1. First we will check that \( d(x,y) \) is, indeed, a metric. Clearly the first axiom is satisfied from the definition of the metric. If \( x = y \) then \( d(x,y) = d(x,x) = d(y,y) = 0 \) which satisfies the second axiom. If \( x \neq y \) then \( d(x,y) = 1 = d(y,x) \) so it satisfies the second axiom. Also, \( S = d(x,y) + d(y,z) \) can be one of three values: 0, 1, 2. If \( S = 0 \) then \( x = y \) and \( y = z \) so \( x = z \) and the third axiom is satisfied. If \( S = 1 \) or 2 then the triangle inequality is satisfied since \( d(x,z) \) is at most 1. So \( d(x,y) \) is, in fact, a metric.

Now we will show that all subsets of \( X \) are both open and closed, and that the only dense set is all of \( X \).

Let \( E \subseteq X \), and \( x \in X \). Let \( N = N_{1/2}(x) \). There are no points other than \( x \) in \( N \), since all other points are at a distance 1 from it. Thus \( x \) cannot be a limit point of any subset of \( X \). However, if \( x \in E \) it is an interior point of \( E \), since \( N \) does not contain any points not in \( E \), so trivially lies completely inside \( E \). Thus any subset \( E \) of \( X \) is open, since every point in it is an interior point. Also, since there are no limit points in \( E \), \( E \) contains all of its limit points, and so \( E \) is also closed.

Now suppose that \( E \) is dense in \( X \), and suppose \( x \in X \), \( x \notin E \). Then, since \( N \) only contains \( x \), \( E \cap N = \emptyset \). Thus \( E \) is not dense in \( X \), a contradiction. So there must not exist an \( x \in X \), \( x \notin E \). So the only subset of \( X \) that is dense in \( X \) is all of \( X \).

2. First we will show that \( d_0 \) is a metric. Let \( x, y, z \in X \). Clearly, since \( d(x,y) \geq 0 \) and \( 1 + d(x,y) \geq 1 \) we know that

\[
\frac{d(x,y)}{1+d(x,y)} \geq 0.
\]

Also, since \( d(x,y) = 0 \) iff \( x = y \), \( d_0(x,y) = 0 \) iff \( x = y \). So \( d_0 \) satisfies the first axiom. Also,

\[
d_0(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d_0(x,y).
\]

Thus \( d_0 \) satisfies the second axiom.

Now notice that the function \( f(x) = x/(1+x) \) is everywhere increasing on the positive reals. Thus if \( x \geq y \) we know that \( f(x) \geq f(y) \). Thus

\[
\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = \frac{d(x,y) + d(y,z) + 2d(x,y)d(y,z)}{1+d(x,y) + d(y,z) + d(x,y)d(y,z)} \geq \frac{d(x,y) + d(y,z)}{1+d(x,y) + d(y,z)} \geq \frac{d(x,z)}{1+d(x,z)} = d_0(x,z).
\]

So \( d_0 \) satisfies the third axiom, and is thus a metric.

Let \( N_{r,d}(p) \) denote a neighborhood of \( p \) with radius \( r \) under the metric \( d \). Consider a subset \( E \) of \( X \) and a point \( p \in E \). Let \( M = N_{r,d}(p) \) and \( M' = N_{r/(1+r),d_0}(p) \). Then if \( x \in M \iff d(x,p) < r \iff \frac{d(x,p)}{1+d(x,p)} < \frac{r}{1+r} \iff d_0(x,p) < \frac{r}{1+r} \iff x \in M' \).

This shows that \( M \cap E = M \) if and only if \( M' \cap E = M' \), since every point is in \( M \) if and only if it is in \( M' \). Thus a point is interior under \( d \) iff it is interior under \( d_0 \). So \( E \) is open under \( d \) iff it is open under \( d_0 \).
We know that \[ \frac{x}{1+x} = 1 - \frac{1}{1+x}. \]

In this case, \( x \) is always nonnegative (since \( x = d(y,z) \), \( y,z \in X \)) so \( 1/(1+x) \) is always positive. This the above fraction will always be less than 1, regardless of how large \( x \) gets. Thus the metric space \((X,d_0)\) is bounded.

