ORTHOGRAPHIC PROJECTIONS

HOMEWORK: 5.1: 8, 16, 18, 20, 28, 38., 14.

ORTHOGRAPHICITY. \( \vec{v} \) and \( \vec{w} \) are called orthogonal if \( \vec{v} \cdot \vec{w} = 0 \).

Examples. 1) \( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \) and \( \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \) are orthogonal in \( \mathbb{R}^3 \). 2) \( \vec{v} \) and \( \vec{w} \) are both orthogonal to \( \vec{v} \times \vec{w} \) in \( \mathbb{R}^3 \).

\( \vec{v} \) is called a unit vector if \( ||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = 1 \). \( \vec{v} \) is called orthogonal if they are all unit vectors. A basis is called an orthonormal basis if it is orthonormal. For an orthonormal basis, the matrix \( A_{ij} = \vec{e}_i \cdot \vec{e}_j \) is the unit matrix.

FACT. Orthogonal vectors are linearly independent and in \( \mathbb{R}^n \) form a basis.

Proof. The dot product of a linear relation \( a_1 \vec{v}_1 + \ldots + a_n \vec{v}_n = \vec{0} \) gives \( a_1 \vec{v}_1 \cdot \vec{v}_1 + \ldots + a_n \vec{v}_n \cdot \vec{v}_n = 0 \) so that \( a_k = 0 \). If we have \( n \) linear independent vectors in \( \mathbb{R}^n \) they then automatically span the space.

ORTHOGONAL COMPLEMENT. A vector \( \vec{w} \in \mathbb{R}^n \) is called orthogonal to a linear space \( V \) in \( \mathbb{R}^n \) to every \( \vec{v} \in V \). The orthogonal complement of a linear space \( V \) is the set \( W \) of all vectors which are orthogonal to \( V \). It forms a linear space because \( \vec{v} \cdot \vec{w} = 0 \) for any \( \vec{v}, \vec{w} \in V \). If \( \vec{v} \) and \( \vec{w} \) are linearly independent vectors in \( \mathbb{R}^n \) they then automatically span the space.

EXAMPLE ENTIRE SPACE: for an orthonormal basis \( \vec{e}_1, \ldots, \vec{e}_n \) of the entire space \( \mathbb{R}^n \).

EXAMPLE LINE: if \( \vec{v} \) is a unit vector then \( \text{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{v}) \vec{v} \) is the vector projection we know from multi variable calculus.

ORTHOGONAL PROJECTION. The orthogonal projection onto a linear space \( V \) with orthonormal basis \( \vec{e}_1, \ldots, \vec{e}_n \) is the linear map \( \text{proj}_V(\vec{x}) = \sum (\vec{x} \cdot \vec{e}_i) \vec{e}_i \).

EXAMPLE: \( \text{proj}_{\mathbb{R}^2}(\vec{x}) = (\vec{x} \cdot \vec{e}_1) \vec{e}_1 + (\vec{x} \cdot \vec{e}_2) \vec{e}_2 \).

ORTHOGONALITY. \( \vec{v} \) and \( \vec{w} \) are orthogonal, then \( ||\vec{v} + \vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 \).

PROJECTIONS DO NOT INCREASE LENGTH: \( ||\text{proj}_V(\vec{x})|| \leq ||\vec{x}|| \).

Proof. Use Pythagoras: on \( \vec{x} = \text{proj}_V(\vec{x}) + (\vec{x} - \text{proj}_V(\vec{x})). \) If \( \text{proj}_V(\vec{x}) = \vec{0} \), then \( \vec{x} \in V \).

CAUCHY-SCHWARTZ INEQUALITY: \( ||\vec{x} \cdot \vec{y}|| \leq ||\vec{x}|| ||\vec{y}|| \).

Proof. \( (\vec{x} \cdot \vec{y})(\vec{x} \cdot \vec{y}) = ||\vec{x}||^2 ||\vec{y}||^2 + 2 \vec{x} \cdot \vec{y} \cdot ||\vec{x}|| ||\vec{y}|| \).

ANGLE. The angle between two vectors \( \vec{x}, \vec{y} \) is \( \alpha = \arccos \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| ||\vec{y}||} \).

CORRELATION. \( \cos(\alpha) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| ||\vec{y}||} \) is the correlation of \( \vec{x} \) and \( \vec{y} \).

EXAMPLE. The angle between two orthogonal vectors is 90 degrees or 270 degrees. If \( \vec{x} \) and \( \vec{y} \) represent data showing the deviation from the mean, then \( \frac{\vec{x} \cdot \vec{y}}{||\vec{x}|| ||\vec{y}||} \) is called the statistical correlation of the data.

