MATH 25A – ADDITIONAL HOMEWORK

1. PART A

Recall that a topological space is a pair $(S, \tau)$, where $S$ is a set and $\tau$ a collection of subsets of $S$, called open sets, satisfying:

- $S, \emptyset \in \tau$.
- $\bigcup_{i \in I} U_i \in \tau$ if all $U_i \in \tau$ (here $I$ is a possibly infinite set of indices.)
- $\bigcap_{i=1}^n U_i \in \tau$ if all $U_i \in \tau$.

A subset $C \subset S$ is called closed if $S - C$ is open. A subset $C \subset S$ is compact if every open cover of $C$ has a finite subcover. Finally, a function $f : S \to T$ between topological spaces is continuous if $f^{-1}(U)$ is open in $S$ for any open $U \subset T$.

1. Let $(S,d)$ be a metric space. The metric topology on $S$ is defined by: $U \subset S$ is open if and only if for any $a \in U$ there exists $\varepsilon > 0$ such that the open $\varepsilon$-ball $B_\varepsilon(a)$ lies in $U$.
   (a) Prove that this defines a topology.
   (b) Prove that two metrics $d_1, d_2$ on $S$ are equivalent if and only if they define the same topology.

2. Let $(S,\tau)$ be a topological space, and $T \subset S$. Define the subset topology on $T$ by: $U \subset T$ is open if and only if $U = T \cap V$ for some open $V \subset S$. Prove that this defines a topology.

3. Define a topology on $\mathbb{R}$ by: $U \subset \mathbb{R}$ is open if and only if either $U = \emptyset$ or $\mathbb{R} - U$ is finite. Prove that this defines a topology and that $\mathbb{R}$ is compact in this topology.

4. Prove that a closed subset $D$ of a compact set $C$ in a topological space $S$ is compact.
   (Hint: given an open cover of $D$, add some open sets to get a cover of $C$.)

5. Given any topological space $S$, we construct a compact space by adding one point to $S$. This compact space is called the one point compactification of $S$. The construction is modeled after the following simple case. Let $C$ be the unit circle in $\mathbb{R}^2$ centered at $(0,1)$. Then lines through $P = (0,2)$ give a one-to-one correspondence between points on the circle, different from $P$, and points on the $x$-axis. This is a continuous map with a continuous inverse, hence we may consider the circle as $\mathbb{R}$ plus one extra point $P$. The circle $C$ is the one point compactification of $\mathbb{R}$.

   Let $S$ be a topological space and $T = S \cup \{\infty\}$, where $\infty$ is simply a point not in $S$. Assume that all compact sets in $S$ are closed. Define the topology on $T$ by: $U \subset T$ is open if either $U$ is an open set in $S$, or $\{\infty\} \subset U$ and $S - U$ is compact.
   (a) Prove that this defines a topology.
   (b) Prove that $T$ is compact in this topology.

2. PART B

1. Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}$ a continuous map, such that $f(x) > 0$ for all $x \in C$. Prove that there exists a constant $K > 0$ such that $f(x) \geq K$ for all $x \in C$.

2. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, such that $\text{Ker}(L) = 0$. Show that there exists a constant $K > 0$ such that
   $$|L(v)| \geq K|v|$$
   for all $v \in \mathbb{R}^n$. (Hint: first find a $K$ that works for all $|v| = 1$.)
(3) Let $C \subset \mathbb{R}^n$ be a compact set, and $f : C \to \mathbb{R}^m$ a continuous injective map. Because $f$ is injective, one can define the inverse map $f^{-1} : f(C) \to C$. Prove that $f^{-1}$ is continuous. (Hint: The problem becomes easy once you choose the right definition of continuity.)

(4) Let $S \subset \mathbb{R}^n$. We define the boundary of $S$, $bd(S)$ as the set of $x \in \mathbb{R}^n$ such that for any $\varepsilon > 0$ the open ball $B_\varepsilon(x)$ contains points from $S$ and from $\mathbb{R}^n - S$. We define the closure of $S$, $cl(S)$ to be the intersection of all closed sets in $\mathbb{R}^n$ containing $S$. The interior of $S$, $int(S)$ is defined to be the union of all open sets contained in $S$.

(a) Prove that $S \subset \mathbb{R}^n$ is closed if and only if $bd(S) \subset S$.

(b) Prove that $S \cup bd(S)$ is closed.

(c) Prove that $cl(S)$ is a closed set and $cl(S) = S \cup bd(S)$.

(d) Prove that $int(S)$ is an open set and $int(S) = S - bd(S)$.

(e) Construct the Cantor set as follows. Start with the interval $[0,1]$. Remove the middle third $(1/3,2/3)$, then remove the middle thirds of each of the two remaining intervals, and so on. Is the Cantor set closed? What is its boundary?