Math 55a: Honors Advanced Calculus and Linear Algebra

Metric topology I: basic definitions and examples

**Definition.** Metric topology is concerned with the properties of and relations among metric spaces. In general, “space” is used in mathematics for a set with a specific kind of structure; in Math 55 we’ll also encounter vector spaces, function spaces, inner-product spaces, and more. The structure that makes a set $X$ a metric space is a *distance* $d$, which we think of as telling how far any two points $p, q \in X$ are from each other. That is, $d$ is a function from $X \times X$ to $\mathbb{R}$. [NB: This is often indicated by the notation $d : X \times X \to \mathbb{R}$; to emphasize that $d$ is a function of two variables one might write $d = d(\cdot, \cdot)$. Cf. the old symbol “$\div$” for division!] The *axioms* that formalize the notion of a distance are:

- **Nonnegativity:** $d(p, q) \geq 0$ for all $p, q \in X$. Moreover, $d(p, p) = 0$ for all $p$, and $d(p, q) > 0$ if $p \neq q$.
- **Symmetry:** $d(p, q) = d(q, p)$ for all $p, q \in X$.
- **Triangle inequality:** $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in X$.

Any function satisfying these three properties is called a *distance function*, or *metric*, on $X$. A set $X$ together with a metric becomes a *metric space*. Note that strictly speaking a metric space is thus an ordered pair $(X, d)$ where $d$ is a distance function on $X$. Usually we’ll simply call this space $X$ when $d$ is understood.

**Examples.** The prototypical example of a metric space is $\mathbb{R}$ itself, with the metric $d(x, y) := |x - y|$. (Check that this in fact satisfies all the axioms required of a metric.) Two trivial examples are an empty set, and a one-point set $\{x\}$ with $d(x, x) = 0$.

Having introduced a new mathematical structure one often shows how to construct new examples from known ones. For the structure of a metric space, the easiest such construction is to take an arbitrary subset $Y$ of a known metric space $X$, using the same distance function — more formally, the restriction of $d$ to $Y \times Y \subset X \times X$. It should be clear that this is a distance function on $Y$, which thus becomes a metric space in its own right, known as a metric *subspace* of $X$. So, for instance, the single metric space $\mathbb{R}$ gives as a huge supply of further metric spaces: simply take any subset, use $d(x, y) = |x - y|$ to make it a subspace of $X$.

Another construction is the *Cartesian product* $X \times Y$ of two known metric spaces $X, Y$. This consists of all ordered pairs $(x, y)$ with $x \in X$ and $y \in Y$. There are several choices of metric on $X \times Y$, of which the simplest is the *sup metric* defined by

$$d_{X \times Y}((x, y), (x', y')) = \max(d_X(x, x'), d_Y(y, y')).$$

(1)

(Note that we use subscripts to distinguish the distance functions on $X, Y$, and $X \times Y$.) So, for instance, taking $X = Y = \mathbb{R}$ we obtain a new metric space
\( \mathbb{R} \times \mathbb{R} \), otherwise known as \( \mathbb{R}^2 \). [Warning: the sup metric
\[
d((x,y),(x',y')) = \max(|x-x'|,|y-y'|)
\]
on \( \mathbb{R}^2 \) is not the Euclidean metric you are familiar with. It is a bit tricky to prove analytically that the Euclidean metric satisfies the triangle inequality; we shall do this when we study inner-product spaces a few weeks hence. For an alternative proof, see Rudin, Thms. 1.35 and 1.37e (pages 15,17).] Of course any subset of \( \mathbb{R}^2 \) then becomes a metric space as well. We can also iterate the product construction, obtaining for instance the metric spaces \( \mathbb{R}^2 \times \mathbb{R} \) and \( \mathbb{R} \times \mathbb{R}^2 \).

