Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #7 (1 November 2002):
Linear Algebra III

The expression $\delta_{ij}$ [see below] is called the Kronecker delta (after the mathematician Leopold Kronecker [1823–1891], who made more substantial contributions to mathematics than this).
— Corwin and Szezbar, Calculus in Vector Spaces, p.124

Some basics about linear transformations and their matrices:

1.-2. Solve Exercises 6, 22, 23, 24 from Chapter 3 of the textbook (pages 59 and 61). For #6, if $S_1 \cdots S_n$ is injective, what if anything can be said of $S_1, S_2, \ldots, S_n$? For the other three exercises, note that \( \mathcal{L}(V) \) is Axler’s abbreviation for \( \mathcal{L}(V, V) \) (it is also known as \( \text{End}(V) = \text{Hom}(V, V) \)).

3. Let \( \mathcal{P}_n \) be the (R- or C-)vector space of polynomials of degree at most \( n \), and \( L: \mathcal{P}_n \to \mathcal{P}_n \) be the linear transformation taking any polynomial \( P(x) \) to the polynomial
\[
(L(P))(x) = (x - 3)P''(x)
\]
(here \( P'' \) is the second derivative \( d^2P/dx^2 \)). Exhibit a matrix for \( L \) relative to a suitable basis for \( \mathcal{P}_n \), and determine the kernel, image, and rank of \( L \).

4. Let \( V, W \) be arbitrary vector spaces over the same field. Show that, for any vector \( v \) in \( V \), the evaluation map \( E_v : \mathcal{L}(V, W) \to W \) defined by \( E_v(L) = L(v) \) for all \( L \in \mathcal{L}(V, W) \) is a linear transformation. If \( V, W \) are finite dimensional, what is the dimension of \( \ker E_v \)?

5. Let \( V, W \) be vector spaces over the rational field \( \mathbb{Q} \). Prove that a map \( T : V \to W \) is linear if and only if \( T(v + v') = T(v) + T(v') \) for all \( v, v' \in V \). (Cf. the marginal note to Exercise 2 on p.59 of the textbook.)

More about duality:

6. If \( v_1, \ldots, v_n \) is a basis for \( V \), prove that there is for each \( j = 1, \ldots, n \) a unique \( v^*_j \in V^* \) such that \( v^*_j(v_i) = 1 \) if \( i = j \) and 0 otherwise. [In other words, \( v^*_j(v_i) = \delta_{ij} \), the “Kronecker delta” referred to above, which is also the \((i, j)\) entry of the identity matrix.] Show further that the \( v^*_j \) constitute a basis for \( V^* \). This is called the dual basis to \( (v_1, \ldots, v_n) \).

7. We saw that, for any vector spaces \( V, W \), the dual of \( V \oplus W \) is naturally identified with \( V^* \oplus W^* \). What is the dual of \( \bigoplus_{i \in I} V_i \)? Use this to construct a vector space \( V \) over some field \( F \) such that \( V \) is not isomorphic with \( V^* \).

8. Let \( x_0, \ldots, x_m \) be distinct elements of \( F \). Recall that the \( m+1 \) vectors \( v_i := (x_0, x_1, \ldots, x_m) \) (\( 0 \leq i \leq m \)) constitute a basis of \( F^{m+1} \). Describe the dual basis.

Problem set is due Friday, Nov. 8 in class.