_n \neq \emptyset \iff n \equiv 0 \pmod{8}$

$O(\mathbb{R}^n) \backslash L_m = X_n$ finite set

$X_8 = \{ E_8 \}$, $X_{16} = \{ E_8, E_8 \otimes E_8 \}$, $\# X_{24} = 24$, $\# X_{32} > 10^5$

Motivations to study these lattices?

1. $L \in L_m$, $\varphi_L : L \to \mathbb{Z}$, $x \mapsto \frac{x \cdot x}{2}$, $q$-forms/2, periodic.

$X_n$ is the set of iso. class of such forms (non-deg/12).

2. Relations to modular forms $SL_2(\mathbb{Z})$, $Sp_{2g}(\mathbb{Z})$ (theta series), orthogonal...

[generalizations of all what follows exist for higher deg, get more

and more complicated; useful $L = 1$ good compromise in high dim.]
A word on classify

L integral latt., \( R(L) = \mathbb{Z} \times \mathbb{L}, \ \alpha \cdot \alpha = 2 \beta \) "noeb"

always ADE root system in \( \text{Vec}_R \) \( R(L) \)

e.g. \( R(\text{En}) = R(\text{Dn}) \) if \( n \geq 8, \ E_8 \ n = 8 \Rightarrow E_8 \neq E_6 + E_6 \)

\( R(\text{Dn}) = D_n \) \( \forall n \geq 2 \)

\( \mathbf{D} \) series \( L \in \mathbb{L} \), \( \mathcal{D}(L) = \sum_{\alpha \in \mathcal{L}} q^{\frac{\alpha^2}{2}} \in \prod_{x \in \mathcal{L}} \prod_{x \in \mathcal{L}} \mathbb{Z}(S_2(Z)) \)

\( n = 8 \), \( \mathcal{D}(L) = \mathcal{D}(E_8) \) \( = 1 + 240 \sum_{n \geq 1} \frac{(E_8)^n}{n} \)

\( = \sum_{n \leq 8} h R(L)q^n \) \( \Rightarrow \)

\( L = E_8 = \mathbb{Z}(E_8) \)

Same \( n = 16 \).

\( n = 24 \) more compl., since alg. of Veblen using \( M_{14} \).

unique \( L \in \mathbb{X}_{24} \) with no root; leech lattice (conway)

then \( \mathbb{X}_{24} \to \text{Leech}^3 \to \text{root} \to \text{latt.} \) lifted onto equi-exter.

Thus \( \mathbb{X}_{24} \to \text{Leech}^3 \to \text{root} \to \text{latt.} \) of rank 24.

(Incidental geometry)

2 Knese neighbourhoods

\( p \) prime, \( L, M, N \in \mathbb{L} \) are \( p \)-neighbours if \( L \cap M \cap N \) index \( p \).

(Knese). Useful way to construct e.u.l.'s.

Construction fix \( p, L \in \mathbb{L} \), let \( C_L(\mathbb{F}_{p^2}) \) be the set of isochoric lines of \( L \otimes \mathbb{F}_{p^2} \). If \( L \in L \otimes \mathbb{F}_{p^2} \), and if \( pL \subset H \subset L \)

is st. \( H/pL = l^1 \), \( \exists \text{exactly } 2 \) e.u.l. containing \( H \), namely \( L \) and \( M_p(L; l) \). Conversely \( \forall l \leq \mathbb{L}, \alpha \cdot \alpha = 2 \Rightarrow \alpha = (p^2) \)

and \( M_p(L; l) \). Clearly \( \forall l \leq \mathbb{L} \)

Thus \( \forall l = H + \mathbb{Z} \frac{\alpha}{p^2} \).

Fact \( L \to M_p(L; l) \) bij. between \( C_L(\mathbb{F}_{p^2}) \) and set of \( p \)-neighbours of \( L \).
Main question: Given $L, M \in \mathcal{L}_n$, can we determine $N_p(L, M)$?

First, inter case $m=16$, but before I come to $n=27$.

Leech $(\mathbb{A}_1^{24})^+$ (explain).

$X_{24}(2)$ dim 5 graph, Leech and $E_{24}$ distance 5.

$M_{47}(E_{24}, <x>) = \text{Leech (Thompson)}$ \hspace{1cm} (Holy construction of C.S!)\hspace{1cm}

Thm $(\mathbb{A}, \mathbb{L})$: $X_{24}(p)$ known $p \neq 2, 3, 5$. $X_{24}(p)$ complete if $p \geq 7$.

