1. The verification that \( \cos(nx), \sin(nx), 1/\sqrt{2} \) form an orthonormal family is an integration computation, when using the identities provided. For example, \( \langle \cos(nx), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin((n-m)x) \cos((n+m)x) \, dx \) which is equal to 1 if \( n = m \) and equal to 0 if \( n \neq m \). The computations can be abbreviated by noting that integrating an odd 2\( \pi \)-periodic function over \([-\pi, \pi]\) is zero because the integral on \([0, \pi]\) cancels with the integral on \([-\pi, 0]\).

2. First get the Fourier series of the function \( f(x) = |x| \). This is an even function so that it has a cos series. We compute

\[
\begin{align*}
a_0 &= \langle f, 1/\sqrt{2} \rangle = \frac{2}{\pi} \int_{0}^{\pi} \frac{x}{\sqrt{2}} \, dx = \frac{\pi \sqrt{2}}{2}.
\end{align*}
\]

\[
\begin{align*}
a_n &= \langle f, \cos(nx) \rangle = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) \, dx = \frac{2}{\pi} \left[ \frac{\cos(n\pi) - 1}{n^2} \right].
\end{align*}
\]

The Fourier coefficients of \( f(x) = 5 - |2x| \) are given as follows:

\[
\begin{align*}
a_0 &= \frac{3\pi \sqrt{2}}{2} + 5\sqrt{2}
\end{align*}
\]

and

\[
\begin{align*}
a_n &= -\frac{4}{\pi} \left[ \frac{\cos(n\pi) - 1}{n^2} \right].
\end{align*}
\]

3. The Fourier series of \( 4\cos^2(3x) + 5\sin^2(11x) + 90 \) is with

\[
\begin{align*}
\cos^2(3x) &= \frac{1 + \cos(6x)}{2}, \\
\sin^2(11x) &= \frac{1 - \cos(22x)}{2}
\end{align*}
\]

given as \( \frac{4\cos(6x)/2 - 5\cos(22x)/2 + 9/2 + 90}{2} \). All Fourier coefficients are zero except \( a_0 \) and \( a_6 \) and \( a_{11} \).

4. To find the Fourier series of the function \( f(x) = |\sin(x)| \), we first note that this is an even function so that it has a cos-series. If we integrate from 0 to \( \pi \) and multiply the result by 2, we can take the function \( \sin(x) \) instead of \( |\sin(x)| \) so that

\[
\begin{align*}
a_0 &= \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin(x)/\sqrt{2}}{\sqrt{2}} \, dx = \frac{2\sqrt{2}}{\pi},
\end{align*}
\]

\[
\begin{align*}
a_n &= \frac{2}{\pi} \int_{0}^{\pi} \sin(nx) \cos(nx) \, dx = \frac{4}{\pi} \frac{1}{1 - n^2}
\end{align*}
\]

for even \( n \) and \( a_n = 0 \) for odd \( n \). To do the integral, use the trigonometric identities \( 2\sin(x)\cos(nx) = \sin(x + nx) + \sin(x - nx) \). We have

\[
\begin{align*}
f(x) &= \frac{2}{\pi} - \frac{4}{\pi} \left( \frac{\cos(2x)}{2^2 - 1} + \frac{\cos(4x)}{4^2 - 1} + \frac{\cos(6x)}{6^2 - 1} + \ldots \right).
\end{align*}
\]

5. The square of the length of the function \( f(x) \) is 1. The Parseval identity shows that

\[
1 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 = \left( \frac{\sqrt{2}}{\pi} \right)^2 + \frac{16}{\pi^2} \left( \frac{1}{(2^2 - 1)^2} + \frac{1}{(4^2 - 1)^2} + \frac{1}{(6^2 - 1)^2} + \cdots \right)
\]

so that the sum is \( \pi^2/16 - 1/2 \).