3. \( d_1(x,y) \) is not a metric, since it does not satisfy the triangle inequality. Letting \( x = 1, y = 2, z = 3 \) we get \((1-3)^2 \leq (1-2)^2 + (2-3)^2\) which is clearly false.

\( d_2(x,y) \) is a metric. It obviously satisfies axioms 1 and 2. To show axiom 3 we do the following:

\[
\begin{align*}
|x - z| & \leq |x - y| + |y - z| \\
|x - z| & \leq |x - y| + |y - z| + 2\sqrt{|x - y||y - z|} \\
\sqrt{|x - z|^2} & \leq (\sqrt{|x - y| + \sqrt{|y - z|}})^2 \\
\sqrt{|x - z|} & \leq \sqrt{|x - y| + |y - z|}.
\end{align*}
\]

Thus \( d_2(x,y) \) is a metric.

\( d_3(x,y) \) does not satisfy axiom 1, since if \( x = 1 \) and \( y = -1 \) then \( x \neq y \) but \( d_3(x,y) = |1 - 1| = 0 \).

\( d_4(x,y) \) is a metric. It trivially satisfies axioms 1 and 2. We wish to check the third axiom, \( d(x, z) \leq d(x, y) + d(y, z) \). Without loss of generality, \( x \leq z \).

Case 1: \( y < x \). Then \( |x^3 - z^3| \leq |x^3 - y^3| + |z^3 - y^3| \iff y^3 < x^3 \) which is true.

Case 2: \( x \leq y \leq z \). Then \( |x^3 - z^3| \leq |x^3 - y^3| + |z^3 - y^3| \iff 0 \leq 0 \) which is also true.

Case 3: \( y > z \). Then \( |x^3 - z^3| \leq |x^3 - y^3| + |z^3 - y^3| \iff 3 \leq y^3 \) which is true. So \( d_4(x,y) \) satisfies the third axiom and so is a metric.

\( d_5(x,y) \) is not a metric because it does not satisfy axiom 2, since \( d_5(1,2) = 3 \) but \( d_5(2,1) = 0 \). (Incidentally, this shows it does not satisfy axiom 1, also).

\( d_6(x,y) \) is a metric. We know that \( |x - y| \) is a metric, because it is the standard metric for the real numbers. From problem 2 we know that if \( d(x,y) \) is a metric then so is \( d(x,y)/(1 + d(x,y)) \). Thus \( |x - y|/(1 + |x - y|) \) is a metric.

4. (a) Clearly, \( f(x) = x \) is injective, because if \( f(x) = f(y) \) then trivially \( x = y \). Also, for all \( x \in X \) \( f(x) = x \) so the map is surjective. \( d(x,y) = d(f(x), f(y)) \) because \( f(x) = x \). Thus the identity map is always an isometry.

(b) Suppose that \( i^{-1}(x) = i^{-1}(y) \). Then, \( i(i^{-1}(x)) = i(i^{-1}(y)) \Rightarrow x = y \). Thus \( i^{-1} \) is injective.

Consider an \( x \in X \). \( i(x) \in Y \) and \( i^{-1}(i(x)) = x \) so \( i^{-1} \) is surjective. Thus it is a bijection.

Let the metric on \( X \) be \( d_X \) and the metric on \( Y \) be \( d_Y \). Then, for \( x, y \in Y \)

\[
d_X(i^{-1}(x), i^{-1}(y)) = d_Y(i(i^{-1}(x)), i(i^{-1}(y))) = d_Y(x, y).
\]

Thus \( i^{-1} \) is also an isometry.

(c) Suppose \( x, y \in X \) and \( j(i(x)) = j(i(y)) \). Then, since \( j \) is injective, \( i(x) = i(y) \). But since \( i \) is injective, that means that \( x = y \). So \( j \circ i \) is injective. Consider \( z \in Z \). Since \( j \) is surjective, we know there exists a \( z' \in Y \) such that \( j(z') = z \). Also, we know that there exist a \( z'' \in X \) such that \( i(z'') = z' \) because \( i \) is injective. Thus \( j(i(z'')) = j(z') = z \), so \( j \circ i \) is surjective. Thus it is a bijection.

