1 Lemniscate of Bernoulli

The *lemniscate of Bernoulli* is a curve defined by the equation

\[
(x^2 + y^2)^2 = x^2 - y^2. \tag{1}
\]

The graph of this curve is a figure eight (Figure 1).

![Figure 1: Lemniscate of Bernoulli](image)

Suppose that we wish to find the $x$-coordinates of points on the curve that have a horizontal tangent line. To do this, we will use implicit differentiation and differentiate both sides of equation (1) with respect to $x$ to obtain

\[
2(x^2 + y^2)(2x + 2yy') = 2x - 2yy'.
\]
This equation is equivalent to
\[ 2x^3 + 2xy^2 + 2x^2yy' + 2y^3y' = x - yy'. \]
Solving for \( y' \), we obtain
\[ y' = \frac{x(1 - 2x^2 - 2y^2)}{y(1 + 2x^2 + 2y^2)}. \]  (2)

We know that we have a horizontal tangent line if \( y' = 0 \), and this will occur exactly when the numerator of the right-hand side of equation (2) is zero. That is, when
\[ x(1 - 2x^2 - 2y^2) = 0. \]
We have two cases to consider, \( x = 0 \) and \( 1 - 2x^2 - 2y^2 = 0 \).

If \( x = 0 \), then equation (1) becomes \( y^4 = -y^2 \). This is only possible if \( y = 0 \). However, if \( y = 0 \), then the derivative (2) is undefined since \( y \) occurs as a factor in the denominator.

The other possible case we must consider is \( 1 - 2x^2 - 2y^2 = 0 \). If we solve this expression for \( y^2 \), then
\[ y^2 = \frac{1 - 2x^2}{2}. \]
We can substitute the right-hand side of this equation for \( y^2 \) in equation (1) to get
\[ \left( x^2 + \frac{1 - 2x^2}{2} \right)^2 = x^2 - \frac{1 - 2x^2}{2}. \]
Simplifying both sides of this equation, we get
\[ \frac{1}{4} = \frac{4x^2 - 1}{2}. \]
We can solve this equation for \( x^2 \) to get \( x^2 = 3/8 \). Thus, we have horizontal tangent lines at \( x = \pm\sqrt{3/8} \).

## 2 Folium of Descartes

As a second example of finding tangent lines to a curve using implicit differentiation, we can consider the folium of Descartes, which is defined by the equation
\[ x^3 + y^3 = 6xy. \]  (3)
The graph of this curve is shown in Figure 2.
First, let us compute $y'$. Differentiating both sides of equation (3), we have

$$3x^2 + 3y^2 y' = 6(y + xy').$$

If we solve this equation for $y'$, we obtain

$$y' = \frac{2y - x^2}{y^2 - 2x}.$$

It is easy to check that the point $(3, 3)$ is on the curve. In order to find an equation of the line that is tangent to the curve at this point, we must compute the derivative at this point,

$$y'|_{(3, 3)} = \frac{2(3) - 3^2}{3^2 - 2(3)} = -1.$$
Since we know a point on the line and the slope of the line, it is now easy to find an equation of the line,
\[ y - 3 = -1(x - 3) \text{ or } y = -x + 6. \]

Finally, let us determine the points on the curve that have a horizontal tangent line. For a point to have a horizontal tangent line, we must have
\[ y' = \frac{2y - x^2}{y^2 - 2x} = 0. \]
This will occur when the numerator of the fraction on the right-hand side of the equation is zero,
\[ 2y - x^2 = 0 \text{ or } y = \frac{x^2}{2}. \]
Substituting \( y = \frac{x^2}{2} \) into equation (3), we have
\[ x^3 + \left( \frac{x^2}{2} \right)^3 = 6x \left( \frac{x^2}{2} \right) \]
or if we simplify,
\[ x^6 = 16x^3. \]
Consequently, \( x^6 - 16x^3 = x^3(x^3 - 16) = 0 \), and either \( x = 0 \) or \( x^3 = 16 \). Thus, we have horizontal tangent lines when \( x = 0 \) or \( x = \sqrt[3]{16} \).