- $N_p(R^4, \text{Leech}) \neq 0 \implies p \geq h(R)$

Thm $(\mathbb{A}, \mathbb{L})$: $N_p(E_8 \oplus E_8, E_16) = \frac{405}{691} \frac{8^n - 1}{p - 1}$

It remains!

First goal of the lecture: explain (several) steps of this Thm.

- explain $2n$ dual analogue (need to phrase it differently)

3. Beginning of $\mathfrak{p}$

$n=0(8)$, non-cyclic. "Hecke op." $T_p: \mathbb{Z}[X_n] \to \mathbb{Z}[X_n]$ \hspace{1cm} $[L] \to \sum [M]$ \hspace{1cm} $M$ $p$-neighbor of $L$.

Easy facts:

1. $T_p$ commute with others.

2. $N_p(L, M) 10(M) = N_p(M, L) 10(L)$, i.e. $T_p$ self-adj.

3. $<L, [M]> = S_{10(M)} 10(L)$

4. $\sum [L] \in [X_n]$ "trivial exp." $T_p e = e^{w(p)} e^{V_p}$. 

5. $e = \sum [L] \in [X_n]$. 

6. $\mathfrak{p}$.
\[ n = 16 \quad O(E_{16}) = W(D_{24}), \quad O(E_8 \otimes E_8) = W(E_8)^2 \times \mathbb{Z}_2, \quad \text{gcd} \text{-card} = \frac{256}{405} \]

get ev: \[ 405 \{E_{16}\} + 256 \{E_8 \otimes E_8\} \text{ and orth. } \{E_{16}\} - \{E_8 \otimes E_8\} \]

\[ \text{mod. } \pi \]

Check that \( \pi \) is ev of \( T_\pi \) on \( 2 \)

\[ \sum \frac{p_{1}^{(m-1)}}{p_{1}^{m-1}} + \cdots + \frac{p_{1}^{(m-1)}}{p_{1}^{m-1}} + 1 \]

A: on recognizes trace of \( T_{\ell p} \) on an \( \ell \)-adic Gal. rep. dim. 16.

It turns out \( E = \mathbb{Q} \) and \( \ell \) is a prime number, \( \ell \neq 3 \)

General fact: \( E \) on \( \ell \)-adic Gal. reps. gives rise to \( m \)-dim.

Gal. rep. \( \ell \)-adic in \( \mathbb{Q} \) and \( T_{\ell p} = T_{\ell p} \), and at \( \ell \)

HT \[ 0, 1, \ldots, m-2, \quad m - 2 \]

\( m = 24 \) \( T_{24} \) has been easy without significant effort by Blankers (+ Nebe-Venkatesh)

It turns out that \( \neq ev \), all in \( \mathbb{Z} \).

I will explain the rep. in next lecture.

A look \( p \)-series

\[ \mathbb{Z}[X_n] \rightarrow M \left( \mathbb{Z}_p \right) \]

relevance: \( \ell \)-adic com. relation

\[ \mathcal{Y} \circ T_{\ell} = \left( \frac{p^{m-2}}{p^{m-1}} \right) \quad \text{mc} \]

"relation between Hecke \( T_{\ell} \) on both side".

\( m = 16 \) \( \{E_{16}\} - \{E_8 \otimes E_8\} \text{ generates kernel, see p.109 for more.}\)

\( m = 24 \) gives ev \( \mathbb{Z}(p^2) - p + p \cdots + p^{24} \)

To go further, Siegel \( \begin{array}{l} \mathbb{Z}[X_n] \rightarrow M \left( \mathbb{Z}_p \right) \left( \mathbb{S}_{2g}(Z) \right) \uparrow \end{array} \]

\[ \{L\} \rightarrow \sum \frac{q}{(\psi, \cdots, \psi) \epsilon L} \]

we can view it as generating series of Gram matrices of \( g \)-vectors in \( L \).

Siegel modular form follows from Poisson formula again.
Again a form of Eichler’s formula with respect to certain 
Hecke op on Siegel mod. forms:
map \psi^{(g)} \text{ is injective for } g \geq n, \text{ surjective } g \leq \frac{n}{4}

pl difficult to say something on Siegel side when \( g \) grows.

\text{compatibility } \Phi : M_k(S_{2g}(Z)) \longrightarrow M_k(S_{2g}^0(\mathbb{Z})) \quad g \geq 2.

