5. Let \( V = \{(a_0, a_1, a_2, \ldots) \mid a_i \in \mathbb{R}\} \) be the vector space of all infinite sequences of real numbers. Let \( W \) be the subspace of \( V \) consisting of all arithmetic sequences. Find a basis for \( W \), and determine the dimension of \( W \). (A sequence is arithmetic if there is some constant \( c \) such that \( a_{n+1} = a_n + c \) for all \( n \geq 0 \).)

Arithmetic sequences have the form \((a_0, a_0 + c, a_0 + 2c, \ldots)\) where \( a_0 \in \mathbb{R} \) and \( c \in \mathbb{R} \). There are of course infinitely many ways of choosing a basis, but perhaps the most straightforward one is \((1, 1, 1, 1, \ldots), (0, 1, 2, 3, \ldots)\).

Both \((1, 1, 1, 1, \ldots)\) and \((0, 1, 2, 3, \ldots)\) are arithmetic sequences and so are in \( W \). Any sequence \((a_0, a_0 + c, a_0 + 2c, \ldots)\) in \( W \) can be expressed as a linear combination of these two vectors:

\[
(a_0, a_0 + c, a_0 + 2c, \ldots) = a_0(1, 1, 1, 1, \ldots) + c(0, 1, 2, 3, \ldots)
\]

As linear combinations of these two vectors form all arithmetic sequences in \( V \) and nothing but arithmetic sequences in \( V \), they span \( W \). Furthermore, since they are not scalar multiples of each other, \((1, 1, 1, 1, \ldots), (0, 1, 2, 3, \ldots)\) is linearly independent and therefore forms a basis for \( W \).

Since we have a basis for \( W \) consisting of two elements, \( \text{dim}(W) = 2 \). This is consistent with the fact that we need only two pieces of information to specify any arithmetic sequence in \( V \), i.e. the start value \( a_0 \) and the step value \( c \). The basis we chose is convenient since it makes this fact explicit, as we multiply the first basis element by \( a_0 \) and the second by \( c \) to obtain any arithmetic sequence of start value \( a_0 \) and step value \( c \). In fact, however, any two arithmetic sequences in \( W \) form a basis for that subspace as long as they are linearly independent; the proof is left as an exercise.

8. Let \( U \) and \( W \) be subspaces of a vector space \( V \). We define two new subspaces as follows:

- \( U + W = \{u + w \mid u \in U, w \in W\} \)
- \( U \cap W = \{v \in V \mid v \in U \text{ and } v \in W\} \)

(a) (*) Convince yourself that both \( U + W \) and \( U \cap W \) are, in fact, subspaces of \( V \).
(b) Show that if \( \text{dim}(V) < \infty \), then

\[
\text{dim}(U + W) = \text{dim}(U) + \text{dim}(W) - \text{dim}(U \cap W).
\]
(a) The proof that $U + W$ and $U \cap W$ are subspaces of $V$ involves showing closure.

(b) Let $B_{U \cap W} = \{v_1, \ldots, v_k\}$ be a basis for $U \cap W$. Note that if $U \cap W = \{0\}$, then $B_{U \cap W} = \emptyset$. We can obtain bases for $U$ and $W$ by extending $B_{U \cap W}$:

$$B_U = \{v_1, \ldots, v_k, u_1, \ldots, u_n\}$$
$$B_W = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}$$

Note that if $U \cap W = U$, no new basis elements are added in the extended basis for $U$, and similarly, if $U \cap W = W$, no new basis elements are added in the extended basis for $W$.

Now, consider the set $B_U \cup B_W = \{v_1, \ldots, v_k, u_1, \ldots, u_n, w_1, \ldots, w_m\}$. Any vector in $U + W$ can be expressed as a linear combination of vectors in this set, and the set is linearly independent since the $u_i$s and $w_j$s cannot non-trivially combine to equal $0$ (otherwise they would be in $U \cap W$ in the first place). This set is therefore a basis for $U + W$. Therefore we have:

$$\dim(U + W) = k + n + m$$
$$= (k + n) + (k + m) - k$$
$$= \dim(U) + \dim(W) - \dim(U \cap W).$$

Note that this solution works because we built bases for $U$ and $W$ from $U \cap W$ up, not the other way around; it is very difficult in general to extract a basis for $U \cap W$ from bases $U$ and $W$.

There is an alternative solution that works even in the case that $V$ is infinite-dimensional, which will make sense now that you have learned direct sums. Recall that the direct sum $U \oplus W$ is the result of imposing a natural vector space structure on the cartesian product $U \times W$.

Define a linear map $L : U \oplus W \to V$ such that $L(u, w) = u - w$ for all $u \in U$ and $w \in W$.

Then $\text{Im}(L) = \{u - w \mid u \in U, w \in W\} = \{u + w \mid u \in U, w \in W\} = U + W$.

Also, $\text{Ker}(L) = \{(u, w) \mid u - w = 0\}$. Now, $u - w = 0$ implies $u = w$ and therefore $u \in U \cap W$, $w \in U \cap W$. Hence $\text{Ker}(L) = \{(v, v) \mid v \in U \cap W\}$ and hence is isomorphic to $U \cap W$, with the isomorphism $\varphi(v, v) = v$.

By the rank-nullity theorem, $\dim(\text{Ker}(L)) + \dim(\text{Im}(L)) = \dim(U \oplus W)$. Now $\text{Ker}(L) \cong U \cap W$ and $\text{Im}(L) = U + W$, and it is left as an exercise to see that $\dim(U + W) = \dim(U) + \dim(W)$. Therefore, we conclude that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$ as required.