STOKES THEOREM II

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\[ \text{curl}(\nabla \times \mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) \]

**Fundamental theorem of line integrals**

\[ \int_{r(a)}^{r(b)} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \times dS \]

**STOKES THEOREM**

\[ \int_{C} \mathbf{F} \times dS = \int_{S} (\nabla \times \mathbf{F}) \cdot dS \]

**REMINDERS:**

- **CURL:** \( \mathbf{A} \times \mathbf{B} \)
- **GRAD:** \( \nabla f \)
- **FTL:** \( f \circ \mathbf{F} = 0 \)
- **STOKES:** \( \int_{C} \mathbf{F} \times dS = \int_{S} (\nabla \times \mathbf{F}) \cdot dS \)

**Proof:**

We already know that \( F = \nabla f \) implies \( \text{curl}(F) = 0 \). To show the converse, we verify that the line integral along any closed curve \( C \) in \( D \) is zero. This is equivalent to the path independence and allows the construction of the potential \( f \) with \( F = \nabla f \).

By assumption, we can deform the curve to a point: if \( r_0(t) \) is the original curve and \( r_1(t) \) is the curve \( r_1(t) = P \) which stays at one point, define a parametrized surface \( S \) by \( r(t, s) = r_1(t) \). By assumption, \( \text{curl}(F) = 0 \) and therefore the flux of \( \text{curl}(F) \) through \( S \) is zero. By Stokes theorem, the line integral along the boundary \( C \) of the surface \( S \) is zero too.

**THE NASH PROBLEM.** Nash challenged his multivariable class in the movie "A beautiful mind" with a problem, where the region is not simply connected.

Find a region \( X \) of \( \mathbb{R}^3 \) with the property that if \( V \) is the set of vector fields \( F \) on \( \mathbb{R}^3 \) which satisfy \( \text{curl}(F) = 0 \) and \( W \) is the set of vector fields \( F \) which are conservative: \( F = \nabla f \). Then, the space \( V/W \) should be 8 dimensional.

**You solve this problem as an inclass exercise (ICE). The problem is to find a region \( D \) in space, in which one can find 8 different closed paths \( C_i \) so that for every choice of constants \( (c_1, ..., c_8) \), one can find a vector field \( F \) which has zero curl in \( D \) and for which \( \int_{C_i} F \cdot dr = c_i \).

One of the many solutions is cut out 8 tori from space. For each torus, there is a vector field \( F_i \) (a vortex ring), which has its vorticity located inside the ring and such that the line integral of a path which winds once around the ring is 1. The vector field \( F = c_1 F_1 + ... + c_8 F_8 \) has the required properties.

**CLOSED SURFACES.** Surfaces with no boundaries are called closed surfaces. For example, the surface of a donut, or the surface of a sphere are closed surfaces. A half sphere is not closed, its boundary is a circle. Half a doughnut is not closed. Its boundary consists of two circles.

**THE ONE MILLION DOLLAR QUESTION.** One of the Millennium problems is to determine whether any three dimensional space which is simply connected is deformable to a sphere. This is called the Poincare conjecture.

**LINE INTEGRAL IN HIGHER DIMENSIONS.** Line integrals are defined in the same way in higher dimensions. \( \int_{C} F \cdot dS \), where is the dot product in \( d \) dimensions and \( dr = x^i dx^i \).

**CURL IN HIGHER DIMENSIONS.** In \( d \) dimensions, the curl is the field \( \text{curl}(F) = \partial_i F_j - \partial_j F_i \) with \( \binom{d}{2} \) components. In 4 dimensions, it has 6 components. In 2 dimensions it has 1 component, in 3 dimensions, it has 3 components.

**SURFACE INTEGRAL IN HIGHER DIMENSIONS.** In \( d \) dimensions, a surface element in the \( ij \)-plane is written as \( ds_{ij} \). The flux integral of the curl of \( F \) through \( S \) is defined as \( \int_{S} (\text{curl}(F)) \cdot ds \), where the dot product is \( \sum_{i<j} \text{curl}(F)_{ij} ds_{ij} \).

**STOKES THEOREM IN HIGHER DIMENSIONS.** If \( S \) is a two dimensional surface in \( d \)-dimensional space and \( C \) is its boundary, then \( \int_{C} \text{curl}(F) \cdot ds = \int_{S} (\nabla \times \mathbf{F}) \cdot dS \).