Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #2 (27 September 2002):
Metrics, topology, continuity, and sequences

Sketch of a proof. I couldn’t verify all the details, so I’ll break it down into the parts I couldn’t prove.\footnote{Definitions of Terms Commonly Used in Higher Math, R. Glover et al.}

Please avoid merely “sketching” (as defined in the above quote) a proof. In all problem sets, you may use the result in one problem (or problem part) to solve another, even if you have not proved the first one, unless this becomes circular [ EXCEPTION: when problem B is clearly a generalization of A, don’t use B to solve A unless you’ve solved B!]. NB the problems are generally not in order of difficulty. Problem set is due Friday, Oct. 4, at the beginning of class.

Two different notions of distance between subsets of a metric space:

1. [Distance between subsets of a metric space] For any two subsets $A, B$ of a metric space $X$, define the distance $d(A, B)$ between $A$ and $B$ by

   \[ d(A, B) := \inf \{d(x, y) : x \in A, y \in B\}. \]

   Prove that for any subsets $A, B, C$ of $X$ and any element $x \in X$ we have:

   i) $d(\overline{A}, \overline{B}) = d(A, B)$ (where $\overline{A}, \overline{B}$ are the closures of $A, B$ respectively);
   ii) $d(\{x\}, A) = 0$ if and only if $x \in \overline{A}$;
   iii) $d(A, B \cup C) = \min\{d(A, B), d(A, C)\}$;
   iv) $d(A, \{x\}) + d(\{x\}, B) \geq d(A, B)$.

   Must the triangle inequality $d(A, C) + d(C, B) \geq d(A, B)$ also hold?

2. [Minkowski distance between nonempty bounded closed subsets] For a subset $A$ of a metric space $X$, and a positive real number $r$, define

   \[ N_r(A) := \bigcup_{x \in A} N_r(x). \]

   (Recall that $N_r(x)$ is the radius-$r$ neighborhood of $x$, a.k.a. the open ball of radius $r$ about $x$; one may visualize $N_r(A)$ as the radius-$r$ neighborhood of $A$. For instance, $N_r(\emptyset) = \emptyset$; $N_r(\{x\}) = N_r(x)$; $N_r(X) = X$; and $r' \geq r \Rightarrow N_{r'}(A) \supseteq N_r(A)$.) For two nonempty, bounded, closed subsets $A, B$ of a metric space $X$, define the Minkowski distance $\delta(A, B)$ between $A$ and $B$ by

   \[ \delta(A, B) := \inf \{r : N_r(A) \supseteq B \text{ and } N_r(B) \supseteq A\}. \]

   Prove that this defines a metric on the space of nonempty, bounded, closed subsets of $X$.

More about the topology of $\mathbb{R}$, and relation with continuity:

3. Prove that the only subsets of $\mathbb{R}$ that are simultaneously open and closed are $\emptyset$ and $\mathbb{R}$. 

\footnote{Definitions of Terms Commonly Used in Higher Math, R. Glover et al.}
4. Suppose $X,Y$ are metric spaces, and that $X$ has the discrete metric. Find all continuous maps from $X$ to $Y$. Find all continuous maps from $\mathbb{R}$ to $X$.

Some more topological notions:

5. A topological space is said to be Hausdorff if, for any two distinct elements $p, q$ of the space, there are disjoint open sets $U, V$ with $U \ni p$ and $V \ni q$. For instance, a metric space is automatically Hausdorff, since we may take $U$ and $V$ to be the open balls of radius $\frac{1}{2}d(p, q)$ about $p$ and $q$.

- i) Prove that in a Hausdorff space every single-point set is closed.
- ii) Now let $X,Y$ be topological spaces with $Y$ Hausdorff, and let $f, g$ be any continuous functions from $X$ to $Y$. If $S \subseteq X$ is a dense subset such that $f(x) = g(x)$ for all $x \in S$, prove that $f = g$, i.e., that $f$ and $g$ induce the same topology on $X$.

6. [Non-metrizable topologies] Recall that a topology on a set $X$ is a family $\mathcal{T}$ of subsets of $X$ which contains $\emptyset, X$, and the finite intersection and arbitrary union of any sets in $\mathcal{T}$. We noted that the open sets in a metric space constitute a topology, but not all topologies arise in this way; for instance, for any set $X$ with more than one element, $\{\emptyset, X\}$ is a non-metric topology, because in a metric topology all one-point sets are closed. Suppose now that $\mathcal{T}$ is a non-metric topology on $X$ containing all complements of one-point sets (so that all one-point sets are closed). Show that $X$ is infinite, and construct such a topology on a countably infinite set.

7. [Homeomorphism] A homeomorphism between two topological spaces\(^2\) $X,Y$ is a bijection $f : X \to Y$ such that both $f$ and the inverse function $f^{-1} : Y \to X$ are continuous. Show that a bijection $f : X \to Y$ is a homeomorphism if and only if $f$ identifies the topologies of $X$ and $Y$, i.e., the open sets of $Y$ are precisely the images of open sets of $X$. Two topological spaces $X,Y$ are said to be homeomorphic if there is a homeomorphism between them. Show that this is an equivalence relation. Show that any isometry is a homeomorphism. Prove that every open ball in $\mathbb{R}$ is homeomorphic with $\mathbb{R}$ but not isometric with $\mathbb{R}$. (Warning: for this last part it is not enough to exhibit a non-isometric homeomorphism; you must show that no bijection between the ball and $\mathbb{R}$ is an isometry.)

Convergence and sequences:

8. [Rudin, p.78, Exercise 1] Suppose $s_n \in \mathbb{R}$. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

9. [Another characterization of convergence] Let $E$ be the subset $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ of $\mathbb{R}$. A sequence $\{s_n\}$ in an arbitrary metric space $X$ is equivalent to the map $\bar{s} : E \to X$ that takes $1/n$ to $s_n$. Show that $\bar{E} = E \cup \{0\}$, and prove that $\{s_n\}$ converges if and only if $\bar{s}$ extends to a continuous function on $\bar{E}$.

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\(^2\) Naturally a “topological space” is a set $X$ endowed with a topology $\mathcal{T}$ of subsets of $X$. 

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