Diagonalization

Let \( L \) be the line \( y = 2x \) in \( \mathbb{R}^2 \). Let \( \text{ref}_L \) be reflection over \( L \), and let \( A \) be the standard matrix of \( \text{ref}_L \).

1. Find an eigenbasis \( \mathfrak{B} \) for \( A \).

**Solution.** Let \( \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). Then, \( \vec{v}_1 \) lies on the line \( L \), so \( A\vec{v}_1 = \text{ref}_L(\vec{v}_1) = \vec{v}_1 \).

On the other hand, \( \vec{v}_2 \) is perpendicular to \( L \), so \( A\vec{v}_2 = \text{ref}_L(\vec{v}_2) = -\vec{v}_2 \). Thus, \( \vec{v}_1 \) and \( \vec{v}_2 \) are both eigenvectors of \( A \). Since \( (\vec{v}_1, \vec{v}_2) \) is clearly a basis of \( \mathbb{R}^2 \), \( \mathfrak{B} = (\vec{v}_1, \vec{v}_2) \) is an eigenbasis for \( A \).

2. Find the \( \mathfrak{B} \)-matrix of \( \text{ref}_L \).

**Solution.** If \( D \) is the \( \mathfrak{B} \)-matrix of \( \text{ref}_L \), the columns of \( D \) are \( [\text{ref}_L(\vec{v}_1)]_{\mathfrak{B}} \) and \( [\text{ref}_L(\vec{v}_2)]_{\mathfrak{B}} \). Since \( \text{ref}_L(\vec{v}_1) = \vec{v}_1 = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 \), \( [\text{ref}_L(\vec{v}_1)]_{\mathfrak{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Since \( \text{ref}_L(\vec{v}_2) = -\vec{v}_2 = 0 \cdot \vec{v}_1 + (-1) \cdot \vec{v}_2 \), \( [\text{ref}_L(\vec{v}_2)]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \). So, the \( \mathfrak{B} \)-matrix of \( \text{ref}_L \) is \( D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \).

3. Find \( A \) (the standard matrix of \( \text{ref}_L \)).

**Solution.** If \( S = [\vec{v}_1 \, \vec{v}_2] \), then \( A = SDS^{-1} \). In this case, this says that

\[
A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}.
\]

True / False

1. If \( A \) is diagonalizable, then \( A^2 \) is diagonalizable.

**Solution.** True. Since \( A \) is diagonalizable, there is an invertible matrix \( S \) such that \( S^{-1}AS \) is a diagonal matrix. Then, \( S^{-1}A^2S = (S^{-1}AS)(S^{-1}AS) \) is the product of two diagonal matrices, which is a diagonal matrix. Therefore, \( A^2 \) is diagonalizable.

2. If \( A \) and \( B \) are \( n \times n \) diagonalizable matrices, then \( A + B \) is diagonalizable.

**Solution.** False. For example, \( A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \) are both diagonalizable, but \( A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) is not.

3. If \( A \) and \( B \) are \( n \times n \) diagonalizable matrices with the same eigenvectors, then \( AB \) is diagonalizable.

**Solution.** True. Since \( A \) is diagonalizable, there is an eigenbasis for \( A \), say \( (\vec{v}_1, \ldots, \vec{v}_n) \). Since \( B \) has the same eigenvectors as \( A \), \( (\vec{v}_1, \ldots, \vec{v}_n) \) is also an eigenbasis for \( B \). Therefore, if \( S = [\vec{v}_1 \ldots \vec{v}_n] \), \( S^{-1}AS \) and \( S^{-1}BS \) are both diagonal matrices. If we multiply two diagonal matrices, we get another diagonal matrix. Thus, \( (S^{-1}AS)(S^{-1}BS) = S^{-1}ABS \) is a diagonal matrix, so \( AB \) is diagonalizable.

4. If \( A \) is diagonalizable, then \( A^T \) is diagonalizable.

**Solution.** True. Since \( A \) is diagonalizable, there is an invertible matrix \( S \) such that \( S^{-1}AS \) is a diagonal matrix \( D \). Then, \( (S^{-1}AS)^T \) is equal to \( D^T \), which is the same as \( D \). On the other hand, \( (S^{-1}AS)^T \) is just \( S^T A^T (S^{-1})^T \). Thus, if \( R = (S^T)^{-1} \), then \( R^{-1}A^TR = D \), so \( A^T \) is diagonalizable.
5. If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues, then $A$ is diagonalizable.

Solution. True. The geometric multiplicity of any eigenvalue is at least 1, so, if $A$ has $n$ distinct eigenvalues, then the sum of the geometric multiplicities of the eigenvalues is $n$. Therefore, $A$ has a basis of eigenvectors, so $A$ is diagonalizable.

6. If $A$ is a diagonalizable matrix and $\lambda$ is an eigenvalue of $A$, then the algebraic multiplicity of $\lambda$ is equal to the geometric multiplicity of $\lambda$.

Solution. True. Let's say $A$ is an $n \times n$ matrix, and let $\lambda$ be an eigenvalue. Then, we know:

- The algebraic multiplicity of $\lambda$ is greater than or equal to the geometric multiplicity of $\lambda$.
- The sum of the algebraic multiplicities of all eigenvalues is at most $n$ (the degree of the characteristic polynomial of $A$).
- Since $A$ is diagonalizable, the geometric multiplicities of the eigenvalues of $A$ must add up to $n$.

Thus, the algebraic multiplicity of $\lambda$ must be the same as the geometric multiplicity of $\lambda$ (otherwise the sum of the algebraic multiplicities would be greater than $n$).

7. If $A$ and $B$ are both diagonalizable and if $A$ and $B$ have the same eigenvalues with the same geometric multiplicities, then $A$ is similar to $B$.

Solution. True. Since $A$ is diagonalizable, the geometric multiplicities of its eigenvalues add up to $n$. That is, $A$ has $n$ eigenvalues $\lambda_1, \ldots, \lambda_n$ (if we count the eigenvalues with their geometric multiplicities). Then, $A$ is similar to the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$ 

Since $B$ has the same eigenvalues with the same geometric multiplicities and $B$ is diagonalizable, $B$ is also similar to $D$. Thus, $A$ and $B$ are both similar to $D$, so they must be similar to each other.