Shouldn’t both of these simply be called \( \mathbb{R}^3 \)? True, both sets can be identified with ordered triples \((x,y,z)\) of real numbers, arising as \(((x,y),z)\) in \( \mathbb{R}^2 \times \mathbb{R} \) and as \((x,(y,z))\) in \( \mathbb{R} \times \mathbb{R}^2 \). But we haven’t defined a metric on the Cartesian product \( X \times Y \times Z \) of three metric spaces \( X,Y,Z \), and meanwhile we have two metrics coming from the definition (1): one from \((X \times Y) \times Z\), the other from \(X \times (Y \times Z)\). Fortunately the two metrics coincide: both tell us that the distance between \((x,y,z)\) and \((x',y',z')\) should be
\[
\max(d_X(x,x'),d_Y(y,y'),d_Z(z,z')).
\]
In other words, the function \( i : (X \times Y) \times Z \to X \times (Y \times Z) \) taking \(((x,y),z)\) to \((x,(y,z))\) is not only a bijection of sets but an isomorphism of metric spaces, a.k.a. an isometry. In general an isometry is a bijection \( i : X \to X' \) between metric spaces such that \( d_X(p,q) = d_{X'}(i(p),i(q)) \) for all \( p,q \in X \). This definition captures the notion that \( X,X' \) are “the same” metric space, and \( i \) effects an identification between \( X,X' \). This justifies our identification of \((X \times Y) \times Z\) with \(X \times (Y \times Z)\) as metric spaces, and calling them both \(X \times Y \times Z\). In particular, we have a natural isometry between \( \mathbb{R}^2 \times \mathbb{R} \) and \( \mathbb{R} \times \mathbb{R}^2 \) and may call them both \( \mathbb{R}^3 \). Likewise we may inductively construct \( \mathbb{R}^n \) \((n = 2,3,4,\ldots)\) as \( \mathbb{R}^m \times \mathbb{R}^{n-m} \) for any integer \( m \) with \(0 < m < n\); the choice does not matter, because we always get the same distance function
\[
d(((x_1,x_2,\ldots,x_n),(x'_1,x'_2,\ldots,x'_n))) = \max_{1 \leq i \leq n} |x_i - x'_i|.
\]
We shall give several further basic constructions of metric spaces and examples of isometries in the first problem set.
Bounded metric spaces and function spaces. Perhaps the simplest property a metric space might have is boundedness. A metric space $X$ is said to be bounded if there exists a real number $B$ such that $d(p, q) < B$ for all $p, q \in X$. Note that this is not quite the definition given by Rudin (2.18i, p.32). However, the two definitions are equivalent by the following easy

**Proposition.** Let $E$ be a nonempty subset of a metric space $X$. The following are equivalent:

i) $E$, considered as a subspace of $X$, is bounded.

ii) There exists $p \in E$ and a real number $M$ such that $d(p, q) < M$ for all $q \in E$.

iii) There exists $p \in X$ and a real number $M$ such that $d(p, q) < M$ for all $q \in E$.

**Proof:** We show (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). (i)$\Rightarrow$(ii) is clear: let $M = B$, and choose for $p$ an arbitrary point of $E$. The implication (ii)$\Rightarrow$(iii) is even easier, for we may use the same $M, p$. Finally (iii)$\Rightarrow$(i) is a consequence of the triangle inequality, with $B = 2M$: for $q, q' \in E$ we have $d(q, q') \leq d(q, p) + d(p, q') < M + M = 2M = B$. □

Why did we require $E$ to be nonempty? Note that the empty metric space is bounded by our definition. It is also bounded by Rudin’s definition, except when regarded as a subset of the empty metric space — which is surely an oversight. We shall always regard $\emptyset$ as bounded regardless of where we found it, even if nowhere!

Further examples: a finite metric space is bounded; so is an interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$, considered as a subspace of $\mathbb{R}$. If $X$ is bounded then so is any subspace; if $X, Y$ are bounded, so is $X \times Y$. The metric space $\mathbb{R}$ is not bounded. If $X, Y$ are metric spaces, and $X$ is not bounded, then neither is $X \times Y$, unless $Y$ is empty. (Verify all these!)

Given a bounded metric space $X$ and any set $S$ we may construct a new kind of metric space, a *function space*. We shall call it $X^S$. As a set, this is simply the space of functions $f : S \rightarrow X$. (Do you see why we use the notation $X^S$ for this?) To make it a metric space we define the distance between two functions $f, g$ by

$$d_{X^S}(f, g) := \sup_{s \in S} d_X(f(s), g(s)).$$

[NB this doesn’t quite work when $S = \emptyset$; what is $X^\emptyset$ then, and what goes wrong with the definition of $d_{X^\emptyset}$? How should we fix it?] This makes sense because $d_X(f(s), g(s)) < B$ for all $s$, so the set $\{d_X(f(s), g(s)) : s \in S\}$ is bounded and thus has a least upper bound. (See Rudin, Chapter 1 to review this notion if necessary.) Verify that this is in fact a metric space. Actually this is not an entirely new example, since if $S$ is the finite set $\{1, 2, \ldots, n\}$ the space $X^S$ is isometric with $X^n$. The more general $X^S$ will be a starting point for many important constructions later in the course.