Siegel op. ker \Phi = S_k(S_{2g}(\mathbb{Z})) \text{ rep. forms } \Phi \psi^{(g)} = \psi^{(g-1)}.

\Rightarrow \text{ ker } \psi^{(g)} \text{ descending filtration on } A[\mathbb{Z}[x]], \text{ graded piece}

\text{ker } \psi^{(0)} \subset S_2(S_{2g}(\mathbb{Z})).

\[ n = 16 \text{ Will had conjectured that } \psi^{(g)}(X_1) = Y^{(g)}(X_1) \neq 0 \text{ for } g \geq 1, \text{ proved by himself } g = 2, \text{ Igusa } g = 3.

F = \psi^{(g)}(X_1) - \psi^{(g)}(X_1) \neq 0 \text{ and in } S_8(S_{18}(\mathbb{Z})).

\text{Igusa: this is socle form!}, \text{ actually has dim } 1 \text{ (Pan-Yuen)}.

So our pf is ev. of socle form, still not clear why \( \psi^{(p)} \) occurs.

One way to conclude: use a special construction of cusp forms due to Ikeda

\text{socle prop. } \text{ "Dual immanant cong" }, \text{ "Sato Kurokawa"}.

\text{long story!} \quad \text{\[ 1 \]}

\textbf{4) Orthogonal modular forms}

\[ n = 6 \times 1 \]

\[ L^2(\mathbb{R}^n) \text{, } L^2(\mathbb{R}^n) \text{, finite adjoint } \text{gps.}

\[ L^2(\mathbb{R}^n) = \bigoplus_{U \in \text{Fin}(\text{On}(1))} U^* \otimes M_0(\text{On}) \text{, } M_0(\text{On}) = \{ f : \mathbb{R}^n \longrightarrow \mathbb{C} \}

\text{ker } T_p \longrightarrow \mathbb{R}^n

\[ T_p(f)(1) \text{ dim } = \sum_{M \in M_1} f(M) \]
\( \text{VI: } M^*_n(\mathbb{Q}_n) = C[X_n]^* \cong T_p \)

\( n = 8, \quad M^*_n(\mathbb{Q}_n) \xrightarrow{\sim} U_0(\mathbb{E}_8) \) need not be explicitly computed (with \( n = 20 \) and \( n = 24 \)).

\[ U = H_d(R^n) = \int P: R^n \rightarrow \mathbb{R}, \text{ hom. d \times d, harmonic} \]

\[ \sum_{d > 0} \dim M^*_n(O_0, d) \cdot t^d = \frac{1 - t^2}{(1 - t^d)^2} = 1 + ... \]

\( d = 8 \), first interesting modular form.

Harmonic \( \mathfrak{O} \)-series:

\[ a_2 \leq 2d \]

\[ V(\mathfrak{O}, P) = \sum P(x) \cdot q^{a_2} \in \mathbb{S}_{2d}(\mathbb{S}_2(\mathbb{Z})) \]

\( \text{Hecke} \)

may view it as:

\[ \mathbb{Z}[\mathfrak{O}_n] \otimes H_\mathfrak{O}(R^n) \xrightarrow{\sim} S_{2d}(\mathbb{S}_2(\mathbb{Z})) \]

get Eichler sel.

\( n = 8, \quad d = 8 \), check isom

nice forms of \( \Delta \)

\[ T_p \uparrow \Delta(\mathfrak{O}, P) = p + p^2 + p^6 + p^{12} + 13 \]

We may use this form to prove this!

\[ \text{two ingredients: } 1 - \text{some more } \mathfrak{O} \text{-series} \]

\[ 2 - \text{triviality} \]

\[ \text{fact: let } \mathfrak{O}_n \text{ be the nth quaternion of } (E_n, \mathfrak{g}_n) \]

\[ \text{then } \mathfrak{g}_n \cong (\mathfrak{g}(\mathfrak{O}) \backslash G(\mathfrak{O}) \mathfrak{g}(\mathfrak{O})) / G(\mathfrak{O}) \]

\[ G : \mathfrak{O}_n, \text{ indeed: if } L \subseteq \mathfrak{O}_n, \text{ then } L \otimes \mathbb{Z}_p \cong \mathbb{E}_p \otimes \mathbb{Z}_p \]

\( \mathfrak{O} \) (single genus), so \( \exists g_0 \in \mathfrak{O}(R) = G(R), \quad g_0^{-1} L \in \mathfrak{g}_n @ \mathbb{Q} \)

\( g_0 \in G(\mathfrak{O}) \)