Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #2 (27 September 2002):
Metrics, topology, continuity, and sequences

Sketch of a proof n. I couldn’t verify all the details, so I’ll break it
down into the parts I couldn’t prove.¹

Please avoid merely “sketching” (as defined in the above quote) a proof. In all
problem sets, you may use the result in one problem (or problem part) to solve
another, even if you have not proved the first one, unless this becomes circular
[exception: when problem B is clearly a generalization of A, don’t use B to
solve A unless you’ve solved B!]. NB the problems are generally
not in order of
difficulty. Problem set is due Friday, Oct. 4, at the beginning of class.
Two different notions of distance between subsets of a metric space:

1. [Distance between subsets of a metric space] For any two subsets
   A, B of a
   metric space X, define the distance d(A, B) between A and B by
   \[d(A, B) := \inf \{d(x, y) : x \in A, y \in B\}\].
   
   Prove that for any subsets A, B, C of X and any element x ∈ X we have:
   i) d(\overline{A}, \overline{B}) = d(A, B) (where \overline{A}, \overline{B} are the closures of A, B respectively);
   ii) d(\{x\}, A) = 0 if and only if x ∈ A;
   iii) d(A, B ∪ C) = \min\{d(A, B), d(A, C)\};
   iv) d(A, \{x\}) + d(\{x\}, B) ≥ d(A, B).
   
   Must the triangle inequality d(A, C) + d(C, B) ≥ d(A, B) also hold?

2. [Minkowski distance between nonempty bounded closed subsets] For a sub-
   set A of a metric space X, and a positive real number r, define
   \[N_r(A) := \bigcup_{x \in A} N_r(x)\].
   
   (Recall that N_r(x) is the radius-r neighborhood of x, a.k.a. the open ball of
   radius r about x; one may visualize N_r(A) as the radius-r neighborhood
   of A. For instance, N_r(∅) = ∅; N_r(\{x\}) = N_r(x); N_r(X) = X; and
   r’ ≥ r ⇒ N_r’(A) ⊇ N_r(A).) For two nonempty, bounded, closed subsets
   A, B of a metric space X, define the Minkowski distance δ(A, B) between A
   and B by
   \[δ(A, B) := \inf\{r : N_r(A) ⊇ B \text{ and } N_r(B) ⊇ A\}\].
   
   Prove that this defines a metric on the space of nonempty, bounded, closed
   subsets of X.

More about the topology of R, and relation with continuity:

3. Prove that the only subsets of R that are simultaneously open and closed
   are ∅ and R.

¹Definitions of Terms Commonly Used in Higher Math, R. Glover et al.
4. Suppose \( X, Y \) are metric spaces, and that \( X \) has the discrete metric. Find all continuous maps from \( X \) to \( Y \). Find all continuous maps from \( \mathbb{R} \) to \( X \).

Some more topological notions:

5. A topological space is said to be Hausdorff if, for any two distinct elements \( p, q \) of the space, there are disjoint open sets \( U, V \) with \( U \ni p \) and \( V \ni q \). For instance, a metric space is automatically Hausdorff, since we may take \( U \) and \( V \) to be the open balls of radius \( \frac{1}{2}d(p, q) \) about \( p \) and \( q \).
   i) Prove that in a Hausdorff space every single-point set is closed.
   ii) Now let \( X, Y \) be topological spaces with \( Y \) Hausdorff, and let \( f, g \) be any continuous functions from \( X \) to \( Y \). If \( S \subset X \) is a dense subset such that \( f(s) = g(s) \) for all \( s \in S \), prove that \( f = g \), i.e., that \( f(x) = g(x) \) for all \( x \in X \). [Naturally you must use the topological definition of denseness: “\( S \) is dense in \( X \)” means that the only open set in \( X \) disjoint from \( S \) is \( \emptyset \).]

6. [Non-metrizable topologies] Recall that a topology on a set \( X \) is a family \( T \) of subsets of \( X \) which contains \( \emptyset, X \), and the finite intersection and arbitrary union of any sets in \( T \). We noted that the open sets in a metric space constitute a topology, but not all topologies arise in this way; for instance, for any set \( X \) with more than 1 element, \( \{\emptyset, X\} \) is a non-metric topology, because in a metric topology all one-point sets are closed. Suppose now that \( T \) is a non-metric topology on \( X \) containing all complements of one-point sets (so that all one-point sets are closed). Show that \( X \) is infinite, and construct such a topology on a countably infinite set.

7. [Homeomorphism] A homeomorphism between two topological spaces\(^2\) \( X, Y \) is a bijection \( f : X \to Y \) such that both \( f \) and the inverse function \( f^{-1} : Y \to X \) are continuous. Show that a bijection \( f : X \to Y \) is a homeomorphism if and only if \( f \) identifies the topologies of \( X \) and \( Y \), i.e., the open sets of \( Y \) are precisely the images of open sets of \( X \). Two topological spaces \( X, Y \) are said to be homeomorphic if there is a homeomorphism between them. Prove that this is an equivalence relation. Show that any isometry is a homeomorphism. Prove that every open ball in \( \mathbb{R} \) is homeomorphic with \( \mathbb{R} \) but not isometric with \( \mathbb{R} \). (Warning: for this last part it is not enough to exhibit a non-isometric homeomorphism; you must show that no bijection between the ball and \( \mathbb{R} \) is an isometry.)

Convergence and sequences:

8. [Rudin, p.78, Exercise 1] Suppose \( s_n \in \mathbb{R} \). Prove that convergence of \( \{s_n\} \) implies convergence of \( \{|s_n|\} \). Is the converse true?

9. [Another characterization of convergence] Let \( E \) be the subset \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \) of \( \mathbb{R} \). A sequence \( \{s_n\} \) in an arbitrary metric space \( X \) is equivalent to the map \( \tilde{s} : E \to X \) that takes \( 1/n \) to \( s_n \). Show that \( \tilde{E} = E \cup \{0\} \), and prove that \( \{s_n\} \) converges if and only if \( \tilde{s} \) extends to a continuous function on \( \tilde{E} \).

\(^2\)Naturally a “topological space” is a set \( X \) endowed with a topology \( T \) of subsets of \( X \).