6. To solve the heat equation \( f_t = 5f_{xx} \) on \([0, \pi]\) with the initial condition \( f(x,0) = \max_{x \in [0,\pi]} |\cos(x)| \), 0, we make a Fourier expansion

\[
\begin{align*}
f(x) &= \sum_{n=1}^{\infty} b_n \sin(nx).
\end{align*}
\]

of the later function and get solution

\[
\begin{align*}
f(x,t) &= \sum_{n=1}^{\infty} b_n e^{-\pi^2 n^2 t} \sin(nx).
\end{align*}
\]

Now to the Fourier coefficients:

\[
\begin{align*}
b_n &= \frac{2}{\pi} \int_{0}^{\pi/2} \cos(x) \sin(nx) \, dx.
\end{align*}
\]

We use the trig identity

\[
\cos(x) \sin(ny) = \sin((n + 1)x) + \sin((n - 1)x)
\]

to solve these integrals:

\[
\begin{align*}
b_n &= \frac{2}{\pi} \left( \frac{1 - \cos((n + 1)\pi/2)}{(n + 1)} + \frac{1 - \cos((n - 1)\pi/2)}{(n - 1)} \right).
\end{align*}
\]

7. The operator \( D^4 + D^2 \) has the eigenvectors \( \sin(nx) \) with eigenvalues \( n^4 - n^2 \). With initial condition \( f(x) = b_n \sin(nx) \) we have the solution \( b_n \sin(nx)e^{(n^2 - n^2)t} \). The function \( x^3 \) has the Fourier coefficients \( \frac{2}{\pi} \left( 6 - n^2 \pi^2 \right)^{-1/2} \).

8. Because the initial condition is zero on the interval \([\pi/2, \pi]\), we have to integrate from 0 to \( \pi/2 \) only. The Fourier coefficients of the function \( g(x) \) can be computed using one of the trigonometric identities you find on the first page of the handout:

\[
\begin{align*}
\frac{2}{\pi} \int_{0}^{\pi/2} \sin(2x) \sin(nx) \, dx &= \frac{-4}{\pi(n^2 - 4)} \sin(n\pi/2).
\end{align*}
\]

The Fourier series of the initial position \( f(x) = 0 \) of the string is equal to zero by assumption. The solution of the wave equation is

\[
\begin{align*}
f(x,t) &= \sum_{n=1}^{\infty} \frac{-4}{\pi(n^2 - 4)} \sin(n\pi/2) \sin(nx) \sin(n^2 t)^{1/2}.
\end{align*}
\]

The solution also exists for \( n = 2 \), where it is 1/2 (which can best be seen by evaluating the original integral \( \frac{2}{\pi} \int_{0}^{\pi/2} \sin^2(2x) \, dx = 1/2 \).
The general solution of the homogeneous equation with the function at rest initially is
\[ u_h(t, x) = \sum b_n \sin(nx) \cos(nt) = 4 \cos(5t) \sin(5x) + 10 \cos(6t) \sin(6x). \] A particular solution which is zero at 0 is
\[ u_p(t, x) = -\cos(t) - \cos(3t)/9 + (1 - 1/9). \] Now fix the Fourier coefficients. We end up with
\[ u(t, x) = 4 \sin(5x) \cos(5t) + 10 \sin(6x) \cos(6t) - \cos(t) - \cos(3t)/9 + 8/9 \]
which is the solution to the differential equation.

10. a) If the function is \(\text{sign}(xy)\), we have
\[ b_{nm} = \frac{4}{\pi^2} \int_0^\pi \int_0^\pi \sin(nx) \sin(ny) \, dy \, dx \]
which is
\[ \frac{4}{\pi^2} \left( -\frac{\cos(nx) n}{n} \right) \left( -\frac{\cos(ny) m}{m} \right) = \frac{16}{\pi^2} \frac{1}{nm}. \]
The Fourier coefficients are \(\frac{16}{\pi^2} \) if \( n, m \) are both odd and zero else.

b) Since every initial condition \( u = b_{nm} \sin(nx) \sin(ny) \) satisfies the ordinary differential equation \( u_t = (-n^2 - m^2) u \) with solution \( u(t) = e^{-n^2 - m^2 t} u(0) = e^{-n^2 - m^2 t} b_{nm} \sin(nx) \sin(ny) \), we can add up a linear combination of such solutions and get
\[ u(x, y, t) = \sum_{n, m=1}^N b_{nm} e^{-(n^2 + m^2) t} \sin(nx) \sin(ny). \]
With the Fourier coefficients computed in part a), we have the final answer
\[ u(x, y, t) = \sum_{n, m=odd}^\infty \frac{16 e^{-(n^2 + m^2) t}}{\pi nm} \sin(nx) \sin(ny). \]