Let the metrics on \( X, Y, Z \) be \( d_X, d_Y, d_Z \), respectively. Then, for \( x, y \in X \) we know that \( d_X(x, y) = d_Y(i(i(x)), y(y)) \) because \( i \) is an isometry. Also, we know that \( d_Y(i(x), i(y)) = d_Z(j(i(x)), j(i(y))) \), because \( j \) is an isometry. Thus we know that \( d_X(x, y) = d_Z(j(i(x)), j(i(y))) \) so \( j \circ i \) is an isometry.
5. First we will show that \( \sim \) is an equivalence relation. We know that \( d(p,p) = 0 \) by the first axiom of a metric. Thus \( p \sim p \). Also, since \( d(p,q) = d(q,p) \), if \( d(p,q) = 0 \) then \( q \sim p \). Also, if \( d(p,q) = 0 \) and \( d(q,r) = 0 \), by the triangle inequality we know that \( 0 \leq d(p,r) \leq d(p,q) + d(q,r) = 0 \) so \( d(p,r) = 0 \). Thus if \( p \sim q \) and \( q \sim r \) then \( p \sim r \). Thus \( \sim \) is an equivalence relation. To show that \( \sim \) is well defined we need to show that if \( p \sim p' \) and \( q \sim q' \) then \( d(p,q) = d(p',q') \), because that would mean that for two different representatives of \([p]\) and \([q]\) the distance between them is the same, since all elements \( p' \in [p] \) have \( p' \sim p \) and \( q' \in [q] \) has \( q' \sim q \). So we apply the triangle inequality: \( d(p,q) \leq d(p,p') + d(p',q') + d(q',q) = d(p',q') \). Also, \( d(p',q') \leq d(p,p') + d(p,q) + d(q',q) = d(p,q) \). Since \( d(p,q) \leq d(p',q') \) and \( d(p',q') \leq d(p,q) \) we get that \( d(p,q) = d(p',q') \). This shows that the function \( \overline{d} \) is well-defined. It trivially satisfies axiom 2, since \( d \) is symmetric. For \([p] \neq [q]\) we know that \( \overline{d}([p],[q]) = d(p,q) \). Thus, since \( d(p,q) > 0 \) (by definition of \([p]\) and \([q]\)) we know that \( \overline{d}([p],[q]) > 0 \). Also, if \([p] = [q]\) we know that \( \overline{d}([p],[q]) = d(p,q) = 0 \) by definition of \([p]\) and \([q]\). So \( \overline{d} \) satisfies axiom 1. Lastly, \( \overline{d}([p],[q]) + \overline{d}([q],[r]) = d(p,q) + d(q,r) \geq d(p,r) = \overline{d}([p],[r]) \). Thus \( \overline{d} \) satisfies axiom 3. Since it satisfies all of the axioms it is a metric.

In \( \mathbb{R}^3 \) with the example metric, two points are at distance zero if their \( z \)-coordinates are equal. Also, their distance apart otherwise is the \( d_{\infty} \) metric in \( \mathbb{R}^2 \). Thus \( \mathbb{R}^3 \) is equivalent to \( \mathbb{R}^2 \) with the \( d_{\infty} \) metric.

