PROBLEM SET #8 SOLUTIONS
PART A
December 6, 2002

(1) Consider the function \( f_y(x) = f(x) - y \), where \( y \in \text{Image} f \) is fixed. As in the proof of the Inverse Function Theorem given in the text, we would like to see if we can solve \( f_y(x) = 0 \) in some neighborhood of \( a_0 \), i.e. if Newton’s method with initial guess \( a_0 \) converges, for any \( y \in \mathbb{R} \). Note that \( Df(x) = Df_y(x) \), so that by the first condition of the theorem \( Df_y(x)^{-1} \) exists and \( |Df_y(X)^{-1}| \leq K \) for every \( x \in B_R(a_0) \). Furthermore if \( Df(x) \) is Lipschitz with constant \( M \) on \( B_R(a_0) \), then so is \( Df_y(x) \), and \( |f_y(x) - f_y(a_0)| = |f(x) - f(a_0)| \leq \frac{1}{4MK^2} \) for all \( x \in B_R(a_0) \). Choosing \( y \) such that \( |f_y(a_0)| = |f(a_0) - y| \leq \frac{1}{4MK^2} \) (i.e. \( y \in B_{1/2MK^2}(f(a_0)) \) – this can be done since \( f \) is continuous), we have \( |f_y(a_0)|K^2M \leq \frac{1}{2} \), and Kantorovich’s Theorem applies. Then, as before, the solution \( g(y) \) to \( f_y(x) = 0 \) is a local inverse to \( f \) at \( a_0 \), and \( f \) is locally invertible at \( a_0 \), as desired.

(2) (Problem 2.7.12)

(a) We find \( f_y(x) \), where \( f_y(x) = \frac{y - x^2 + 8 + \cos x}{x - y^2 + 9 + 2 \cos y} \). One iteration of Newton’s method starting at \( a_0 = \left( \frac{\pi}{\pi} \right) \) gives

\[
\begin{align*}
    a_1 &= a_0 - Df(a_0)^{-1}f(a_0) \\
    &= \left( \frac{\pi}{\pi} \right) - \left[ \begin{array}{cc} -2\pi & 1 \\
        1 & -2\pi \end{array} \right]^{-1} \left( \begin{array}{c} \pi - \pi^2 + 7 \\
        \pi - \pi^2 + 7 \end{array} \right) \\
    &= \left( \frac{\pi}{\pi} \right) - \pi^2 - \pi - 7 \cdot \frac{1}{2\pi - 1} \left( \begin{array}{c} 1 \\
        1 \end{array} \right) \approx \left( \begin{array}{c} 3.193 \\
        3.193 \end{array} \right).
\end{align*}
\]

(b) We would like to apply Kantorovich’s Theorem here. Let \( h_0 = -|Df(a_0)|^{-1}f(a_0) \approx 0.0514 \left( \frac{1}{\pi} \right) \); then \( |h_0| \approx 0.0728 \). Since \( U = \mathbb{R}^2 \) in this case, we have trivially that \( U_0 = B_{|h_0|}(a_1) \subset U \). Furthermore, \( \frac{\pi}{2} < \pi - 0.0729 < \pi + 0.0729 < \frac{3\pi}{2} \), so that \( \cos x \) and \( \cos y \) are bounded above by 0 in \( U_0 \). Now, \( f \) is clearly \( C^2 \), so we may apply Proposition 2.7.10, we have that \( Df \) is Lipschitz in \( U_0 \) with constant

\[
M = \sqrt{\sum_{1 \leq i, j, k \leq 2} (c_{ij,k})^2} = \max_{x,y \in U_0} \sqrt{(-2 - \cos y)^2 + (-2 - \cos x)^2} \leq \sqrt{2 \cdot 2} = 2\sqrt{2}.
\]

So we have

\[
|f \left( \frac{\pi}{\pi} \right) | \cdot |Df \left( \frac{\pi}{\pi} \right)|^{-1}^2 \cdot M \approx 0.272 \cdot 2 \cdot 2\sqrt{2} \approx 0.0596 < \frac{1}{2}.
\]

Thus the conditions for Kantorovich’s Theorem are satisfied, and so Newton’s Method with initial guess \( a_0 \) converges within \( U_0 \).
(c) We know by Kantorovich’s Theorem that Newton’s Method converges in $U_0$. Note that the radius of $U_0$ is at $|h_0| = |a_0 - a_1| \approx 0.0728 < 0.0729 = r$, so that $B_{2r}(a_0) \supset B_r(a_1) = U_0$ contains a root of the equations, and $R = 2r = 0.1458 < 1$, as desired.

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