1. Consider two rows \( R \) and \( S \). Using only the given two row operations, we can exchange the rows as follows: 
\[
\begin{pmatrix} R \\ S \end{pmatrix} \rightarrow \begin{pmatrix} R \\ S - R \end{pmatrix} \rightarrow \begin{pmatrix} S \\ R - S \end{pmatrix} \rightarrow \begin{pmatrix} S \\ R \end{pmatrix}.
\]

2. Use row reduction. We begin with the matrix
\[
\begin{bmatrix}
1 & -1 & -1 & -3 & 1 & 1 \\
1 & 1 & -5 & -1 & 7 & 2 \\
-1 & 2 & 2 & 2 & 1 & 0 \\
-2 & 5 & -4 & 9 & 7 & \beta
\end{bmatrix}
\]
and row reducing gives
\[
\begin{bmatrix}
1 & 0 & 0 & -4 & 3 & 2 \\
0 & 1 & 0 & -1/3 & 7/3 & 5/6 \\
0 & 0 & 1 & -2/3 & -1/3 & 1/6 \\
0 & 0 & 0 & 0 & 0 & 1 + 2\beta
\end{bmatrix}
\]
(You were expected to show the work of row reducing, but it is omitted here because I wrote on your papers where you made mistakes if you didn’t get it right.) For this system to have solutions, we must have \( 1 + 2\beta = 0 \), so \( \beta = -1/2 \). Then, we can solve for the \( x \) in terms of two independently chosen variables as follows:
\[
\begin{align*}
x_1 &= 2 + 4\alpha - 3\beta \\
x_2 &= 5/6 + 1/3\alpha - 7/3\beta \\
x_3 &= 1/6 + 2/3\alpha + 1/3\beta \\
x_4 &= \alpha \\
x_5 &= \beta
\end{align*}
\]

3. Use row reduction. We begin with the matrix
\[
\begin{bmatrix}
1 & 1 & a & 1 \\
1 & a & 1 & 1 \\
a & 1 & 1 & a
\end{bmatrix}
\]
and row reducing (watch out for division by 0!) gives
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
when \(a \neq -2, 1\) (in these cases, there would be division by zero when row reducing). (Again, you were expected to show the work, but I corrected the work you submitted, so the row reduction is omitted here.) In this case, we have \(x = 1, y = 0, z = 0\) as the unique solution.

When \(a = 1\), all three equations become \(x + y + z = 1\). So, we can pick \(y\) and \(z\) arbitrarily and let \(x = 1 - y - z\). Thus, there are infinitely many solutions.

When \(a = -2\), we substitute for \(a\) and then row reduce to get
\[
\begin{bmatrix}
1 & 0 & -1 & 1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
So, we can pick \(z\) arbitrarily and set \(y = z\) and \(x = 1 + z\). Thus, there are infinitely many solutions.

4. Again, since I pointed out where mistakes were made in the row reducing, those steps are omitted here and just the answers are given. If \(A\) is the given matrix, row reducing \((A|I)\) gives the following:

a. \[
\begin{bmatrix}
1 & 0 & 1/6 & 5/54 \\
0 & 1 & -1/6 & 1/54
\end{bmatrix},
\]
and the inverse exists and is
\[
\begin{bmatrix}
1/6 & 5/54 \\
-1/6 & 1/54
\end{bmatrix}
\]
b. \[
\begin{bmatrix}
1 & 3 & 0 & 1/3 \\
0 & 0 & 1 & -1/3
\end{bmatrix},
\]
The left half does not reduce to the identity, so the matrix has no inverse.

c. \[
\begin{bmatrix}
1 & 0 & 0 & 3/2 & -1/4 & -9/4 \\
0 & 1 & 0 & -1 & 1/2 & 3/2 \\
0 & 0 & 1 & 1/2 & -1/4 & -1/4
\end{bmatrix},
\]
and the inverse exists and is the right half of this matrix.

d. The inverse does not exist because this matrix is not square.

\[
\begin{bmatrix}
1 & 0 & 0 & 1/7 & -1/2 & 1/14 \\
0 & 1 & 0 & 4/21 & 5/6 & -1/14 \\
0 & 0 & 1 & -4/21 & 1/6 & 1/14
\end{bmatrix},
\]
and the inverse exists and is the right half of this matrix.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 1 & -1 & 1
\end{bmatrix},
\]
and the inverse exists and is the right half of this matrix.
g. \[
\begin{bmatrix}
1 & 0 & 0 & 4 & -6 & 1 & -1 \\
0 & 1 & 0 & -6 & 14 & -11 & 3 \\
0 & 0 & 1 & 0 & -11 & 10 & -3 \\
0 & 0 & 0 & 1 & -1 & 3 & -3 & 1
\end{bmatrix},
\]
and the inverse exists and is the right half of this matrix.

5a. Row reducing gives
\[
A' = \begin{bmatrix}
1 & 0 & 0 & 3a - b - 4c \\
0 & 1 & 0 & a - b - c \\
0 & 0 & 1 & -2a + b + 3c
\end{bmatrix}
\]

5b. The inverse is
\[
B^{-1} = \begin{bmatrix}
3 & -1 & 4 \\
1 & -1 & -1 \\
-2 & 1 & 3
\end{bmatrix}
\]

5c. The matrix \(B^{-1}\) consists of the coefficients of \(a, b,\) and \(c\) in the row-reduced matrix \(A'\). Let \(A = (B|x)\), where \(x\) is the vector \((a \ b \ c)\). Then, row-reducing \(A\) is equivalent to applying a matrix \(E\), the product of the matrices representing the row operations. Looking at the left three columns of \(A' = (EB|Ex)\), we know \(EB = I\), so \(E = B^{-1}\). Then, the rightmost column is \(Ex = B^{-1}x\), so \(B^{-1}\) must be the coefficients of \(a, b,\) and \(c\), as expected.