6. Let \( z \) be a limit point of \( E' \). Then we know that, for \( r > 0 \) there exists a \( y \in E' \) such that \( d(y,z) < r \). Let \( d(z,y) = h \). Let \( r' = \min(h,r-h) \). \( y \in E' \) implies that there exists \( x \in E \) such that \( d(y,x) < r' \). Then \( d(x,z) \leq d(x,y) + d(y,z) < r' + h \leq r - h + h = r \). Thus for any \( r > 0 \) there exists \( x \in E \) such that \( d(x,z) < r \), so \( z \in E' \). Thus \( E' \) is closed. If all limit points of \( E \) are limit points of either \( E \) or \( E' \) then we are done, since \( E' \) contains all of those limit points, so \( E' \) would simply have \( E' \) as its limit points, which would mean that it has the same limit points as \( E \). So suppose \( E \) has a limit point \( z \notin E' \). Consider a sequence of points \( \{a_n\} \) such that \( d(z,a_n) < 1/n \). An infinite subsequence of these must be in either \( E \) or \( E' \), since \( E = E \cup E' \). But then \( z \) is a limit point of that set. However, we know that \( E' \) is the set of limit points of \( E \) and \( E' \) is closed. Thus \( z \) is in \( E \), contradicting our conjecture. Thus \( E \) and \( E' \) have the same limit points.

However, \( E \) and \( E' \) do not necessarily have the same limit points. The set \( A_{0.1} \) (look in 7 for definition of \( A_{x,y} \)) has only one limit point, 0. Thus \( A'_{0.1} = 0 \), which has no limit points because it is finite.

7. First I will construct a bounded subset \( A_{x,y} \) of \( \mathbb{R} \) that has only one limit point. \( A_{x,y} \) will be a sequence of points \( \{a_n\} \) such that

\[
a_n = x + \frac{1}{2^n}|x-y|.
\]

Then clearly no point outside of \([x,y]\) is a limit point. Also no point in \((x,y)\) is a limit point, since the sequence is monotonically decreasing with limit \( x \) as \( n \to \infty \). Thus every number is between two
members of the sequence (which means it is not a limit point) or is a member of the sequence (and is thus also not a limit point. However, $x$ is a limit point, since for $r > 0$, if $n = \lceil \log_2(|x - y|/r) \rceil$, $d(x, a_n) < r$.

(a) Consider the set $A = A_{0,1} \cup A_{2,3} \cup A_{4,5} \cup \{0, 2, 4\}$. From the demonstration above it is clear that this set only has three limit points. Also, the set is clearly closed.

(b) Consider the set

$$A = \bigcup_{n=0}^{\infty} A_{2^{-n}, 2^{-n-1}} \cup \bigcup_{n=0}^{\infty} \{2^{-n}\}$$

This set will be contained in the closed interval $[0, 1]$, so is bounded. Clearly, no point outside this interval is a limit point. Also, each of the intervals $[2^{-n}, 2^{-n-1})$ clearly has only one limit point, by the definition of $A_{x,y}$. Also, 0 is a limit point, since the sequence $2^{-n}$ converges to 0. Thus this set has an infinite, though countable, set of limit points. Clearly this set is also bounded.

8. (a) Consider any limit point $z$ of $B$, and consider an infinite sequence of points $a_n$ such that $d(z, a_n) < 1/n$. An infinite number of these must be in one of the sets $A_i$. Thus $z$ will be a limit point of that $A_i$. Thus, since any limit point of $B$ is in some $A_i$, we know that $\overline{B} \subseteq \bigcup_{k=1}^{m} \overline{A_k}$. Now consider any limit point $z$ of some $A_i$. We know that the sequence of points $\{a_n\}$ such that $d(z, a_n) < 1/n$ is in $A_i$, and is thus also in $B$. Thus $z$ is a limit point of $B$. So $\bigcup_{k=1}^{m} \overline{A_k} \subseteq \overline{B}$. Thus $\overline{B} = \bigcup_{k=1}^{m} \overline{A_k}$.

(b) We know that $B$ contains all of the points in each $A_i$. Consider a limit point $z$ of some $A_i$. Since $A_i \subseteq B$ we know that $z$ must be a limit point of $B$. Thus all of the points in each $A_i$ and all of the limit points of every $A_i$ are in $\overline{B}$, so $\bigcup_{n=0}^{\infty} \overline{A_i} \subseteq \overline{B}$.

Consider the sets $A_i = [2^{-i}, 2^{-i-1}]$. The point 0 is not contained in the closure of any of the $A_i$, since each is already a closed set that does not contain 0. However, the closure of their union contains the point 0.