QUESTION. Express the fact that \( \vec{x} \) is in the kernel of a matrix \( A \) using orthogonality.

\( \text{ANSWER: } A \vec{x} = 0 \) means that \( \vec{x} \cdot \vec{w} = 0 \) for every row vector \( \vec{w} \) of \( A \).

REMARK. We will call later the matrix \( A^T \) obtained by switching rows and columns of \( A \) the transpose of \( A \). You see already that the image of \( A^T \) is orthogonal to the kernel of \( A \).

QUESTION. Find a basis for the orthogonal complement of the linear space \( V \) spanned by \( \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \).

\( \text{ANSWER: } \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \) and \( \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix} \) to the two vectors means solving the linear system of equations \( x + 2y + 3z + 4w = 0 \) and \( 4x + 5y + 6z + 7w = 0 \). An other way to solve it: the kernel of \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix} \) is the orthogonal complement of \( V \). This reduces the problem to an older problem.

ON THE RELEVANCE OF ORTHOGONALITY.

1) During the pyramid age in Egypt (from -2800 til -2300 BC), the Egyptians used ropes divided into length ratios 3 : 4 : 5 to build triangles. This allowed them to triangulate areas quite precisely: for example to build irrigation needed because the Nile was reshaping the land constantly or to build the pyramids: for the great pyramid at Giza with a base length of 230 meters, the error average on each side is less then 20cm, an error of less then 1/1000. A key to achieve this was orthogonality.

2) During one of Thales (-624 BC to -548 BC) journeys to Egypt he used a geometrical trick to measure the height of the great pyramid. He measured the size of the shadow of the pyramid. Using a stick, he found the relation between the length of the stick and the length of its shadow. The same length ratio applies to the pyramid (orthogonal triangles). Thales found also these triangles inscribed into a circle and having as the base as the diameter must have a right angle.

3) The Pythagoreans (-572 until -507) were interested in the discovery that the squares of a lengths of a triangle with two orthogonal sides would add up as \( a^2 + b^2 = c^2 \). They were puzzled in assigning a length to the diagonal of the unit square. This number is irrational because \( \sqrt{2} = \frac{p}{q} \) would imply that \( q^2 = 2p^2 \). While the prime factorization of \( q^2 \) contains an even power of 2, the prime factorization of \( 2p^2 \) contains an odd power of 2.

4) Eratosthenes (-274 until 194) realized that while the sun rays were orthogonal to the ground in the town of Scen, this did no more do so at the town of Alexandria, where they would hit the ground at 7.2 degrees). Because the distance was about 500 miles and 7.2 is 1/50 of 360 degrees, he measured the circumference of the earth at 25000 miles - pretty close to the actual value 24874 miles.

5) Closely related to orthogonality is parallelism. For a long time mathematicians tried to prove Euclid’s parallel axiom using other postulates of Euclid (-325 until -265).

These attempts had to fail because there are geometries in which parallel lines always meet (like on the sphere) or geometries, where parallel lines never meet (the Poincaré half plane). Also these geometries can be studied using linear algebra. The geometry on the sphere with rotations, the geometry on the half plane uses Möbius transformations, \( 2 \times 2 \) matrices with determinant one.

6) The question whether the angles of a right triangle are in reality always add up to 180 degrees became an issue when geometries where discovered, in which the measurement depends on the position in space. Riemannian geometry, found 150 years ago, is the foundation of general relativity a theory which describes gravity geometrically: the presence of mass bends space-time, where the dot product can depend on space. Orthogonality becomes relative too.

7) In probability theory the notion of independence or decorrelation is used. For example, when throwing a dice, the number shown by the first dice is independent and decorrelated from the number shown by the second dice. Decorrelation is identical to orthogonality, when vectors are associated to the random variables. The correlation coefficient between two vectors \( \vec{v}, \vec{w} \) is defined as \( \vec{v} \cdot \vec{w} / (||\vec{v}|| ||\vec{w}||) \). It is the cosine of the angle between these vectors.

8) In quantum mechanics, states of atoms are described by functions in a linear space of functions. The states with energy \( E \) are orthogonal states of atoms are described by functions in a linear space of functions. The states with energy \( E \) are orthogonal. Two states of two different atoms which don’t interact are orthogonal. One of the challenges in quantum computing, where the computation deals with qubits (=vectors) is that orthogonality is not preserved during the computation. Different states can interact. This coupling is called decoherence.